Poincaré constant on manifolds with ends

Alexander Grigor'yan*
Department of Mathematics
University of Bielefeld
33501 Bielefeld, Germany
grigor@math.uni-bielefeld.de

Satoshi Ishiwata†
Department of Mathematical Sciences
Yamagata University
Yamagata 990-8560, Japan
ishiwata@sci.kj.yamagata-u.ac.jp

Laurent Saloff-Coste ‡
Department of Mathematics
Cornell University
Ithaca, NY, 14853-4201, USA
lsc@math.cornell.edu

June 8, 2022

Abstract

We obtain optimal estimates of the Poincaré constant of central balls on manifolds with finitely many ends. Surprisingly enough, the Poincaré constant is determined by the *second* largest end. The proof is based on the argument by Kusuoka-Stroock where the heat kernel estimates on the central balls play an essential role. For this purpose, we extend earlier heat kernel estimates obtained by authors to a lager class of parabolic manifolds with ends.

2010 Mathematics Subject Classification: Primary 58C40, Secondary 35K08, 58J65 and 58J35

Keywords: Poincaré inequality, Poincaré constant, manifold with ends, model manifold, heat kernel

Contents

*Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - SFB 1283/2 2021 - 317210226

†Partially supported by JSPS KAKENHI 17K05215 and 22K03280

‡Partially supported by NSF grant DMS-1707589 and DMS-2054593

1 Introduction

Let M be a Riemannian manifold. Denote by μ the Riemannian measure on M and by ∇ the gradient. For a precompact connected open set $U \subset M$, define the *Poincaré constant* $\Lambda(U)$ as the smallest number such that the following inequality holds for all $f \in C^1(\overline{U})$:

$$\int_U |f - f_U|^2 d\mu \le \Lambda(U) \int_U |\nabla f|^2 d\mu,$$

where $f_U := \frac{1}{\mu(U)} \int_U f d\mu$. Equivalently, we have

$$\Lambda(U) = \frac{1}{\lambda(U)},$$

where $\lambda(U)$ is the smallest positive eigenvalue of $-\Delta$ in U with the Neumann condition on ∂U. Here Δ is the Laplace-Beltrami operator on M. Estimating the Poincaré constant has many applications. See [1], [4], [5], [6], [8], [26] for examples and references therein.

Denote by $d(x,y)$ the geodesic distance on M and by $B(x,r)$ – open geodesic balls on M. In this paper we are concerned with estimating the Poincaré constant $\Lambda(B(x,r))$. It is well-known that in $\mathbb{R}^n$

$$\Lambda(B(x,r)) = C_n r^2.$$

It is also known by [23] that on complete non-compact manifolds with non-negative Ricci curvature

$$\Lambda(B(x,r)) \simeq r^2. \tag{1.1}$$

There are other class of manifolds satisfying (1.1), for example, Lie groups of polynomial volume growth (see [9] and [10]), the tube manifold around the square lattice $\mathbb{Z}^d$ (or jungle gym).

However, there are natural examples of manifolds where (1.1) does not hold, for example, the hyperbolic spaces where $\Lambda(B(x,r))$ grows exponentially in r. Another example that is more relevant for this paper is the connected sum $\mathbb{R}^n \# \mathbb{R}^n$, where $n \ge 2$. By $\mathbb{R}^n \# \mathbb{R}^n$ we denote any manifold that is obtained by gluing together two copies of $\mathbb{R}^n$ over a compact tube (see Fig. 1).

It follows from the results of this paper (Theorems 2.7 and 2.10) that on such a manifold

$$\Lambda(B(o,r)) \simeq \begin{cases} r^n & \text{if } n > 2, \\ r^2 \log r & \text{if } n = 2, \end{cases}$$

that is,

$$\lambda(B(o,r)) \simeq \begin{cases} \frac{1}{r^n} & \text{if } n > 2, \\ \frac{1}{r^2 \log r} & \text{if } n = 2 \end{cases}$$

for all large r and for a central reference point $o \in \mathbb{R}^n \# \mathbb{R}^n$ (see Section 3). In these simple cases, these results are well-known to the specialists of the subject.

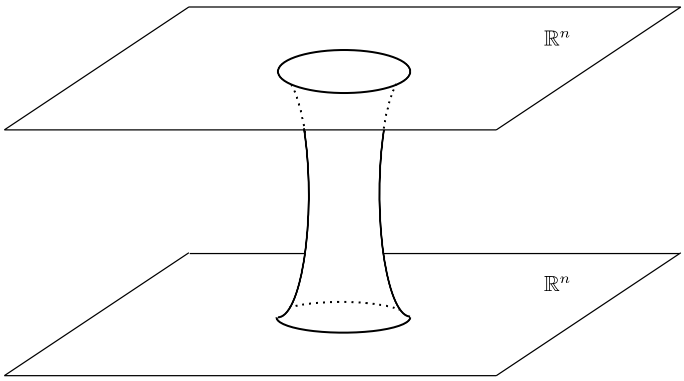

Figure 1: $\mathbb{R}^n\#\mathbb{R}^n$

Consider now a connected sum $M = M_1\#\cdots\#M_k$ of k model manifolds $M_1, \ldots, M_k$ with a same dimension N (see Section 3 for details of this construction). For example, M_i can be a surface of revolution (see Fig. 2). Assume that the volume growth function $V_i(r)$ of M_i satisfies for some $\alpha_i > 0$ and $\beta_i \in \mathbb{R}$

$$V_i(r) \simeq r^{\alpha_i}(\log r)^{\beta_i}, \quad (i = 1, \ldots, k, \ r \gg 1).$$

We assume that

$$(N, 0) \succeq (\alpha_1, \beta_1) \succeq (\alpha_2, \beta_2) \succeq \cdots \succeq (\alpha_k, \beta_k)$$

in the sense of the lexicographical order which implies that

$$V_1(r) \gtrsim V_2(r) \gtrsim \cdots \gtrsim V_k(r)$$

(see Fig. 2). It follows from the main results of this paper (Theorems 2.7 and 2.10) that the Poincaré constant $\Lambda(B(o, r))$ on M is determined, quite surprisingly, solely by the ***second largest*** end M_2:

$$\Lambda(B(o,r)) \simeq \begin{cases} r^{\alpha_2}(\log r)^{\beta_2} & \text{if } (\alpha_2, \beta_2) \succ (2, 1), \\ r^2 (\log r) (\log \log r) & \text{if } (\alpha_2, \beta_2) = (2, 1), \\ r^2 \log r & \text{if } \alpha_2 = 2, \beta_2 < 1, \\ r^2 & \text{if } \alpha_2 < 2 \end{cases}$$

for all large r (see Example 2.12 for details).

Let us mention for comparison that if $V_i(r) \simeq r^{\alpha_i}$ then the heat kernel long time behavior is determined by the end M_i having α_i nearest to 2 (see Example 2.20).

In the next section, we describe a more general class of manifolds with ends where we obtain two sided estimates of the Poincaré constant. The main results are stated in Theorems 2.7 and 2.10. Proofs are given in Sections 5 and 6 and use crucially heat kernel estimates.

Notation. The notation $f \simeq g$ for two non-negative functions f, g means that there are two positive constants c_1, c_2 such that $c_1 g \le f \le c_2 g$ for the specified range of the arguments of f and g. Similarly, the notation $f \gtrsim g$ (resp. $f \lesssim g$) means that there is a positive constant c such that $cf \ge g$ (resp. $f \le cg$) . Throughout this article, the letters $c, C, b, \ldots$ denote positive constants whose values may be different at different instances. When the value of a constant is significant, it will be explicitly stated.

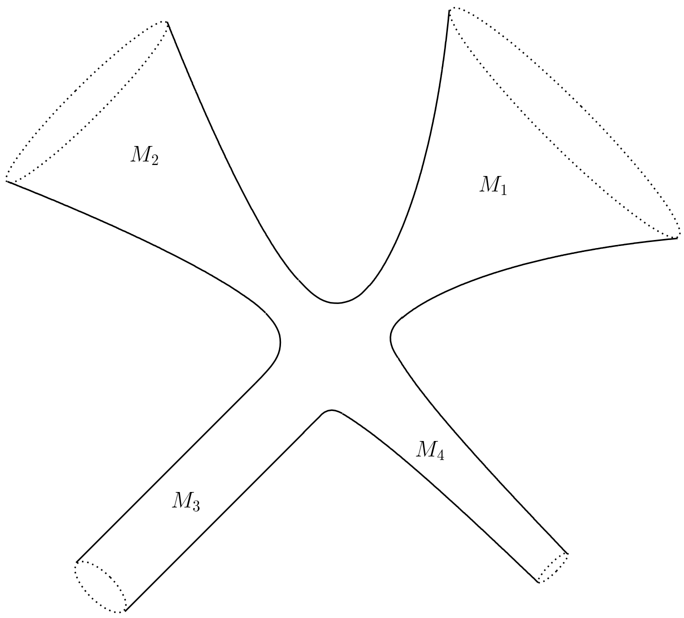

Figure 2: Connected sum of model manifolds M_1, M_2, M_3, M_4

2 Main results

2.1 Heat kernels

Let us recall some known results about the heat kernel on manifolds. Let M be a Riemannian manifold. Denote by vol the Riemannian measure on M. Given a smooth positive function σ on M, define a measure μ on M by $d\mu = \sigma d\text{vol}$. The pair (M, μ) is called a *weighted manifold*. Any Riemannian manifold can be considered as a weighted manifold with $\sigma = 1$. The Laplace operator Δ of the weighted manifold (M, μ) is defined by

$$\Delta = \frac{1}{\sigma}\text{div}(\sigma\nabla),$$

where div and ∇ are the divergence and the gradient of the Riemannian metric of M. The operator Δ is known to be symmetric with respect to the measure μ (see [12]).

Set $V(x, r) = \mu(B(x, r))$.

Definition 2.1. We say that a manifold M satisfies the *volume doubling* condition (VD) if there exists a constant C such that, for all $x \in M$ and $r > 0$,

$$V(x, 2r) \leq CV(x, r).$$

Definition 2.2. We say that a manifold M admits the *scale invariant Poincaré inequality* (PI) if there exist constants $C > 0$ and $\kappa \geq 1$ such that, for all $x \in M$ and $r > 0$, and for all $f \in C^1(\overline{B(x, \kappa r)})$,

$$\int_{B(x,r)} |f - f_{B(x,r)}|^2 d\mu \leq Cr^2 \int_{B(x,\kappa r)} |\nabla f|^2 d\mu.$$

Denote by $p(t, x, y)$ the heat kernel of M, that is, the minimal positive fundamental solution of the heat equation $\partial_t u = \Delta u$. The following theorem is a combined result of [10], [25] based on previous contributions of Moser [24], Kusuoka–Stroock [21] et al.

Theorem 2.3. *On a geodesically complete, non-compact weighted manifold M, the following conditions are equivalent:*

(i) (PI) *and* (VD)*.*

(ii) The Li-Yau type heat kernel estimates:

$$p(t,x,y) \asymp \frac{C}{V(x,\sqrt{t})} \exp\left(-b\frac{d^2(x,y)}{t}\right), \tag{2.1}$$

where the sign $\asymp$ means that both $\le$ and $\ge$ hold but with different values of the positive constants C and b.

(iii) The uniform parabolic Harnack inequality (for the definition see [18, Section 2.1]).

The estimate (2.1) was proved for the first time by Li and Yau [23] on manifolds of non-negative Ricci curvature.

2.2 Manifold with ends

Fix a natural number $k \ge 2$. Let $M_1, ..., M_k$ be a sequence of geodesically complete, non-compact weighted manifolds of the same dimension.

Definition 2.4. Let M be a weighted manifold. We say that M is a manifold with k ends $M_1, \ldots, M_k$ and write

$$M = M_1 \# ... \# M_k \tag{2.2}$$

if there is a compact set $K \subset M$ so that $M \setminus K$ consists of k connected components $E_1, E_2, \ldots, E_k$ such that each E_i is isometric (as a weighted manifold) to $M_i \setminus K_i$ for some compact set $K_i \subset M_i$ (see Fig. 3). Each E_i will be referred to as an *end* of M.

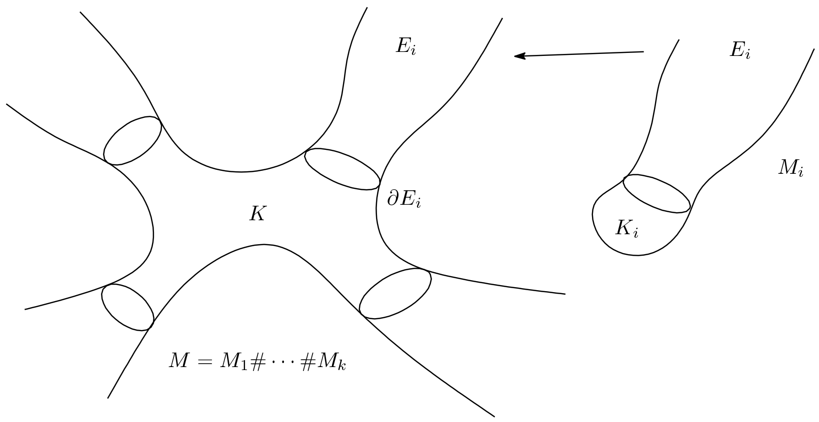

Figure 3: Manifold with ends

For the manifold (2.2) we will also use the notation $M = \#_{i \in I} M_i$ where

$$I = \{1, 2, ..., k\}.$$

It will be important for us to distinguish between *parabolic* and *non-parabolic* manifolds.

Definition 2.5. We say that a weighted manifold N is *parabolic* if the weighted Laplace operator Δ has no positive Green function. Equivalently N is parabolic if and only if

$$\int^{\infty} p(t,x,y)\, dt = \infty \tag{2.3}$$

for all/some $x, y \in N$.

If the integral in (2.3) converges then it determines the minimal positive Green function

$$g(x,y) = \int_0^{\infty} p(t,x,y) dt.$$

If N satisfies (VD) and (PI) then N is parabolic if and only if

$$\int^{\infty} \frac{r dr}{V(x,r)} = \infty$$

for all/some $x \in N$.

For a manifold with ends (2.2), we say that an end E_i is parabolic (or non-parabolic) if M_i is parabolic (resp., non-parabolic). It is easy to verify that M is parabolic if and only if all the ends E_i are parabolic.

In the sequel, we always assume that each end M_i admits the Poincaré inequality (PI) and volume doubling condition (VD), so that M_i satisfies each of the conditions of Theorem 2.3. Besides, if M_i is parabolic then we assume in addition that M_i satisfies the following condition (RCA).

Definition 2.6 (RCA)**.** We say that a Riemannian manifold N has *relatively connected annuli* (shortly (RCA)) with respect to a reference point $o \in N$ if there exist a constant $A > 1$ such that, for any $r > A^2$, any two points $x, y \in N$ satisfying $d(x,o) = d(y,o) = r$, are connected by a continuous path in $B(o, Ar) \setminus B(o, A^{-1}r)$.

For each $i = 1, ..., k$, let μ_i be the reference measure on M_i and d_i be the geodesic distance on M_i. Denote by $B_i(x,r)$ geodesic balls on M_i and set $V_i(x,r) = \mu_i(B_i(x,r))$. Fix a reference point $o_i \in K_i$, set

$$V_i(r) = V_i(o_i, r) \tag{2.4}$$

and refer to V_i as the (pointed) volume function of M_i. Note that for small r, we have $V_i(r) \simeq V_j(r) \simeq r^n$ where $n = \dim M$.

Clearly, M_i is parabolic if and only if

$$\int^{\infty} \frac{r dr}{V_i(r)} = \infty.$$

Define a function $h_i(r)$ for all $r > 0$ by

$$h_i(r) = 1 + \left(\int_1^r \frac{s ds}{V_i(s)} \right)_+ . \tag{2.5}$$

Then M_i is parabolic if $h_i(r) \to \infty$ as $r \to \infty$ and non-parabolic if $h_i \simeq 1$.

For example, if $V_i(r) \simeq r^{\alpha}$ for large r then

$$h_i(r) \simeq \begin{cases} r^{2-\alpha} & \text{if } \alpha < 2, \\ \log r & \text{if } \alpha = 2, \\ 1 & \text{if } \alpha > 2, \end{cases}$$

for large r. In this case M_i is parabolic if and only if $\alpha \leq 2$.

If $V_i(r) \simeq r^2 (\log r)^\beta$ for large r then

$$h_i(r) \simeq \begin{cases} (\log r)^{1-\beta} & \text{if } \beta < 1, \\ \log\log r & \text{if } \beta = 1, \\ 1 & \text{if } \beta > 1, \end{cases}$$

for large r. In this case M_i is parabolic if and only if $\beta \leq 1$.

In the next section we state the main results of this paper about the Poincaré constant on manifolds with ends. For all Poincaré type estimates obtained in this article, the range of interest is $r \gg 1$. Similarly, for all heat kernel estimates, the range of interest is $t \gg 1$.

2.3 Poincaré constant

First let us state a general definition of Poincaré constant. Let (M, μ) be a weight manifold. For precompact connected open sets $U \subset U' \subset M$, define the *Poincaré constant* $\Lambda(U, U')$ of the couple (U, U') as the smallest number such that the following inequality holds for all $f \in C^1(\overline{U'})$:

$$\int_U |f - f_U|^2 d\mu \leq \Lambda(U, U') \int_{U'} |\nabla f|^2 d\mu,$$

where $f_U := \frac{1}{\mu(U)} \int_U f d\mu$. Equivalently, we have

$$\Lambda(U, U') = \sup_{\substack{f \in C^1(\overline{U'}) \\ f \neq \text{const}}} \frac{\inf_{\xi \in \mathbb{R}} \int_U |f - \xi|^2 d\mu}{\int_{U'} |\nabla f|^2 d\mu}. \tag{2.6}$$

We note that the Poincaré inequality (PI) is equivalent to hold for all $x \in M$, and $r > 0$,

$$\Lambda(B(x, r), B(x, \kappa r)) \lesssim r^2.$$

The function $(U, U') \to \Lambda(U, U')$ is clearly monotone in the following sense: if $W \subset U \subset U' \subset W'$ then $\Lambda(W, W') \leq \Lambda(U, U')$. If $U = U'$, then we use a shorter notation

$$\Lambda(U) := \Lambda(U, U).$$

In this case the number

$$\lambda(U) := \frac{1}{\Lambda(U)} = \inf_{\substack{f \in C^1(\overline{U}) \\ f \neq \text{const}}} \frac{\int_U |\nabla f|^2 d\mu}{\int_U |f - f_U|^2 d\mu} \tag{2.7}$$

is the spectral gap of $-\Delta$ on U, that is, the smallest positive eigenvalue of $-\Delta$ in U with the Neumann condition on ∂U.

Let $M = \#_{i \in I} M_i$ be a manifold with ends as described above. For any $r > 0$ define natural numbers $m = m(r)$ and $n = n(r)$ so that

$$V_m(r) = \max_{i \in I} V_i(r) \tag{2.8}$$

and

$$V_n(r) = \max_{i \in I \setminus \{m\}} V_i(r). \tag{2.9}$$

That is, V_m is the largest volume function at scale r, and V_n is the *second largest* volume function at scale r.

Fix a central reference point $o \in K$. We first state our result in the case when all M_i are non-parabolic.

Theorem 2.7 (All ends are non-parabolic)**.** *Let $M = \#_{i\in I} M_i$ be a manifold with ends where each M_i is a geodesically complete, non-compact weighted manifold that satisfies* (PI) *and* (VD)*. Assume that all M_i are non-parabolic. Then, for all $r \gg 1$,*

$$\Lambda(B(o,r)) \lesssim V_n(r), \tag{2.10}$$

where n is defined in (2.9). If, in addition each V_i satisfies for all $r \gg 1$

$$rV_i'(r) \lesssim V_i(r),$$

then, for all $r \gg 1$,

$$\Lambda(B(o,r)) \simeq V_n(r).$$

As this theorem shows, the Poincaré constant is determined by the *second largest* volume function at scale r via the simple formula $\Lambda(B(o,r)) \simeq V_n(r)$.

The case when some ends are parabolic is more challenging and our results are less complete. In this case, we will need the following definition which requires to subdivide the set of ends into three subsets I_{super}, I_{middle} and I_{sub}. The subset I_{super} is made of "large ends". The set of I_{sub} is made of "small ends". Here, large and small are defined by two fixed positive parameter ϵ and δ. Restrictive hypothesis are made on the ends in I_{middle}.

Definition 2.8 (COE)**.** We say that a manifold $M = \#_{i\in I} M_i$ has *critically ordered ends* (COE) if there exist $\varepsilon, \delta, \gamma_1, \gamma_2 > 0$ such that

$$\gamma_1 < \varepsilon, \quad \gamma_1 + \gamma_2 < \delta < 2, \quad 2\gamma_1 + \gamma_2 < 2, \tag{2.11}$$

and a decomposition

$$I = I_{super} \sqcup I_{middle} \sqcup I_{sub} \tag{2.12}$$

such that the following conditions are satisfied:

(a) For each $i \in I_{super}$ and all $r \geq 1$,

$$V_i(r) \gtrsim r^{2+\epsilon}\ .$$

(b) For each $i \in I_{sub}$, V_i is *subcritical*, i.e., for all $r \geq 1$

$$h_i(r) \simeq \frac{r^2}{V_i(r)}, \tag{2.13}$$

and

$$V_i(r) \lesssim r^{2-\delta}\ . \tag{2.14}$$

(c) For each $i \in I_{middle}$,

$$\left(\frac{R}{r}\right)^{2-\gamma_2} \lesssim \frac{V_i(R)}{V_i(r)} \lesssim \left(\frac{R}{r}\right)^{2+\gamma_1} \quad \text{for all } 1 \leq r \leq R. \tag{2.15}$$

For any pair $i, j \in I_{middle}$, $V_i \gtrsim V_j$ or $V_j \gtrsim V_i$ (i.e., the ends in I_{middle} can be ordered according to their volume growth uniformly over $r \in [1, \infty)$) and $V_i \gtrsim V_j$ implies that $V_i h_i \gtrsim V_j h_j$. Finally, if M is parabolic (i.e., all ends are parabolic) then we assume that $V_i \gtrsim V_j$ also implies $V_i h_i^2 \lesssim V_j h_j^2$.

Note that the subcriticality condition (2.13) is equivalent to

$$h_i(r) \lesssim \frac{r^2}{V_i(r)},$$

since the opposite inequality

$$h_i(r) \gtrsim \frac{r^2}{V_i(r)} \tag{2.16}$$

follows trivially from (2.5) and the doubling property of $V_i(r)$. Also, (2.13) implies (2.14) with *some* $\delta > 0$ (see Section 4.1), but here we need δ also to satisfy (2.11).

Example 2.9. Let all functions $V_i(r)$ have for $r \gg 1$ the form

$$V_i(r) = r^{\alpha_i} (\log r)^{\beta_i} \tag{2.17}$$

for some $\alpha_i > 0$ and $\beta_i \in \mathbb{R}$. Let us show that M satisfies (COE). Define I_{super} to consist of all indices i such that $\alpha_i > 2$, I_{sub} to consist of all i such that $\alpha_i < 2$, and I_{middle} to consists of all i with $\alpha_i = 2$. Clearly, (a) and (b) are satisfied. Let us verify (c). Indeed, in the case of $\alpha_i = 2$, we have

$$h_i(r) \simeq \begin{cases} 1, & \text{if } \beta_i > 1 \\ \log\log r & \text{if } \beta_i = 1 \\ (\log r)^{1-\beta_i} & \text{if } \beta_i < 1 \end{cases}$$

and, hence,

$$V_i(r)h_i(r) \simeq \begin{cases} r^2 (\log r)^{\beta_i} & \text{if } \beta_i > 1 \\ r^2 \log r \log\log r & \text{if } \beta_i = 1 \\ r^2 \log r & \text{if } \beta_i < 1. \end{cases}$$

We see that

$$V_i \gtrsim V_j \Leftrightarrow \beta_i \geq \beta_j \Leftrightarrow V_i h_i \gtrsim V_j h_j.$$

Moreover, if M is parabolic then all $\beta_i \leq 1$ and in this case

$$V_i(r)h_i^2(r) \simeq \begin{cases} r^2 \log r (\log\log r)^2 & \text{if } \beta_i = 1 \\ r^2 (\log r)^{2-\beta_i} & \text{if } \beta_i < 1 \end{cases}$$

so that

$$V_i \gtrsim V_j \Leftrightarrow \beta_i \geq \beta_j \Leftrightarrow V_i h_i^2 \lesssim V_j h_j^2.$$

Hence, M satisfies (COE). Further examples of such manifolds can be found in Section 3.

Now we can state our main result in the case when the ends may be parabolic.

Theorem 2.10 (At least one end is parabolic)**.** *Let $M = \#_{i\in I} M_i$ be a manifold with ends such that each end M_i is geodesically complete, non-compact weighted manifold that satisfies* (PI) *and* (VD)*. Assume that each parabolic end M_i satisfies also* (RCA) *with a reference point o_i. Assume further that M satisfies* (COE)*. Then, for all $r \gg 1$, we have*

$$\Lambda(B(o,r)) \lesssim V_n(r)h_n(r), \tag{2.18}$$

where n is defined in (2.9). If, in addition each V_i satisfies for all $r \gg 1$

$$rV_i'(r) \lesssim V_i(r), \tag{2.19}$$

then, for all $r \gg 1$,

$$\Lambda(B(o,r)) \simeq V_n(r)h_n(r).$$

Hence, in this case the Poincaré constant $\Lambda(B(o,r))$ is again determined by the second largest volume function V_n but the formula also involves the associated function h_n.

Remark 2.11. To obtain the upper bound of $\Lambda(B(o,r))$ in (2.10) and (2.18), we first prove the upper bound of $\Lambda(B(o,r), B(o,\kappa r))$ for some $\kappa > 1$ (See section 5.3). To reduce $\kappa > 1$ to $\kappa = 1$, we need an additional argument presented in Section 6.

Example 2.12. Assume that, under the hypotheses of Theorem 2.10, all M_i have volume functions (2.17) as in Example 2.9. Define on the set of all pair (α_i, β_i) the lexicographical order $\succeq$, that is,

$$(\alpha_i, \beta_i) \succeq (\alpha_j, \beta_j)$$

if $\alpha_i > \alpha_j$ or $\alpha_i = \alpha_j$ and $\beta_i \geq \beta_j$. Then we have

$$V_i \gtrsim V_j \Leftrightarrow (\alpha_i, \beta_i) \succeq (\alpha_j, \beta_j) \,.$$

Clearly, each of the function (2.17) satisfies (2.19). By Theorem 2.10, we conclude that the Poincaré constant on M is determined by the second largest pair (α_n, β_n), that is

$$\Lambda(B(o,r)) \simeq V_n(r) h_n(r) \simeq \begin{cases} r^{\alpha_n} (\log r)^{\beta_n} & \text{if } (\alpha_n, \beta_n) \succ (2,1), \\ r^2 \log r \log\log r & \text{if } (\alpha_n, \beta_n) = (2,1), \\ r^2 \log r & \text{if } \alpha_n = 2, \beta_n < 1, \\ r^2 & \text{if } \alpha_n < 2. \end{cases}$$

Corollary 2.13. *Under the hypotheses of Theorem 2.10, if*

$$|I_{super}| + |I_{middle}| \leq 1, \tag{2.20}$$

then

$$\Lambda\left(B(o,r)\right) \lesssim r^2,$$

that is,

$$\lambda(B(o,r)) \gtrsim \frac{1}{r^2}$$

for all large r. Consequently, M satisfies (PI).

Proof. Indeed, under the hypothesis (2.20) the second largest end M_n belongs to I_{sub}. By Definition 2.8, for a subcritical end we have

$$h_n(r) \simeq r^2 / V_n(r)$$

whence

$$\Lambda(B(o,r)) \lesssim V_n\left(r\right) h_n\left(r\right) \simeq r^2.$$

Note also that if $B(x,\kappa r) \subset E_i$ then, by (PI) on M_i, we have

$$\Lambda(B(x,r), B(x,\kappa r)) \leq C r^2. \tag{2.21}$$

Hence, using the terminology of [18, Sect. 4], (PI) holds for *anchored* and *remote* balls in M. By [18, Prop. 4.2], (PI) holds for all balls in M. ∎

For some applications it is desirable to get rid of the hypothesis (COE). We state the target result here as a conjecture.

Conjecture 2.14. *Let $M = \#_{i\in I} M_i$ be a manifold with ends such that each end M_i satisfies* (PI) *and* (VD)*. Assume also that each parabolic end satisfies* (RCA)*. For any $r \gg 1$, define $m = m(r)$ and $n = n(r)$ so that*

$$V_m(r)h_m(r) \gtrsim V_n(r)h_n(r) \gtrsim V_i(r)\, h_i(r) \quad \textit{for all } i \neq n, m.$$

Then

$$\Lambda(B(o, r)) \lesssim V_n(r)h_n(r).$$

Obviously, Theorems 2.7 and 2.10 support this conjecture. A typical case that is not covered by these theorems is when $M = M_1 \# M_2$ where the both volume functions V_1, V_2 are close to the critical case $V(r) = r^2$ but not ordered in the sense of $\gtrsim$. See [14] for such examples.

2.4 Heat kernel estimates on manifolds with ends

Our strategy of the proof of Theorems 2.7 and 2.10 is inspired by the argument of Kusuoka-Stroock [21] (see also [26]), where (PI) was deduced from the heat kernel estimates (2.1).

Let $M = \#_{i\in I} M_i$ be a manifold with ends as above. We obtain the estimates of the Poincaré constant on M by using heat kernel bounds on M. In the case when M is non-parabolic, matching upper and lower estimates of the heat kernel on M were obtained in [19]. To state this result, let us introduce the following notation:

$$\widetilde{V}_i := V_i h_i^2,$$

where $V_i(r)$ and $h_i(r)$ were defined in (2.4) and (2.5), respectively.

Theorem 2.15. *([19, Cor. 6.8], [20]) Let $M = \#_{i\in I} M_i$ be a manifold with ends, where each M_i satisfies* (PI) *and* (VD) *and each parabolic M_i satisfies* (RCA)*. If M is non-parabolic then, for all $t > 0$, we have*

$$p(t, o, o) \simeq \frac{1}{\min_{i\in I} \widetilde{V}_i(\sqrt{t})}.$$

In particular, when all M_i are non-parabolic then $\widetilde{V}_i \simeq V_i$ and we obtain

$$p(t, o, o) \simeq \frac{1}{\min_{i\in I} V_i(\sqrt{t})},$$

that is, $p(t, o, o)$ is determined by the end with the smallest volume function.

In the case when M is parabolic, similar estimates were obtained in [13], however, with certain restriction on the volume functions $V_i(r)$ near the critical case $V_i(r) \simeq r^2$. For example, the result of [13] includes $V_i(r) \simeq r^\alpha$ with $\alpha \leq 2$ but does not include $V_i(r) \simeq r^2 \log r$ or $r^2/\log r$.

In this paper, we prove matching upper and lower heat kernel bounds on parabolic manifolds with ends under weaker restrictions than in [13].

Definition 2.16. A parabolic end M_i is called *regular* if it satisfies (2.15) with positive exponents γ_1 and γ_2 such that $2\gamma_1 + \gamma_2 < 2$, that is,

$$c\left(\frac{R}{r}\right)^{2-\gamma_2} \leq \frac{V_i(R)}{V_i(r)} \leq C\left(\frac{R}{r}\right)^{2+\gamma_1} \quad \text{for all } 1 \leq r \leq R. \tag{2.22}$$

For example, if M satisfies (COE) (see Definition 2.8) then any M_i with $i \in I_{middle}$ is regular.

Definition 2.17 (DOE)**.** We say that $M = \#_{i \in I} M_i$ has an end that dominates in the order (shortly (DOE)) if there exists $l \in I$ such that, for all $i \in I$

$$V_l \gtrsim V_i \quad \text{and} \quad \widetilde{V}_l \lesssim \widetilde{V}_i. \tag{2.23}$$

Remark 2.18. If a manifold $M = \#_{i \in I} M_i$ satisfies (COE) with $I_{super} = \emptyset$ and $I_{middle} \neq \emptyset$, then M admits (DOE) with $l \in I_{middle}$ and for all $r \gg 1$

$$V_l(r) \simeq V_m(r),$$

where $m = m(r)$ is the index of the largest end defined in (2.8). See Lemma 5.2 for details.

The next theorem is our main result regarding heat kernel estimates.

Theorem 2.19. *Let $M = \#_{i \in I} M_i$ be a manifold with parabolic ends, where each end satisfies* (PI)*,* (VD)*,* (RCA) *and is regular or subcritical. If there exists at least one non-subcritical regular end, assume also that M admits* (DOE)*. Then, for all $t > 0$,*

$$p(t, o, o) \simeq \frac{1}{V_m(\sqrt{t})}, \tag{2.24}$$

where m is defined in (2.8).

Hence, in the situation covered by this theorem, the long time behavior of the heat kernel is determined by the *largest* volume function, in contrast to the case of non-parabolic ends where, as we have seen above, $p\,(t, o, o)$ is determined by the *smallest* volume function. The estimate (2.24) implies sharp matching upper and lower bounds for $p\,(t, x, y)$ for all $x, y \in M$ by means of the gluing techniques of [19] (see also Theorem 4.3 below).

Example 2.20. Let $M = \#_{i \in I} M_i$ be a manifold with ends, where each M_i satisfies (PI), (VD) and each parabolic M_i satisfies (RCA). Assume that the volume growth function $V_i(r)$ of M_i satisfies for some $\alpha_i > 0$

$$V_i(r) \simeq r^{\alpha_i}, \quad (i \in I,\ r \gg 1).$$

Then Theorems 2.15 and 2.19 imply that for all $t > 0$

$$p(t, o, o) \simeq \frac{1}{t^{\alpha/2}},$$

where $\alpha = 2 + \min_{i \in I} |\alpha_i - 2|$ so that $p(t, o, o)$ is determined by α_i nearest to 2, unlike the Poincaré constant $\Lambda(B(o, r))$.

Example 2.21. Let $M = \#_{i \in I} M_i$ be as above. Assume that $V_i(r)$ satisfies for some $\beta_i \in \mathbb{R}$

$$V_i(r) \simeq r^2 \,(\log r)^{\beta_i}, \quad (i \in I,\ r \gg 1).$$

Then Theorems 2.15 and 2.19 imply that for all $t > 0$

$$p(t, o, o) \simeq \begin{cases} \frac{1}{t (\log t)^{\beta}} & \text{if } \beta_i \neq 1 \text{ for all } i \in I, \\ \frac{1}{t (\log t)(\log \log t)^2} & \text{if } \beta_i = 1 \text{ for some } i \in I, \end{cases}$$

where $\beta = 1 + \min_{i \in I} |\beta_i - 1|$ so that $p(t, o, o)$ is determined by β_i nearest to 1.

It seems that the condtion (DOE) in Theorem 2.19 is technical and we conjecture, that in general the following is true:

Conjecture 2.22. *Let $M = \#_{i\in I} M_i$ be a manifold with ends such that each end M_i is a geodesically complete, non-compact weighted manifold that satisfies* (PI) *and* (VD)*, and each parabolic end also satisfies* (RCA)*. Then, for all $t > 0$, we have*

$$p(t,o,o) \simeq \frac{\min_{i\in I} h_i^2(\sqrt{t})}{\min_{i\in I} \widetilde{V}_i(\sqrt{t})}. \tag{2.25}$$

Remark 2.23. Regarding condition (DOE), observe that the first condition $V_l \gtrsim V_i$ in (2.23) does not imply in general the second condition $\widetilde{V}_l \lesssim \widetilde{V}_i$. However, if all ends are regular and there exists $\eta \geq 2\gamma_1$ such that for all i

$$\frac{V_l(r)}{V_i(r)} \geq Cr^\eta \quad \text{for all } r \geq 1, \tag{2.26}$$

then $\widetilde{V}_l \lesssim \widetilde{V}_i$ is satisfied. Indeed, by the regularity of V_l and (2.5), we obtain

$$h_l(r) = 1 + \frac{1}{V_l(r)} \int_1^r \frac{V_l(r)}{V_l(s)} sds \lesssim 1 + \frac{1}{V_l(r)} \int_1^r \left(\frac{r}{s}\right)^{2+\gamma_1} sds \lesssim \frac{r^{2+\gamma_1}}{V_l(r)},$$

which implies

$$\widetilde{V}_l(r) = V_l(r)\, h_l^2(r) \lesssim \frac{r^{4+2\gamma_1}}{V_l(r)}.$$

Since $h_i(r) \gtrsim \frac{r^2}{V_i(r)}$, we obtain

$$\widetilde{V}_i(r) = V_i(r)\, h_i^2(r) \gtrsim \frac{r^4}{V_i(r)}.$$

Hence, (2.26) implies that

$$\frac{\widetilde{V}_l(r)}{\widetilde{V}_i(r)} \lesssim \frac{V_i(r)}{V_l(r)} r^{2\gamma_1} \lesssim r^{2\gamma_1-\eta} \lesssim 1,$$

as was claimed.

2.5 Structure of the paper

In Section 3 we give examples of application of Theorems 2.7 and 2.10 in the case when all ends are model manifolds.

In Section 4, we first survey the previous results of [19] and [13] on heat kernel estimates on manifolds with ends and then prove Theorem 2.19.

In Section 5 we give the proofs of Theorems 2.7 and 2.10. For that, we first obtain in Section 5.2 a lower bound for the Dirichlet heat kernel in balls (Lemma 5.4) using the two-sided off-diagonal bounds of the heat kernel $p(t,x,y)$ on M that follow from Theorem 2.19. By means of this estimate, we obtain in Section 5.3 an upper bound for $\Lambda(B(o,r), B(o,\kappa r))$ for some large enough $\kappa > 1$. We use an additional argument to reduce κ to 1 stated in Section 6. The lower bound for $\Lambda(B(o,r))$ is proved in Section 5.4.

In Section 6, we obtain a general upper bound of the Poincaré constant $\Lambda(B(o,r))$ by using a collection of $\Lambda(B(x,s), B(x,\kappa s))$, where $B(x,s) \subset B(o,r)$ and $\kappa \geq 1$.

3 Model manifold and examples

Let ψ be a smooth positive function on $(0, +\infty)$ such that such $\psi(r) = r$ for $r < 1$. Fix an integer $N \geq 2$ and consider in $\mathbb{R}^N$ a Riemannian metric g_ψ given in the polar coordinates $(r, \theta) \in (0, \infty) \times \mathbb{S}^{N-1}$ by

$$g_\psi = dr^2 + \psi(r)^2 d\theta^2.$$

In a punctured neighborhood of the origin $o \in \mathbb{R}^N$, this metric coincides with the Euclidean one and, hence, extends to the full neighborhood of o, so that g_ψ is defined on the entire $\mathbb{R}^N$. The Riemannian manifold $\left(\mathbb{R}^N, g_\psi\right)$ is called a model manifold. For example, if $\psi(r) = r$ then g_ψ is the canonical Euclidean metric.

Let us assume in addition that, for some $C > 0$ and all $r \gg 1$,

$$\begin{cases} \sup\limits_{[r,2r]} \psi \leq C \inf\limits_{[r,2r]} \psi, \\ \psi(r) \leq Cr, \\ \displaystyle\int_0^r \psi^{N-1}(s)ds \leq Cr\psi^{N-1}(r). \end{cases} \tag{3.1}$$

Since for any bounded range $r \in (0, r_0)$ the condition (3.1) is trivially satisfied, we obtain by [18, Prop. 4.10] that $\left(\mathbb{R}^N, g_\psi\right)$ satisfies (PI) and (VD). It is also obvious that $\left(\mathbb{R}^N, g_\psi\right)$ satisfies (RCA) with respect to the origin o because the geodesic balls $B(o, r)$ coincide with the Euclidean balls.

Let us reformulate conditions (3.1) in terms of the volume $V(r)$ of a ball $B(o, r)$ on $\left(\mathbb{R}^N, g_\psi\right)$:

$$V(r) = V(o, r) = \omega_N \int_0^r \psi^{N-1}(s)ds. \tag{3.2}$$

Lemma 3.1. *The conditions (3.1) are equivalent to the following conditions to be satisfied for all $r \gg 1$:*

$$\begin{aligned} &V(r) \leq Cr^N \\ &V(r) \simeq rV'(r). \end{aligned} \tag{3.3}$$

Proof. The inequality $V(r) \leq Cr^N$ follows from $\psi(r) \leq Cr$ and (3.2). The third condition in (3.1) is equivalent to

$$V(r) \leq CrV'(r)$$

that is the upper bound in the second condition in (3.3). To obtain a similar lower bound, observe that, the first condition in (3.1) is equivalent to

$$V'(s) \simeq V'(r) \quad \text{for all } s \in \left[\frac{1}{2}r, r\right], \tag{3.4}$$

which implies

$$V(r) \geq \int_{r/2}^r V'(s)\,ds \simeq rV'(r).$$

Let us now prove that (3.3) implies (3.1). Clearly, the third condition in (3.1) follows from

$$V(r) \simeq rV'(r). \tag{3.5}$$

The conditions (3.3) imply the second condition in (3.1) as follows:

$$\psi(r)^{N-1} = \omega_N^{-1} V'(r) \simeq \frac{V(r)}{r} \leq Cr^{N-1}. \tag{3.6}$$

Finally, let us prove the first condition in (3.1), or, equivalently, (3.4). In the view of (3.5) and (3.6) it suffices to prove that V satisfies the doubling property, that is,

$$V(2r) \leq CV(r). \tag{3.7}$$

Indeed, applying again (3.5), we obtain

$$\ln V(2r) - \ln V(r) = \int_r^{2r} \frac{V'(t)}{V(t)} dt \simeq \int_r^{2r} \frac{dt}{t} = \ln 2,$$

whence (3.7) follows. ∎

In view of Lemma 3.1, it will be more convenient to describe a model manifold in terms of the volume function $V(r)$ rather than $\psi(r)$. Hence, we denote the model manifold $\left(\mathbb{R}^N, g_\psi\right)$ shortly by $\mathcal{M}_V$.

Given k model manifolds $\mathcal{M}_{V_1}, ..., \mathcal{M}_{V_k}$ satisfying (3.3) (and, hence, (VD) and (PI)), consider their connected sum

$$M = \mathcal{M}_{V_1} \# ... \# \mathcal{M}_{V_k}.$$

Assume that the standing assumptions of either Theorem 2.7 or Theorem 2.10 are satisfied, that is, either all ends are non-parabolic or all ends are (COE). Then, Theorems 2.7 and 2.10 yield that

$$\Lambda(B(o,r)) \simeq V_n(r) h_n(r),$$

that is,

$$\lambda(B(o,r)) \simeq \frac{1}{V_n(r) h_n(r)},$$

where n is the index of the second largest end.

Example 3.2. Assume that, for all $r \gg 1$,

$$V(r) = r^\alpha \prod_{j=1}^{J} (\log_{[j]} r)^{\beta(j)}, \tag{3.8}$$

where $\alpha > 0$, $\beta(1), \ldots, \beta(J) \in \mathbb{R}$ and $\log_{[j]} r$ is the j-times iterated logarithm. Clearly, in this case $\mathcal{M}_V$ is parabolic if and only if

$$(\alpha, \beta(1), \ldots, \beta(J)) \preceq (2, 1, \ldots, 1), \tag{3.9}$$

where $\preceq$ denotes the lexicographical order. Also, it is easy to see that $V(r)$ satisfies (3.3) and hence (PHI) if and only if

$$(\alpha, \beta(1), \ldots, \beta(J)) \preceq (N, 0, \ldots, 0). \tag{3.10}$$

Note for comparison that the function $V(r) = (\log r)^\beta$ with $\beta > 0$ does not satisfy the second condition in (3.3). Some examples of model manifolds with $V(r) = r^\alpha$ are shown on Fig. 4.

Assume now that for each i and all $r \gg 1$

$$V_i(r) = r^{\alpha_i} \prod_{j=1}^{J} (\log_{[j]} r)^{\beta_i(j)}, \tag{3.11}$$

where each $(J+1)$-tuple $(\alpha_i, \beta_i(1), \ldots, \beta_i(J))$ satisfies (3.10). The condition (COE) holds in this case with the following decomposition of the index set I:

$$i \in \begin{cases} I_{super} & \text{if } \alpha_i > 2, \\ I_{sub} & \text{if } \alpha_i < 2, \\ I_{middle} & \text{if } \alpha_i = 2. \end{cases}$$

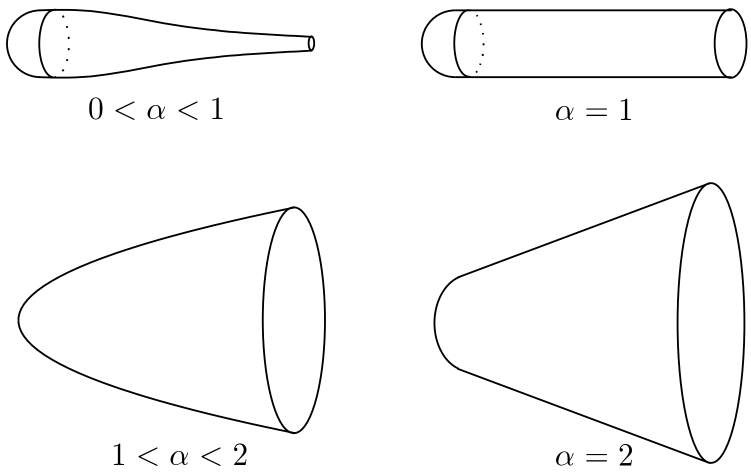

Figure 4: Model manifolds $\mathcal{M}_V$ of dimension $N=2$ with $V(r)=r^{\alpha}$, $r \gg 1$.

Let us compute $V_i h_i$. If V_i is non-parabolic, that is, if

$$(\alpha_i, \beta_i(1), \ldots, \beta_i(J)) \succ (2, 1, \ldots, 1),$$

then by (2.5) $h_i \simeq 1$ and, hence, $V_i h_i \simeq V_i$.

Let now V_i be parabolic, that is,

$$(\alpha_i, \beta_i(1), \ldots, \beta_i(J)) \preceq (2, 1, \ldots, 1). \tag{3.12}$$

If $\alpha_i < 2$ then V_i is subcritical and, hence,

$$V_i h_i(r) \simeq r^2. \tag{3.13}$$

In the case $\alpha_i = 2$, we denote by J_i the smallest index $j = 1, ..., J$ such that $\beta_i(j) < 1$; if such j does not exists then take $J_i = J+1$. The parabolicity of V_i implies that

$$\beta_i(j) = 1 \quad \text{for all } j < J_i.$$

With this notation we obtain, for all $r \gg 1$,

$$h_i(r) \simeq \begin{cases} r^{2-\alpha_i} \prod_{j=1}^{J} \left(\log_{[j]} r\right)^{-\beta_i(j)} & \text{if } \alpha_i < 2 \\ \left(\log_{[J_i]} r\right) \prod_{j=J_i}^{J} \left(\log_{[j]} r\right)^{-\beta_i(j)} & \text{if } \alpha_i = 2. \end{cases}$$

Combining this, (3.13) and (3.11), we obtain

$$V_i(r) h_i(r) \simeq \begin{cases} r^2 & \text{if } \alpha_i < 2, \\ r^2 \prod_{j=1}^{J_i} \log_{[j]} r & \text{if } \alpha_i = 2 \end{cases} \tag{3.14}$$

and

$$\widetilde{V}_i(r) \simeq \begin{cases} r^{4-\alpha_i} \prod_{j=1}^{J} \left(\log_{[j]} r\right)^{-\beta_i(j)} & \text{if } \alpha_i < 2, \\ r^2 \prod_{j=1}^{J_i - 1} \left(\log_{[j]} r\right) (\log_{[J_i]} r)^2 \prod_{j=J_i}^{J} \left(\log_{[j]} r\right)^{-\beta_i(j)} & \text{if } \alpha_i = 2. \end{cases} \tag{3.15}$$

Theorems 2.7, 2.10, 2.19 and 2.15 yield the following. Let m be the index for the largest volume function and let n be the index for the *second largest* volume function, that is, for all $i \neq m, n$,

$$(\alpha_m, \beta_m(1), \ldots, \beta_m(J)) \succeq (\alpha_n, \beta_n(1), \ldots, \beta_n(J)) \succeq (\alpha_i, \beta_i(1), \ldots, \beta_i(J)).$$

Case 1. Let M be non-parabolic. Then

$$p(t, o, o) \simeq \frac{1}{\min_{i \in I} \widetilde{V}_i(\sqrt{t})}, \tag{3.16}$$

that is, $p(t, o, o)$ is determined by the smallest function $\widetilde{V}_i(\sqrt{t})$ (see further examples and comments in [19]). Note that it is not easy to give an explicit expression for $\min_{i \in I} \widetilde{V}_i$ because the form of the result varies depending on the exact nature of $(\alpha_i, \beta_i(1), \ldots, \beta_i(J)), 1 \leq i \leq k$. Of course, in any particular example, one can compute $\min_{i \in I} \widetilde{V}_i$ explicitly.

Case 2. Let M be parabolic. Then

$$p(t, o, o) \simeq \frac{1}{V_m(\sqrt{t})} \simeq \frac{1}{t^{\alpha_m/2} \prod_{j=1}^{J} \left(\log_{[j]} t\right)^{\beta_m(j)}}. \tag{3.17}$$

If $V_i(r) \simeq r^{\alpha_i}$, then the above estimates in (3.16) and (3.17) imply that

$$p(t, o, o) \simeq \frac{1}{t^{\alpha/2}},$$

where

$$\alpha = 2 + \min_{i \in I} |\alpha_i - 2|$$

so that $p(t, o, o)$ is determined by the volume growth function nearest to r^2.

In the both case 1 and case 2, the Poincaré constant $\Lambda\left(B\left(o, r\right)\right)$ is determined by the *second largest* volume function $V_n(r)$, namely, for all $r \gg 1$

$$\Lambda(B(o, r)) \simeq V_n(r) h_n(r).$$

If V_n is non-parabolic, then $V_n h_n \simeq V_n$, whence we obtain

$$\Lambda(B(o, r)) \simeq r^{\alpha_n} \prod_{j=1}^{J} \left(\log_{[j]} r\right)^{\beta_n(j)},$$

that is,

$$\lambda(B(o, r)) \simeq r^{-\alpha_n} \prod_{j=1}^{J} \left(\log_{[j]} r\right)^{-\beta_n(j)}.$$

If V_n is parabolic, then by (3.14) we obtain

$$\Lambda(B(o, r)) \simeq \begin{cases} r^2 & \text{if } \alpha_n < 2, \\ r^2 \prod_{j=1}^{J_n} \log_{[j]} r & \text{if } \alpha_n = 2, \end{cases} \tag{3.18}$$

that is,

$$\lambda(B(o, r)) \simeq \begin{cases} \frac{1}{r^2} & \text{if } \alpha_n < 2, \\ \frac{1}{r^2 \prod_{j=1}^{J_n} \log_{[j]} r} & \text{if } \alpha_n = 2. \end{cases} \tag{3.19}$$

For example, (3.18) holds provided at most one of the ends of M is non-parabolic, because in this case the second largest V_n is parabolic. We see from (3.18) that in this case the estimate of the Poincaré constant exhibits a certain rigidity – it does not depend on the exponents α_i and $\beta_i\left(j\right)$, although it does depend on J_n.

4 Heat kernel estimates (central estimates)

This section is devoted to obtain heat kernel estimates on manifolds with ends, which play a key role in the proof of Theorem 2.10. The main result here is Theorem 2.19 which provides two-sided matching estimates of $p(t,o,o)$ on parabolic manifolds with ends. This result is of interest by itself independently of the application to bounding the Poincaré constant. Theorem 2.19 implies the off-diagonal estimates of $p(t,x,y)$ as in Theorem 4.3.

4.1 Known results

Let us first recall some known results. As before, consider a manifold with ends $M = M_1\#...\#M_k$ where each M_i is a geodescially complete, weighted manifold that satisfies (PI) and (VD). Besides, if M_i is parabolic then we assume that it satisfies (RCA). Fix a reference point $o_i \in M_i$, set $V_i(r) = V_i(o_i, r)$ and define $h_i(r)$ by (2.5).

In the case when M is non-parabolic the matching upper and lower bounds of $p(t,o,o)$ are provided by Theorem 2.15. Let us discuss the case when M is parabolic.

Recall that an end M_i is called *subcritical* if, for all $r \gg 1$,

$$h_i(r) \simeq \frac{r^2}{V_i(r)} \tag{4.1}$$

(see Definition 2.8). For example, if $V_i(r) \simeq r^\alpha$ then M_i is subcritical if and only if $\alpha < 2$. Conversely, if M_i is subcritical, then there exists $\delta > 0$ such that

$$V_i(r) \lesssim r^{2-\delta}. \tag{4.2}$$

Indeed, (4.1) implies that, for some $\delta > 0$,

$$(\log h_i(r))' \geq \frac{\delta}{r},$$

which implies upon integration that $h_i(r) \gtrsim r^\delta$, which together with (4.1) yields (4.2).

Definition 4.1. An end M_i is called *critical* if

$$V_i(r) \simeq r^2.$$

Note that all critical and subcritical ends are parabolic.

Theorem 4.2. *([13, Theorem 2.1]) Let $M = M_1\#\cdots\#M_k$ where each end M_i is either critical or subcritical. Then, for all $t > 0$, we have*

$$p(t,o,o) \simeq \frac{1}{V_m(\sqrt{t})},$$

where $m = m(r)$ is the index of the largest volume function at scale r.

Our main goal in this section is to prove Theorem 2.19 stated in Introduction that provides the same estimates of $p(t,o,o)$ but for a more general class of parabolic manifolds with ends (see Section 3).

Let us first explain how the estimates of $p(t,o,o)$ lead to the estimates of $p(t,x,y)$ for all $x, y \in M$. For $r, t > 0$, set

$$\begin{aligned} H_i(r,t) &= \frac{r^2}{V_i(r)h_i(r)} + \frac{1}{h_i(\sqrt{t})}\left(\int_r^{\sqrt{t}} \frac{sds}{V_i(s)}\right)_+ \\ &= \frac{r^2}{V_i(r)h_i(r)} + \frac{\left(h_i\left(\sqrt{t}\right) - h_i(r)\right)_+}{h_i(\sqrt{t})} \end{aligned}$$

and

$$D_i(r,t) = \frac{h_i(r)}{h_i(r) + h_i(\sqrt{t})}.$$

The following result follows from [19, Theorems 3.5], [17, Thms 4.4, 4.6], [16, Thms 3.3, 4.9], [13, (5.1), (5.6), (5.7)] and the local Harnack inequality near $o \in K$. Set $|x| = d(x,K)+3$.

Theorem 4.3. *Let $M = M_1 \# \cdots \# M_k$ be a manifold with ends, where each M_i satisfies* (PI) *and* (VD)*, and each parabolic end also satisfies* (RCA)*. Then, for all $x \in E_i$, $y \in E_j$, $t \gg 1$ we have*

$$\begin{aligned} p(t,x,y) \quad &\asymp \quad \delta_{ij} \frac{1}{V_i(x,\sqrt{t})} D_i(|x|,t)\, D_j(|y|,t)\, e^{-b\frac{d^2(x,y)}{t}} \\ &+ p(t,o,o) H_i(|x|,t)\, H_j(|y|,t)\, e^{-b\frac{|x|^2+|y|^2}{t}} \\ &+ \int_1^t p(s,o,o)ds \left[\frac{D_i(|x|,t)\, H_j(|y|,t)}{V_i(\sqrt{t}) h_i(\sqrt{t})} + \frac{D_j(|y|,t)\, H_i(|x|,t)}{V_j(\sqrt{t}) h_j(\sqrt{t})} \right] e^{-b\frac{|x|^2+|y|^2}{t}}. \end{aligned}$$

This is a very general estimate. In the view of this result, the on-diagonal value $p(t,o,o)$ of the heat kernel plays a key role in estimating of $p(t,x,y)$. One of the aim of this paper is to improve upon the existing results from [13]. See Theorem 2.19.

4.2 Preliminary estimates

A classical method of obtaining on-diagonal heat kernel bounds is to use a Nash-type functional inequality, which gives a uniform upper bound for $p(t,x,x)$ for all $x \in M$. Indeed, such an inequality can be proved on manifold M in the setting of Theorem 4.3 (see [20]) but in the case when M is parabolic, this upper bound is not optimal. In order to obtain an optimal upper bound, we have developed in [13, Section 3] a different method based on the *integrated resolvent kernel* $\gamma_\lambda(x)$ and its derivative $\dot{\gamma}_\lambda(x)$. However, in [13, Section 3] we could handle only critical and subcritical ends (cf. Theorem 4.2). For the proof of Theorem 2.19, we apply the method of [13, Section 3], but with significant improvements that allow us to handle much more general ends.

Let (M,μ) be a geodesically complete, non-compact weighted manifold. For any $\lambda > 0$, we define the *resolvent operator* G_λ acting on non-negative measurable functions f on M by

$$G_\lambda f(x) = \int_M \int_0^\infty e^{-\lambda t} p(t,x,z) f(z) dz dt.$$

We remark that $u = G_\lambda f$ is the minimal non-negative weak solution of the equation

$$\Delta u - \lambda u = -f \tag{4.3}$$

(see [12]). Fix a compact set $K \subset M$. We define the functions γ_λ and $\dot{\gamma}_\lambda$ on M by

$$\gamma_\lambda(x) := G_\lambda 1_K(x) = \int_K \int_0^\infty e^{-\lambda t} p(t,x,z) dz dt$$

and

$$\dot{\gamma}_\lambda(x) := G_\lambda \gamma_\lambda = -\frac{\partial}{\partial \lambda} \gamma_\lambda(x) = \int_K \int_0^\infty t e^{-\lambda t} p(t,x,z) dz dt.$$

Fix a reference point $o \in K$. The following lemma follows from the estimates in [13, (3.10), (3.11)].

Lemma 4.4. *There exist constants $C > 0$ and $t_0 > 0$ depending on K and such that, for all $x \in K$ and $t \geq t_0$,*

$$p(t,o,o) \leq \frac{C}{t}\gamma_{\frac{1}{t}}(x), \tag{4.4}$$

$$p(t,o,o) \leq \frac{C}{t^2}\dot{\gamma}_{\frac{1}{t}}(x), \tag{4.5}$$

As in [13, Section 4], in order to estimate γ_λ and $\dot{\gamma}_\lambda$, we will use the functions Φ_λ^Ω and Ψ_λ^Ω defined below. Denote by $p_\Omega^D(t,x,y)$ the Dirichlet heat kernel in an open set $\Omega \subset M$. For any $\lambda > 0$, define the resolvent operator G_λ^Ω on non-negative measurable functions f in Ω by

$$G_\lambda^\Omega f(x) = \int_\Omega \int_0^\infty e^{-\lambda t} p_\Omega^D(t,x,z) f(z) dz dt,$$

so that $G_\lambda^\Omega f$ is the minimal non-negative weak solution of the equation $\Delta u - \lambda u = -f$ in Ω.

Now we define the functions Φ_λ^Ω and Ψ_λ^Ω in Ω by

$$\begin{aligned} \Phi_\lambda^\Omega &= \lambda G_\lambda^\Omega 1, \\ \Psi_\lambda^\Omega &= G_\lambda^\Omega(1 - \Phi_\lambda^\Omega). \end{aligned}$$

These functions have certain probabilistic meaning. Indeed, denote by $(\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in M})$ Brownian motion on M. For any open set $\Omega \subset M$, let τ_Ω be the first exit time of X_t from Ω, that is,

$$\tau_\Omega = \inf\{t > 0 : X_t \notin \Omega\}.$$

Then, by [13, (3.17), (3.25)], for any $x \in \Omega$,

$$\Phi_\lambda^\Omega(x) = \int_0^\infty \lambda e^{-\lambda t} \mathbb{P}_x(\tau_\Omega > t) dt, \tag{4.6}$$

$$\Psi_\lambda^\Omega(x) = \int_0^\infty t e^{-\lambda t} \partial_t \mathbb{P}_x(\tau_\Omega \leq t) dt. \tag{4.7}$$

Now let us assume that the manifold M satisfies (PI), (VD) and (RCA). Let A be a precompact open set containing

$$K_\varepsilon := \{z \in M : d(K,z) < \varepsilon\}$$

for some $\varepsilon > 0$. Define function $h(r)$ for $r > 0$ by

$$h(r) = 1 + \left(\int_1^r \frac{sds}{V(s)}\right)_+, \tag{4.8}$$

where $V(r) = V(o,r)$. In the next lemmas, we obtain some new estimates of $\Phi_\lambda^{K^c}$ and $\Psi_\lambda^{K^c}$.

Lemma 4.5. *There exists $c, \lambda_0 > 0$ depending on K and A, and such that, for all $\lambda \in (0, \lambda_0)$,*

$$\inf_{\partial A} \Phi_\lambda^{K^c} \geq \frac{c}{h(\frac{1}{\sqrt{\lambda}})}. \tag{4.9}$$

Proof. By [17, (4.47)], we have, for all $x \in \partial A$ and $t \geq t_1$,

$$\mathbb{P}_x(\tau_{K^c} > t) \approx \frac{h(|x|)}{h(\sqrt{t})}$$

where $t_1 > 0$ depends on A. Substituting this into (4.6) and using the fact that $\inf_{\partial A} h(|x|) > 0$, we obtain, for all $\lambda < 1/(2t_1)$ and $x \in \partial A$,

$$\Phi_\lambda^{K^c}(x) \geq \int_{1/(2\lambda)}^{1/\lambda} \lambda e^{-\lambda t} \frac{c}{h(\sqrt{t})} dt \geq \frac{c}{h(\frac{1}{\sqrt{\lambda}})} \int_{1/(2\lambda)}^{1/\lambda} \lambda e^{-\lambda t} dt \geq \frac{c}{h(\frac{1}{\sqrt{\lambda}})}.$$

■

Remark 4.6. If M is critical, then the estimate in (4.9) implies that

$$\inf_{\partial A} \Phi_\lambda^{K^c} \geq \frac{c}{\log \frac{1}{\sqrt{\lambda}}},$$

which coincides with [13, (3.34)]. If M is subcritical, the estimate in (4.9) implies that

$$\inf_{\partial A} \Phi_\lambda^{K^c} \geq c\lambda V(\frac{1}{\sqrt{\lambda}}),$$

which is identical to [13, (3.33)].

Recall that $V(r)$ is regular if it satisfies (2.22) that is,

$$c\left(\frac{R}{r}\right)^{2-\gamma_2} \leq \frac{V(R)}{V(r)} \leq C\left(\frac{R}{r}\right)^{2+\gamma_1} \quad \text{for all } 1 \leq r \leq R$$

for some positive γ_1, γ_2 such that $2\gamma_1 + \gamma_2 < 2$.

Set $\widetilde{V}(r) = V(r)h^2(r)$. Then we obtain the following upper estimate of $\widetilde{V}(r)$ on manifold with regular volume function.

Lemma 4.7. *Let M be regular, that is, the volume function V is regular. Then, for all $1 \leq r \leq R$,*

$$\widetilde{V}(r) \lesssim r^{2+\gamma_1+\gamma_2} \lesssim r^4. \tag{4.10}$$

$$\frac{\widetilde{V}(R)}{\widetilde{V}(r)} \lesssim \left(\frac{R}{r}\right)^{2+2\gamma_1+\gamma_2}. \tag{4.11}$$

Proof. By (4.8) and regularity, we have

$$V(r)h(r) = V(r) + \int_1^r \frac{V(r)}{V(s)} sds \leq C\frac{V(r)}{V(1)} + C\int_1^r \left(\frac{r}{s}\right)^{2+\gamma_1} sds \leq C' r^{2+\gamma_1}. \tag{4.12}$$

Similarly, we have

$$h(r) = 1 + \int_1^r \frac{sds}{V(s)} \leq 1 + \frac{1}{V(1)} \int_1^r \frac{V(1)}{V(s)} sds \leq 1 + C\int_1^r \frac{sds}{s^{2-\gamma_2}} \leq C' r^{\gamma_2},$$

whence (4.10) follows.

Now we prove (4.11). By the regularity of V and

$$h(r) \geq c\frac{r^2}{V(r)},$$

we obtain

$$
\begin{aligned}
\frac{\widetilde{V}(R)}{\widetilde{V}(r)} &= \frac{V(R)h^2(R)}{V(r)h^2(r)} = \frac{V(R)\left(h(r)+\int_r^R \frac{sds}{V(s)}\right)^2}{V(r)h^2(r)} = \frac{V(R)}{V(r)}\left(1+\frac{\int_r^R \frac{sds}{V(s)}}{h(r)}\right)^2 \\
&\leq \frac{V(R)}{V(r)}\left(1+C\frac{V(r)}{r^2}\frac{\int_r^R \frac{V(R)}{V(s)}sds}{V(R)}\right)^2 \\
&\leq \frac{V(R)}{V(r)}\left(1+C\frac{V(r)}{r^2}\frac{\int_r^\infty \left(\frac{R}{s}\right)^{2+\gamma_1}sds}{V(R)}\right)^2 \\
&\leq C\frac{V(R)}{V(r)}\left(1+\frac{V(r)}{V(R)}\left(\frac{R}{r}\right)^{2+\gamma_1}\right)^2 \\
&\leq C\left(\frac{R}{r}\right)^{2+\gamma_1}+C\left(\frac{R}{r}\right)^{2+2\gamma_1+\gamma_2} \leq C\left(\frac{R}{r}\right)^{2+2\gamma_1+\gamma_2}
\end{aligned}
$$

which proves (4.11). ■

Lemma 4.8. *Assume that the manifold M is either subcritical or regular. Then there exist $C, \lambda_0 > 0$ depending on K and A, such that, for all $\lambda \in (0, \lambda_0)$,*

$$\sup_{\partial A} \Psi_\lambda^{K^c} \leq \frac{C}{\lambda^2 \widetilde{V}(\frac{1}{\sqrt{\lambda}})}. \tag{4.13}$$

Proof. Set $\Omega = K^c$. By [17, (4.48)] we have, for all $x \in \partial A$ and $t \geq t_0 > 0$

$$\partial_t \mathbb{P}_x(\tau_\Omega \leq t) \leq \frac{C}{\widetilde{V}(\sqrt{t})}.$$

Taking $\lambda_0 \leq \frac{1}{t_0}$, we obtain by (4.7) for any $0 < \lambda < \lambda_0$

$$
\begin{aligned}
\Psi_\lambda^\Omega(x) &= \int_0^{t_0} te^{-\lambda t}\partial_t \mathbb{P}_x(\tau_\Omega \leq t)dt + \int_{t_0}^\infty te^{-\lambda t}\partial_t \mathbb{P}_x(\tau_\Omega \leq t)dt \\
&\leq t_0 + C\int_{t_0}^\infty e^{-\lambda t}\frac{t}{\widetilde{V}(\sqrt{t})}dt. \qquad (4.14)
\end{aligned}
$$

Observe that of M is regular then by (4.10) we have for all $r \geq 1$

$$\widetilde{V}(r) \leq Cr^4. \tag{4.15}$$

If M is subcritical, we have

$$\widetilde{V}(r) = V(r)h^2(r) \simeq V(r)\left(\frac{r^2}{V(r)}\right)^2 = \frac{r^4}{V(r)}, \tag{4.16}$$

which again implies (4.15).

In particular, it follows from (4.15) for $r = \frac{1}{\sqrt{\lambda}}$ that

$$\frac{1}{\lambda^2 \widetilde{V}(\frac{1}{\sqrt{\lambda}})} \geq c \tag{4.17}$$

so that the constant term t_0 in (4.14) can be estimated from above as follows:

$$t_0 \lesssim \frac{1}{\lambda^2 \widetilde{V}(\frac{1}{\sqrt{\lambda}})}. \tag{4.18}$$

Next, let us decompose the integral in (4.14) into two intervals: $[t_0, 1/\lambda]$ and $[1/\lambda, \infty)$. For the second interval, we obtain

$$\int_{1/\lambda}^{\infty} e^{-\lambda t} \frac{t}{\widetilde{V}\left(\sqrt{t}\right)} dt \leq \frac{1}{\widetilde{V}(\frac{1}{\sqrt{\lambda}})} \int_{1/\lambda}^{\infty} t e^{-\lambda t} dt \simeq \frac{1}{\lambda^2 \widetilde{V}(\frac{1}{\sqrt{\lambda}})}. \tag{4.19}$$

Let us estimate the first integral. If M is regular then, by using (4.11), we obtain

$$\begin{aligned} \int_{t_0}^{1/\lambda} e^{-\lambda t} \frac{t}{\widetilde{V}\left(\sqrt{t}\right)} dt &\leq \frac{1}{\widetilde{V}(\frac{1}{\sqrt{\lambda}})} \int_{t_0}^{1/\lambda} \frac{\widetilde{V}\left(\frac{1}{\sqrt{\lambda}}\right)}{\widetilde{V}\left(\sqrt{t}\right)} t dt \\ &\lesssim \frac{1}{\lambda^2 \widetilde{V}(\frac{1}{\sqrt{\lambda}})} \int_{0}^{1/\lambda} \left(\frac{1}{\lambda t}\right)^{1+\gamma_1+\gamma_2/2} (\lambda t)\, d(\lambda t) \\ &= \frac{1}{\lambda^2 \widetilde{V}(\frac{1}{\sqrt{\lambda}})} \int_{0}^{1} s^{-(\gamma_1+\gamma_2/2)} ds \\ &\lesssim \frac{1}{\lambda^2 \widetilde{V}(\frac{1}{\sqrt{\lambda}})}, \end{aligned} \tag{4.20}$$

because $\gamma_1 + \gamma_2/2 < 1$.

If M is subcritical, then we use the fact that (VD) implies the *reverse volume doubling* (see, for instance, [26, Lemma 5.2.8]), that is, for some $\theta > 0$ and for all $R \geq r \geq 1$,

$$\frac{V(r)}{V(R)} \lesssim \left(\frac{r}{R}\right)^{\theta}. \tag{4.21}$$

Then we obtain by (4.16) and (4.21) that

$$\begin{aligned} \int_{t_0}^{1/\lambda} e^{-\lambda t} \frac{t}{\widetilde{V}\left(\sqrt{t}\right)} dt &\leq \int_{t_0}^{1/\lambda} \frac{t}{\widetilde{V}\left(\sqrt{t}\right)} dt \\ &\simeq \int_{t_0}^{1/\lambda} \frac{t}{t^2/V\left(\sqrt{t}\right)} dt \\ &= V(\frac{1}{\sqrt{\lambda}}) \int_{t_0}^{\frac{1}{\lambda}} \frac{V(\sqrt{t})}{V(\frac{1}{\sqrt{\lambda}})} \frac{1}{t} dt \\ &\lesssim V(\frac{1}{\sqrt{\lambda}}) \int_{0}^{\frac{1}{\lambda}} (\lambda t)^{\theta/2} \frac{1}{\lambda t} d(\lambda t) \\ &\lesssim V(\frac{1}{\sqrt{\lambda}}) \simeq \frac{1}{\lambda^2 \widetilde{V}(\frac{1}{\sqrt{\lambda}})}. \end{aligned} \tag{4.22}$$

Combining (4.18), (4.19), (4.20) and (4.22), we obtain (4.13). ■

Remark 4.9. If M is critical, that is, $V(r) \simeq r^2$, then Lemma 4.8 implies that

$$\sup_{\partial A} \Psi_{\lambda}^{K^c} \leq \frac{C}{\lambda \log^2 \frac{1}{\lambda}},$$

which coincides with [13, (3.39)]. If M is subcritical, then Lemma 4.8 implies that

$$\sup_{\partial A} \Psi_{\lambda}^{K^c} \leq C V(\frac{1}{\sqrt{\lambda}}).$$

Using further (4.2), we obtain

$$\sup_{\partial A} \Psi_\lambda^{K^c} \le C \left(\frac{1}{\lambda}\right)^{1-\delta/2},$$

which is a significant improvement of [13, (3.39)].

4.3 Proof of Theorem 2.19

Here we prove Theorem 2.19. Recall that we consider a manifold with ends $M = \#_{i\in I} M_i$ where each end is parabolic satisfying (VD), (PI) and (RCA) (see Figure 3). Besides, we assume that each end satisfies either regular or subcritical, and also, (DOE) if there exists at least one non-subcritical regular end (see Definition 2.17). Our aim is to prove the estimate (2.24) that is,

$$p(t,o,o) \simeq \frac{1}{V_m(\sqrt{t})}, \tag{4.23}$$

where $V_m(r)$ is the largest volume function at scale r.

As before, let K be the central part of M. Let A be a connected, precompact open subset of M with smooth boundary and such that $K_\varepsilon \subset A$ for some large enough $\varepsilon > 0$. Set

$$\partial A_i := (\partial A) \cap E_i, \quad i = 1, \dots, k.$$

First we prove the following heat kernel upper bound.

Proposition 4.10. *Let $M = \#_{i\in I} M_i$ be a manifold with parabolic ends, where each M_i is either regular (Definition 2.16) or subcritical (Definition 2.8). Then for any $t \gg 1$*

$$p(t,o,o) \lesssim \frac{\min_i h_i^2(\sqrt{t})}{\min_i \widetilde{V}_i(\sqrt{t})}. \tag{4.24}$$

We use here the following two Lemmas from [13] that were proved for arbitrary manifolds with ends.

Lemma 4.11. *([13, Lemma 4.1]) There exists a constant $C = C(A,K) > 0$ such that, for all $\lambda > 0$,*

$$(\sup_{\partial K} \gamma_\lambda) \sum_{i=1}^{k} \inf_{\partial A_i} \Phi_\lambda^{E_i} \le C. \tag{4.25}$$

The estimate (4.25) combined with the estimates in (4.4) and (4.9) implies that

$$p(t,o,o) \le \frac{1}{t} \min_i h_i(\sqrt{t}).$$

As was pointed out in [13, Remark 3.2], this estimate gives the optimal upper bound of $p(t,o,o)$ when all M_i are subcritical. However, if there exists at least one critical end, this estimate yields

$$p(t,o,o) \le C \frac{\log t}{t},$$

instead of the desired bound

$$p(t,o,o) \le \frac{C}{t}.$$

In order to obtain an optimal bound of $p(t,o,o)$ on parabolic manifold with non-subcritical ends, we will use the following result.

Lemma 4.12. *([13, Lemma 4.2]) There exists a constant $C>0$ depending on A,K such that for any $\lambda>0$*

$$(\sup_{\partial K}\dot{\gamma}_\lambda)\sum_{i=1}^{k}\inf_{\partial A_i}\Phi_\lambda^{E_i}\leq C+(\sup_{\partial K}\gamma_\lambda)\left(C+\sum_{i=1}^{k}\sup_{\partial A_i}\Psi_\lambda^{E_i}\right). \tag{4.26}$$

Proof of Proposition 4.10. Substituting (4.9) into (4.25), we obtain

$$\sup_{\partial K}\gamma_\lambda\lesssim\frac{1}{\sum_{i=1}^{k}\inf_{\partial A_i}\Phi_\lambda^{E_i}}\lesssim\frac{1}{\sum_{i=1}^{k}\frac{1}{h_i(\frac{1}{\sqrt{\lambda}})}}\lesssim\min_{i\in I}h_i(\frac{1}{\sqrt{\lambda}}):=w\left(\lambda\right). \tag{4.27}$$

Applying (4.26) with (4.27) and (4.13), we obtain, for small $\lambda>0$,

$$\sup_{\partial K}\dot{\gamma}_\lambda\lesssim w\left(\lambda\right)\left(1+w\left(\lambda\right)\left(1+\sum_{i=1}^{k}\sup_{\partial A_i}\Psi_\lambda^{E_i}\right)\right)\lesssim\frac{w^2\left(\lambda\right)}{\lambda^2\min_i\widetilde{V}_i(\frac{1}{\sqrt{\lambda}})},$$

where we have also used that $w\geq 1$ and, by (4.15),

$$\lambda^2\min_i\widetilde{V}_i(\frac{1}{\sqrt{\lambda}})\lesssim 1.$$

Hence, we have proved that

$$\sup_{\partial K}\dot{\gamma}_\lambda\lesssim\frac{\min_i h_i^2(\frac{1}{\sqrt{\lambda}})}{\lambda^2\min_i\widetilde{V}_i(\frac{1}{\sqrt{\lambda}})}.$$

Taking here $\lambda=t^{-1}$ and substituting this estimate of $\dot{\gamma}_\lambda$ into (4.5), we conclude Proposition 4.10. ■

Proof of Theorem 2.19 (estimate (4.23)). First, if all ends are subcritical, then we have by definition

$$h_i(r)\simeq\frac{r^2}{V_i(r)},\quad \widetilde{V}_i(r)\simeq r^2.$$

Then the estimate in (4.24) implies that

$$p(t,o,o)\lesssim\frac{\min_i\frac{t}{V_i(\sqrt{t})}}{t}=\frac{1}{V_m(\sqrt{t})}\simeq\frac{1}{V(o,\sqrt{t})}, \tag{4.28}$$

where $V(o,r)$ is the measure of the geodesic ball in M with radius r centered at o.

If there exists at least one non-subcritical regular end, then by the assumption of (DOE), we have

$$\min_i h_i^2(r)\simeq h_l^2(r),\quad \min_i\widetilde{V}_i(r)\simeq\widetilde{V}_l(r).$$

Substituting this into the estimate in (4.24), it follows that for all $t\gg 1$

$$p(t,o,o)\lesssim\frac{h_l^2\left(\sqrt{t}\right)}{\widetilde{V}_l\left(\sqrt{t}\right)}=\frac{1}{V_l(\sqrt{t})}. \tag{4.29}$$

Because $V\left(o,r\right)\simeq V_m(r)\simeq V_l\left(r\right)$ (see Lemma 5.2), we obtain, for all $t\gg 1$

$$p(t,o,o)\lesssim\frac{1}{V\left(o,\sqrt{t}\right)},$$

which gives the same upper estimate as in (4.28).

For a bounded range of t, this estimate is trivially satisfied, so that it holds for all $t > 0$. Since $V_m(r)$ is doubling, the volume function $V(o,r)$ on M is also doubling. By [7], we conclude that $p(t,o,o)$ satisfies also a matching lower bound

$$p(t,o,o) \gtrsim \frac{1}{V\left(o,\sqrt{t}\right)} \simeq \frac{1}{V_m\left(\sqrt{t}\right)}.$$

Hence, we obtain the estimate in (4.23), which concludes Theorem 2.19. ■

Remark 4.13. If all ends M_i are either critical (i.e., $V(r) \simeq r^2$) or subcritical, then (2.24) was already obtained in our previous paper [13, Theorem 2.1]. When all ends are subcritical, m may depend on r (see Theorem 4.2).

If (DOE) is not satisfied then the upper estimate in (4.29) might not be optimal. We expect to obtain such an example in the case $M = M_1 \# M_2$ with $V_i(r) = r^2\varphi_i(r)$, where each φ_i is a slowly varying function and V_1, V_2 are not properly ordered in any of the above sense. See [14] for an example of volume functions V_1, V_2 behaving that way.

5 Estimate of the Poincaré constant

5.1 Remarks on the conditions (COE), (DOE) and regularity

In the next two lemmas we collect some already known properties.

Lemma 5.1. *Let V be regular with parameters $\gamma_1, \gamma_2 > 0$ satisfying $2\gamma_1 + \gamma_2 < 2$. Then for all $r \geq 1$ we have*

$$r^{2-\gamma_2} \lesssim V(r) \lesssim r^{2+\gamma_1}, \tag{5.1}$$

$$r^2 \lesssim V(r)h(r) \lesssim r^{2+\gamma_1}, \tag{5.2}$$

$$V(r) \lesssim \widetilde{V}(r) \lesssim r^{2+\gamma_1+\gamma_2}. \tag{5.3}$$

Proof. The estimates (5.1) follow immediately from (2.22). The lower bound in (5.2) is equivalent to (2.16). The upper bound in (5.2) is equivalent to (4.12).

The lower bound in (5.3) is trivial because $h \geq 1$. The upper bound in (5.3) coincides with (4.10). ■

Lemma 5.2. *Let $M = \#_{i\in I} M_i$ be a manifold with ends, where each M_i satisfies (VD) and (PI). Assume that M satisfies (COE) with parameters $\varepsilon, \delta, \gamma_1, \gamma_2$ (see Definition 2.8).*

- *In general, we have:*
  - *(a) If $i \in I_{super}$, then M_i is non-parabolic and $h_i(r) \simeq 1$.*
  - *(b) If $i \in I_{super}$, $j \in I_{middle}$ and $k \in I_{sub}$, then, for all $r \gg 1$,*

$$V_i(r) \gtrsim V_j(r) \gtrsim V_k(r) \quad \text{and } V_i(r) \gtrsim V_j(r)h_j(r) \gtrsim V_k(r)h_k(r) \simeq r^2 \tag{5.4}$$

and

$$\widetilde{V}_j(r) \lesssim \widetilde{V}_k(r). \tag{5.5}$$

- *If all ends are parabolic and $I_{middle} \neq \emptyset$ then the following properties hold:*
  - *(c) $I_{super} = \emptyset$ and M admits (DOE), introduced in Definition 2.17, that is, there exists a dominating volume function V_l with $l \in I_{middle}$, that is a volume function such that for all $i \in I$*

$$V_l \gtrsim V_i, \quad \widetilde{V}_l \lesssim \widetilde{V}_i. \tag{5.6}$$

(d) For all $r \gg 1$,

$$V_l(r) \simeq V_m(r) \text{ and } \quad \min_i h_i(r) \simeq h_l(r) \simeq h_m(r),$$

where $m = m(r)$ is the index of the largest end defined in (2.8).

(e) For all $r \gg 1$,

$$\max_{j \neq l} \{V_j(r)h_j(r)\} \simeq \quad V_n(r)h_n(r),$$

where $n = n(r)$ is the index of the second largest end defined in (2.9).

Proof. (a) For any $i \in I_{super}$ we have by definition $V_i(r) \gtrsim r^{2+\varepsilon}$ and, hence,

$$1 \leq h_i(r) \lesssim 1 + \int_1^r \frac{1}{s^{2+\varepsilon}} s ds \lesssim 1.$$

This means non-parabolicity of M_i because of (2.1).

(b) By (2.11) $\varepsilon > \gamma_1$ and $\gamma_2 < \delta$, we obtain, by using Definition 2.8 that

$$V_i(r) \gtrsim r^{2+\varepsilon} \gtrsim r^{2+\gamma_1} \gtrsim V_j(r) \gtrsim r^{2-\gamma_2} \gtrsim r^{2-\delta} \gtrsim V_k(r).$$

By (5.2), subcriticality of V_k and $\varepsilon > \gamma_1$ we obtain

$$V_k(r)h_k(r) \simeq r^2 \lesssim V_j(r)h_j(r) \lesssim r^{2+\gamma_1} \lesssim r^{2+\varepsilon} \lesssim V_i(r),$$

which proved (5.4).

Since V_k is subcritical, we have

$$\widetilde{V}_k(r) \simeq \frac{r^4}{V_k(r)} \geq cr^{2+\delta}.$$

Because $\gamma_1 + \gamma_2 < \delta$, we obtain by (5.3)

$$\widetilde{V}_j(r) \lesssim r^{2+\gamma_1+\gamma_2} \lesssim r^{2+\delta} \lesssim \widetilde{V}_k(r).$$

Now we assume that all ends are parabolic and $I_{middle} \neq \emptyset$.

(c) (a) implies $I_{super} = \emptyset$ immediately. By the definition of I_{middle} and the estimates in (5.4), there exists an end $l \in I_{middle}$ such that for all $i \in I$

$$V_l \gtrsim V_i. \tag{5.7}$$

By the estimates in (5.5), we obtain

$$\widetilde{V}_l \lesssim \widetilde{V}_i,$$

which concludes (DOE).

(d) $V_l \gtrsim V_m$ and $V_m(r) \geq V_l(r)$ imply $V_l(r) \simeq V_m(r)$ immediately. By the estimates in (5.7), we obtain

$$\min_{i \in I} h_i \simeq h_l.$$

Next, $V_l \gtrsim V_m$ implies that $h_l \lesssim h_m$ and, by the assumption of (COE)

$$V_l h_l \gtrsim V_i h_i.$$

Since $V_l(r) \simeq V_m(r)$ at r, we obtain

$$h_l(r) \gtrsim h_m(r),$$

which concludes $h_l(r) \simeq h_m(r)$.

(e) If $|I_{middle}| = 1$, then for all $r \gg 1$, $l = m(r)$ and the second largest end is subcritical. Hence we can prove (e) easily (see also Lemma 5.2).

If $|I_{middle}| \geq 2$, then for all $r \gg 1$, m and n are in I_{middle}. Assume first that $m = m(r) = l$. Let $j \in I_{middle}$ be the index so that $V_j(r)h_j(r) = \max_{i \in I \setminus \{l\}} \{V_i(r)h_i(r)\}$. By the condition of (COE), Either $V_n \gtrsim V_j$ or $V_j \gtrsim V_n$ holds. If $V_n \gtrsim V_j$, then $V_n h_n \gtrsim V_j h_j$, which implies (e). If $V_j \gtrsim V_n$, we note that $V_n(r) \geq V_j(r)$ by $j \neq m = l$. Then same argument as in (a) concludes $V_j(r) \simeq V_n(r)$ and $h_j(r) \simeq h_n(r)$ which concludes (e).

Next, assume that $m \neq l$. In this case, we have

$$V_l \gtrsim V_m \quad \text{and} \quad V_l \gtrsim V_n,$$
$$V_m(r) \geq V_n(r) \geq V_l(r).$$

Then we obtain $V_l(r) \simeq V_m(r) \simeq V_n(r)$ and, by the same argument as in (d), we obtain

$$h_l(r) \simeq h_m(r) \simeq h_n(r),$$

whence we conclude (e). ■

5.2 Lower bound on Dirichlet heat kernel

Recall that M is a manifold with ends $M_1, \ldots, M_k$, where each end admits (VD), (PI). Moreover, assume (RCA) on each parabolic end. First, we prove the following technical lemmas.

Lemma 5.3. *Let M be as above. Assume also that for all $t > 0$*

$$p(t, o, o) \simeq \frac{\min_i h_i^2(\sqrt{t})}{\min_i \widetilde{V}_i(\sqrt{t})}. \tag{5.8}$$

Then we have for all $r \gg 1$

$$\int_1^{r^2} p(s, o, o)ds \simeq \min_i h_i(r). \tag{5.9}$$

Proof. If M is non-parabolic, although the estimates in (5.8) holds (see Theorem 2.15), we can prove the estimate (5.9) without using (5.8). Indeed, by the definition of the non-parabolicity we have for $r \gg 1$

$$\int_1^{r^2} p(s, o, o)ds \simeq 1.$$

Because $h_i \simeq 1$ for each non-parabolic end, we have

$$\min_i h_i \simeq 1.$$

Then we conclude (5.9).

Next, suppose that M is parabolic. By the assumption (5.8),

$$\int_1^{r^2} p(s, o, o)ds \simeq \int_1^r \frac{\min_i h_i^2(s)}{\min_i \widetilde{V}_i(s)} sds.$$

Observe that

$$h_i(r) = \frac{1}{\int_r^\infty \frac{sds}{\widetilde{V}_i(s)}},$$

which is obtained by integration of the identity

$$\left(\frac{1}{h(r)}\right)' = -\frac{h'(r)}{h^2(r)} = -\frac{r/V(r)}{h^2(r)} = -\frac{r}{\widetilde{V}_i(r)}$$

and

$$\frac{1}{h(r)} \to 0 \text{ as } r \to \infty$$

by the parabolicity. Then we have

$$\begin{aligned} \min_i h_i(r) &= \frac{1}{\max_i \int_r^\infty \frac{sds}{\widetilde{V}_i(s)}} \simeq \frac{1}{\sum_i \int_r^\infty \frac{sds}{\widetilde{V}_i(s)}} \\ &\simeq \frac{1}{\int_r^\infty \frac{sds}{\min_i \widetilde{V}_i(s)}} = \frac{1}{\int_r^\infty \frac{sds}{W(s)}} =: h(r), \end{aligned}$$

where

$$W(r) = \min_i \widetilde{V}_i(r).$$

Because

$$-\frac{h'(r)}{h^2(r)} = \left(\frac{1}{h(r)}\right)' = -\frac{r}{W(r)}$$

and h is a doubling function, we conclude that for all $r \geq 2$

$$\int_1^r \frac{\min_i h_i^2(s)}{\min_i \widetilde{V}_i(s)} sds \simeq \int_1^r \frac{h^2(s)}{W(s)} sds = \int_1^r h'(s)ds = h(r) - h(1) \simeq h(r).$$

■

For $i = 1, \dots, k$, set

$$A_i(r) = (B(o,r) \backslash B(o,r/2)) \cap E_i. \tag{5.10}$$

For $\kappa > 0$, κB means $B(o,\kappa r)$. Recall that $p^D_{\kappa B}(t,x,y)$ is the Dirichlet heat kernel on κB.

The following estimates of $p^D_{\kappa B}(t,x,y)$ has a key role for the estimate of the Poincaré constant:

Lemma 5.4. *Assume that a connected sum $M = M_1 \# \dots \# M_k$ satisfies all the hypothesis of Lemma 5.3. Then there exists $\kappa \geq 1$ such that for all $r \gg 1$, for any $x \in A_i(r)$, $y \in B(o,r) \cap E_j$,*

$$p^D_{\kappa B}(r^2,x,y) \simeq \frac{1}{V_i(r)h_j(r)} \left[\delta_{ij} h_j(|y|) + \frac{\min_\eta h_\eta(r)}{h_i(r)} \left(\int_{|y|}^r \frac{sds}{V_j(s)} + \frac{r^2 h_j(|y|)}{V_j(r)h_j(r)} \right) \right]. \tag{5.11}$$

Moreover, this implies that

$$\inf_{x \in A_i(r), y \in B} p^D_{\kappa B}(r^2,x,y) \gtrsim \frac{r^2 \min_\eta h_\eta(r)}{V_i(r)h_i(r) \max_{j \neq i} \{V_j(r)h_j(r)\}}. \tag{5.12}$$

Proof. We prove the estimate in (5.11) only from below (upper bound follows from the trivial bound $p^D_{\kappa B}(r^2,x,y) \leq p(r^2,x,y)$ and the estimate in (5.14)).

Recall that $\tau_{\kappa B}$ is the first exist time from κB. By the definition of the Dirichlet heat kernel and the strong Markov property of the Brownian motion on M (see also [19]), we obtain

$$p^D_{\kappa B}(r^2,x,y) = \quad p(r^2,x,y) - \mathbb{E}_y(\mathbf{1}_{\{\tau_{\kappa B} \leq r^2\}} p(r^2 - \tau_{\kappa B}, X_{\tau_{\kappa B}}, x)),$$

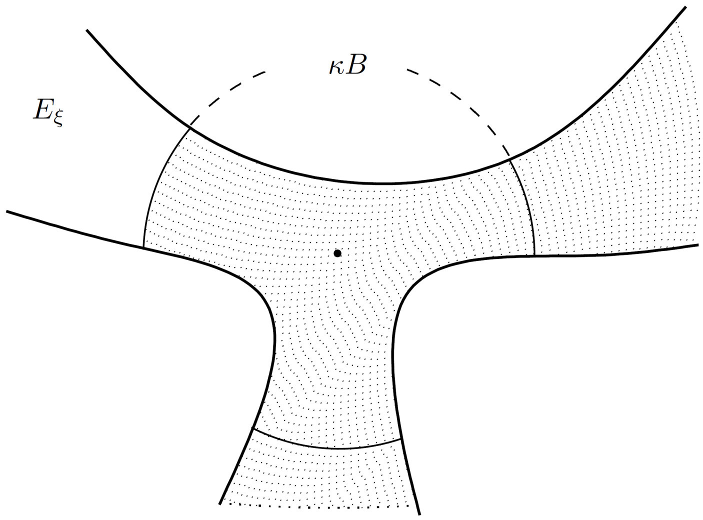

Figure 5: $\Big((\kappa B)^c \cap E_\xi\Big)^c$ (shadow area).

where $\kappa \geq 1$ will be chosen later. By the structure of the connected sum, we observe that

$$\{\tau_{\kappa B} \leq r^2\} = \cup_{\xi=1}^k \{\tau_{\kappa B} \leq r^2,\ X_{\tau_{\kappa B}} \in E_\xi\} \subset \cup_{\xi=1}^k \left\{\tau_{\left((\kappa B)^c \cap E_\xi\right)^c} \leq r^2\right\}$$

(see Figure 5.2).

We note that

$$\partial\Big((\kappa B)^c \cap E_\xi\Big)^c = (\partial \kappa B) \cap E_\xi.$$

Then we obtain

$$\begin{aligned} p^D_{\kappa B}(r^2,x,y) &= p(r^2,x,y) - \sum_{\xi=1}^k \mathbb{E}_y\left(\mathbf{1}_{\{\tau_{\kappa B}\leq r^2,\ X_{\tau_{\kappa B}}\in E_\xi\}} p(r^2-\tau_{\kappa B}, X_{\tau_{\kappa B}}, x)\right) \\ &\geq p(r^2,x,y) - \sum_{\xi=1}^k \mathbb{P}_y\left(\tau_{\left((\kappa B)^c\cap E_\xi\right)^c} \leq r^2\right) \sup_{\substack{0<s\leq r^2 \\ z\in(\partial\kappa B)\cap E_\xi}} p(s,z,x). \end{aligned} \tag{5.13}$$

We estimate $p(r^2,x,y)$, $\mathbb{P}_y\Big(\tau_{\left((\kappa B)^c\cap E_\xi\right)^c} \leq r^2\Big)$ and $\sup_{\substack{0<s\leq r^2 \\ z\in(\partial\kappa B)\cap E_\xi}} p(s,z,x)$ separately.

Step1: Estimate of $p(r^2,x,y)$

We will prove that for any $x \in A_i(r)$, $y \in B(o,r)\cap E_j$

$$p(r^2,x,y) \simeq \frac{1}{V_i(r)h_j(r)}\left[\delta_{ij}h_j(|y|) + \frac{\min_\eta h_\eta(r)}{h_i(r)}\left(\int_{|y|}^r \frac{sds}{V_j(s)} + \frac{r^2 h_j(|y|)}{V_j(r)h_j(r)}\right)\right], \tag{5.14}$$

which is nothing but the same estimate as in (5.11) for the full heat kernel $p(r^2,x,y)$ instead of $p^D_{\kappa B}(r^2,x,y)$. Because $p^D_{\kappa B}(r^2,x,y) \leq p(r^2,x,y)$ is always true, this estimate gives the upper bound in (5.11).

To prove (5.14), we use Theorem 4.3 and the location of x, y to obtain

$$\begin{aligned} p(r^2,x,y) \quad &\simeq \quad \delta_{ij}\frac{1}{V_i(r)}\frac{h_j(|y|)}{h_j(r)} + p(r^2,o,o)\frac{r^2}{V_i(r)h_i(r)}\frac{1}{h_j(r)}\int_{|y|}^{r}\frac{sds}{V_j(s)} \\ &+\int_1^{r^2} p(s,o,o)ds\left(\frac{1}{V_i(r)h_i(r)}\frac{1}{h_j(r)}\int_{|y|}^{r}\frac{sds}{V_j(s)} + \frac{r^2}{V_i(r)h_i(r)}\frac{h_j(|y|)}{V_j(r)h_j^2(r)}\right). \end{aligned} \tag{5.15}$$

Because $p(t,o,o)$ is monotone decreasing, it is easy to see that

$$r^2 p(r^2,o,o) \le C\int_1^{r^2} p(s,o,o)ds, \tag{5.16}$$

which makes the third term of the right hand side in (5.15) dominate the second term. By using Lemma 5.3, we obtain (5.14).

Step2: Estimate of $\mathbb{P}_y\Big(\tau_{\big((\kappa B)^c\cap E_\xi\big)^c} \le r^2\Big)$

We will prove that for any $y \in B(o,r)\cap E_j$

$$\mathbb{P}_y\left(\tau_{\big((\kappa B)^c\cap E_\xi\big)^c} \le r^2\right) \asymp C\Big[\delta_{\xi j}\frac{h_j(|y|)}{h_j(r)} + \frac{\min_\eta h_\eta(r)}{h_\xi(r)}\frac{1}{h_j(r)}\int_{|y|}^{r}\frac{sds}{V_j(s)}\Big]e^{-b\kappa^2}. \tag{5.17}$$

Since $y \in B(o,r)\cap E_j$, applying Theorem 3.5 in [17], we obtain

$$\mathbb{P}_y(\tau_{((\kappa B)^c\cap E_\xi)^c} \le r^2) \le \mathbb{P}_y(\tau_{(F^c)} \le r^2) \le 2\mathrm{cap}(F,\Omega)\int_1^{r^2}\sup_{\omega\in\Omega\setminus F} p(s,\omega,y)ds,$$

where we chose $\kappa > 2$ and

$$F = \left\{\omega \in E_\xi \ : \ \frac{3\kappa r}{4} \le |\omega| \le \frac{5\kappa r}{4}\right\},\ \Omega = \left\{\omega\in E_\xi \ : \ \frac{\kappa r}{2} < |\omega| < \frac{3\kappa r}{2}\right\}.$$

Here we remark that $(\partial\kappa B)\cap E_\xi \subset F \subset \Omega$ and $y \notin \Omega$ because of $\kappa > 2$.

By the estimate in Theorem 4.3,

$$\begin{aligned} \sup_{\omega\in\Omega\setminus F} p(s,\omega,y) \asymp C\Big[&\frac{\delta_{\xi j}}{V_\xi(\sqrt{s})}\frac{h_j(|y|)}{h_j(|y|)+h_j(\sqrt{s})} + p(s,o,o)\frac{r^2}{V_\xi(r)h_\xi(r)}\mathbb{P}_y(\tau_K \le s) \\ +&\int_1^s p(u,o,o)du\Big(\frac{1}{V_\xi(\sqrt{s})h_\xi(\sqrt{s})}\mathbb{P}_y(\tau_K\le s) + \frac{r^2}{V_\xi(r)h_\xi(r)}\partial_s\mathbb{P}_y(\tau_K\le s)\Big)\Big]e^{-b(\kappa r)^2/s}. \end{aligned}$$

Then we obtain

$$\begin{aligned} &\int_1^{r^2}\sup_{\omega\in\Omega\setminus F} p(s,\omega,y)ds \\ &\qquad\le C\Big[\delta_{\xi j}\frac{r^2}{V_\xi(r)}\frac{h_j(|y|)}{h_j(r)} + \int_1^{r^2} p(s,o,o)ds\frac{r^2}{V_\xi(r)h_\xi(r)}\mathbb{P}_y(\tau_K\le r^2)\Big]e^{-b\kappa^2}. \end{aligned}$$

Since

$$\mathrm{cap}(F,\Omega) = \mathrm{cap}\Big(\overline{B_\xi\Big(o_\xi,\frac{\kappa r}{2}\Big)}, B_\xi\Big(o_\xi,\frac{3\kappa r}{4}\Big)\Big) + \mathrm{cap}\Big(\overline{B_\xi\Big(o_\xi,\frac{5\kappa r}{4}\Big)}, B_\xi\Big(o_\xi,\frac{3\kappa r}{2}\Big)\Big),$$

where $B_\xi(x,r)$ is a geodesic open ball in M_ξ. By using the general estimate for any $0 < r < R$

$$\mathrm{cap}(\overline{B(o,r)}, B(o,R)) \le \frac{2}{\int_r^R \frac{sds}{\mu(B(o,s))}}$$

(see [11, Theorem 7.1], [16, (4.5)] for the detail), we obtain

$$\operatorname{cap}(F,\Omega) \le C\frac{V_\xi(r)}{r^2}.$$

By Theorem 4.6 in [17],

$$\mathbb{P}_y(\tau_K \le r^2) \le \frac{C}{h_j(r)}\int_{|y|}^r \frac{sds}{V_j(s)}.$$

Combining these estimate and Lemma 5.3, we conclude (5.17).

Step3: Estimate of $\sup_{0<s\le r^2, z\in(\partial\kappa B)\cap E_\xi} p(s,z,x)$

We will prove that for any $x \in A_i(r)$

$$\sup_{\substack{0<s\le r^2\\ z\in(\partial\kappa B)\cap E_\xi}} p(s,z,x) \le \frac{C}{\kappa^{2\alpha}}\Big[\delta_{i\xi}\frac{1}{V_i(r)} + \min_\eta h_\eta(r)\frac{r^2}{V_i(r)h_i(r)}\frac{1}{V_\xi(r)h_\xi(r)}\Big]. \tag{5.18}$$

By the estimates in Theorem 4.3, for all $0 < s \le r^2$ and $z \in (\partial\kappa B)\cap E_\xi$

$$p(s,x,z) \asymp Ce^{-b(\kappa r)^2/s}\Big[\delta_{i\xi}\frac{1}{V_i(\sqrt{s})} + p(s,o,o)\frac{r^2}{V_i(r)h_i(r)}\frac{r^2}{V_\xi(r)h_\xi(r)}$$
$$+\int_1^s p(u,o,o)du\left(\frac{1}{V_i(\sqrt{s})h_i(\sqrt{s})}\frac{r^2}{V_\xi(r)h_\xi(r)} + \frac{r^2}{V_i(r)h_i(r)}\frac{1}{V_\xi(\sqrt{s})h_\xi(\sqrt{s})}\right)\Big].$$

Since V_i and $\widetilde{V}_i$ are doubling, there exist $C,\alpha>0$ such that

$$\begin{aligned}
\frac{1}{V_i(\sqrt{s})} &\le \frac{C}{V_i(r)}\left(\frac{r^2}{s}\right)^\alpha,\\
p(s,o,o) &\simeq p(r^2,o,o)\frac{\frac{\min_i h_i^2(\sqrt{s})}{\min_i \widetilde{V}_i(\sqrt{s})}}{\frac{\min_i h_i^2(r)}{\min_i \widetilde{V}_i(r)}}\\
&\le p(r^2,o,o)\frac{\min_i \widetilde{V}_i(r)}{\min_i \widetilde{V}_i(\sqrt{s})} \le Cp(r^2,o,o)\left(\frac{r^2}{s}\right)^\alpha,\\
\frac{1}{V_i(\sqrt{s})h_i(\sqrt{s})} &= \frac{h_i(\sqrt{s})}{\widetilde{V}_i(\sqrt{s})} \le \frac{C}{V_i(r)h_i(r)}\left(\frac{r^2}{s}\right)^\alpha.
\end{aligned}$$

Because the inequality

$$t^\alpha e^{-t} \le \alpha^\alpha e^{-\alpha}$$

is always true for all $t>0$, we have

$$\left(\frac{r^2}{s}\right)^\alpha e^{-b\kappa^2\frac{r^2}{s}} \le \frac{1}{b^\alpha\kappa^{2\alpha}}\alpha^\alpha e^{-\alpha}.$$

Then we obtain

$$\begin{aligned}
\sup_{\substack{0<s\le r^2\\ z\in(\partial\kappa B)\cap E_\xi}} p(s,z,x) &\le \frac{C}{\kappa^{2\alpha}}\Big[\delta_{i\xi}\frac{1}{V_i(r)} + p(r^2,o,o)\frac{r^2}{V_i(r)h_i(r)}\frac{r^2}{V_\xi(r)h_\xi(r)}\\
&\quad+\int_1^{r^2} p(u,o,o)du\left(\frac{1}{V_i(r)h_i(r)}\frac{r^2}{V_\xi(r)h_\xi(r)} + \frac{r^2}{V_i(r)h_i(r)}\frac{1}{V_\xi(r)h_\xi(r)}\right)\Big].
\end{aligned}$$

Using (5.16) and Lemma 5.3 again, we conclude (5.18).

Step4: Proof of (5.11)

Combining (5.17) and (5.18), we obtain

$$\begin{aligned}
&\sum_{\xi=1}^{k} \mathbb{P}_y(\tau_{(\kappa B)^c\cap E_\xi} \le r^2) \sup_{\substack{0<s\le r^2\\ z\in(\partial\kappa B)\cap E_\xi}} p(s,z,x)\\
&\le \frac{Ce^{-b\kappa^2}}{\kappa^{2\alpha}} \sum_{\xi=1}^{k} \left(\delta_{\xi j}\frac{h_j(|y|)}{h_j(r)} + \frac{\min_\eta h_\eta(r)}{h_\xi(r)}\frac{1}{h_j(r)}\int_{|y|}^{r}\frac{sds}{V_j(s)}\right)\\
&\qquad\times\left(\delta_{i\xi}\frac{1}{V_i(r)} + \min_\eta h_\eta(r)\frac{r^2}{V_i(r)h_i(r)}\frac{1}{V_\xi(r)h_\xi(r)}\right)\\
&= \frac{Ce^{-b\kappa^2}}{\kappa^{2\alpha}}\frac{1}{V_i(r)h_j(r)}\sum_{\xi=1}^{k}\left(\delta_{\xi j}h_j(|y|) + \frac{\min_\eta h_\eta(r)}{h_\xi(r)}\int_{|y|}^{r}\frac{sds}{V_j(s)}\right)\\
&\qquad\times\left(\delta_{i\xi} + \frac{\min_\eta h_\eta(r)}{h_i(r)}\frac{r^2}{V_\xi(r)h_\xi(r)}\right)\\
&= \frac{Ce^{-b\kappa^2}}{\kappa^{2\alpha}}\frac{1}{V_i(r)h_j(r)}\Big[\delta_{ij}h_j(|y|)\\
&\qquad+\frac{\min_\eta h_\eta(r)}{h_i(r)}\Big(\int_{|y|}^{r}\frac{sds}{V_j(s)}\Big(1+\frac{r^2\min_\eta h_\eta(r)}{\min_\eta \widetilde{V}_\eta(r)}\Big) + \frac{r^2}{V_j(r)h_j(r)}h_j(|y|)\Big)\Big].
\end{aligned}$$

By using Lemma 5.3 and a trivial lower bound $V_\eta(r)h_\eta(r) \gtrsim r^2$, we obtain

$$r^2\min_\eta h_\eta(r) \lesssim \min_\eta \widetilde{V}_\eta(r)$$

which implies that

$$1+\frac{r^2\min_\eta h_\eta(r)}{\min_\eta \widetilde{V}_\eta(r)} \simeq 1.$$

Then we have

$$\begin{aligned}
&\sum_{\xi=1}^{k} \mathbb{P}_y(\tau_{(\kappa B)^c\cap E_\xi} \le r^2) \sup_{\substack{0<s\le r^2\\ z\in(\partial\kappa B)\cap E_\xi}} p(s,z,x)\\
&\le \frac{Ce^{-b\kappa^2}}{\kappa^{2\alpha}}\frac{1}{V_i(r)h_j(r)}\Big[\delta_{ij}h_j(|y|) + \frac{\min_\eta h_\eta(r)}{h_i(r)}\Big(\int_{|y|}^{r}\frac{sds}{V_j(s)} + \frac{r^2}{V_j(r)h_j(r)}h_j(|y|)\Big)\Big],
\end{aligned} \tag{5.19}$$

which is same as (5.14) up to $Ce^{-b\kappa^2}/\kappa^{2\alpha}$. Choosing $\kappa > 2$ large enough and substituting (5.14) and (5.19) into (5.13), we conclude (5.11).

Step5: Proof of (5.12)

If $i = j$, then we obtain by the estimate in (5.11) and the fact that $h_j(|y|) + \int_{|y|}^{r}\frac{sds}{V_j(s)} = h_j(r)$

$$p^D_{\kappa B}(r^2,x,y) \ge \frac{c}{V_i(r)h_i(r)}\frac{\min_\eta h_\eta(r)}{h_i(r)}h_i(r) = c\frac{\min_\eta h_\eta(r)}{V_i(r)h_i(r)}. \tag{5.20}$$

If $i \neq j$, then the function

$$F(t) = \int_t^r \frac{sds}{V_j(s)} + \frac{Cr^2}{V_j(r)h_j(r)}h_j(t)$$

is monotone decreasing for some constant $C > 0$ and for all $r \gg 1$, $t \simeq r$ gives the lower bound of F, namely we obtain

$$F(r) \ge c\frac{r^2}{V_j(r)}.$$

Then we obtain by the estimate in (5.11)

$$\begin{aligned} p^D_{\kappa B}(r^2,x,y) &\geq \frac{c}{V_i(r)h_j(r)}\frac{\min_\eta h_\eta(r)}{h_i(r)}\frac{r^2}{V_j(r)} \\ &= c\frac{\min_\eta h_\eta(r)}{V_i(r)h_i(r)}\frac{r^2}{V_j(r)h_j(r)}. \end{aligned} \tag{5.21}$$

Because $V_j(r)h_j(r) \geq cr^2$ is always true, combining estimates in (5.20) and (5.21), we conclude the estimate in (5.12). ■

5.3 Upper bound on Poincaré constant

We prove here the upper bounds of the Poincaré constant: (2.10) of Theorem 2.7 and (2.18) of Theorem 2.10 with some large $\kappa \geq 1$. The following lemma inspired by Kusuoka-Stroock [21, Theorem 5.10] (see also [26, 5.5.1]) plays a key role to obtain an upper bound of the Poincaré constant from a lower bound of the Dirichlet heat kernel.

Lemma 5.5. *Let (M,μ) be a weighted manifold. Let $U \subset U' \subset M$ be precompact connected open sets. Then for any $t > 0$ and any open set $A \subset U'$,*

$$\Lambda(U,U') \leq \frac{2t}{\mu(A)\inf_{x\in A, y\in U} p^D_{U'}(t,x,y)}.$$

Proof. Let $\{P^{N,U'}_t\}$ be the Neumann-heat semigroup on U' and $p^N_{U'}(t,x,y)$ its kernel function (Neumann heat kernel). By a well-known fact for Dirichlet forms, for any $f \in C^1(\overline{U'})$, and for any $t > 0$,

$$\int_{U'} |\nabla f|^2 d\mu = \mathcal{E}(f,f)_{U'} \geq \left(\frac{f - P^{N,U'}_{2t} f}{2t}, f\right)_{L^2(U')}.$$

Then we obtain for any open set $A \subset U'$

$$\begin{aligned} 2t\int_{U'} |\nabla f|^2 d\mu &\geq \left(f - P^{N,U'}_{2t} f, f\right)_{L^2(U')} \\ &= \int_{U'} P^{N,U'}_t \left[f - P^{N,U'}_t f(x)\right]^2 (x) d\mu(x) \\ &\geq \int_A P^{N,U'}_t \left[f - P^{N,U'}_t f(x)\right]^2 (x) d\mu(x) \\ &\geq \mu(A) \inf_{x\in A} P^{N,U'}_t \left[f - P^{N,U'}_t f(x)\right]^2 (x). \end{aligned} \tag{5.22}$$

Because the Neumann heat kernel is always larger than the Dirichlet heat kernel, we obtain for all $x \in A$

$$\begin{aligned} P^{N,U'}_t \left[f - P^{N,U'}_t f(x)\right]^2 (x) &= \int_{U'} p^N_{U'}(t,x,y) \left[f(y) - P^{N,U'}_t f(x)\right]^2 d\mu(y) \\ &\geq \int_U p^D_{U'}(t,x,y) \left[f(y) - P^{N,U'}_t f(x)\right]^2 d\mu(y) \\ &\geq \inf_{y\in U} p^D_{U'}(t,x,y) \int_U |f(y) - P^{N,U'}_t f(x)|^2 d\mu(y) \\ &\geq \inf_{y\in U} p^D_{U'}(t,x,y) \int_U |f(y) - f_U|^2 d\mu(y). \end{aligned} \tag{5.23}$$

Here the final line follows from the fact that f_U minimizes the function $\xi \mapsto \int_U |f-\xi|^2 d\mu$. Combining the estimates in (5.22) and (5.23), we complete the proof. ■

Proof of Theorems 2.7 and 2.10. Now we start to prove the upper bound of the Poincaré constant in the main theorems. For the lower bound, see section 5.4.

Recall that M is the connected sum of manifolds $M_1,\dots,M_k$, where each M_i satisfies (VD) and (PI) and, (RCA) for parabolic end. If there is at least one parabolic end, we assume also that M admits (COE) defined in Definition 2.8. By the results in Theorems 2.15 and 2.19, M satisfies the estimate in (5.8) which allows us to use Lemma 5.4. Let $\kappa > 1$ be the constant from (5.11). Apply Lemma 5.5 to the case of $U = B = B(o,r)$, $U' = \kappa B = B(o,\kappa r)$, $t = r^2$ and $A = A_i(r)$ given in (5.10). Then we obtain

$$\Lambda(B,\kappa B) \le \frac{2r^2}{\mu(A_i(r)) \inf_{x\in A_i(r), y\in B} p^D_{\kappa B}(r^2,x,y)}.$$

Since $\mu(A_i(r)) \simeq V_i(r)$ by (VD) on M_i, substituting the estimates in (5.12) to the above, we obtain for all $i = 1,\dots,k$

$$\Lambda(B,\kappa B) \lesssim \frac{h_i(r)}{\min_\eta h_\eta(r)} \max_{j\neq i}\{V_j(r)h_j(r)\}. \tag{5.24}$$

If all ends are non-parabolic, then $h_i(r) \simeq 1$ for all $i = 1,\dots,k$. By taking $i = m = m(r)$, the index of the largest end at r, we obtain

$$\Lambda(B,\kappa B) \lesssim \max_{j\neq m} V_j(r) \simeq V_n(r).$$

Hence, the estimate in (2.10) holds for some $\kappa \ge 1$.

Next, let us assume that $M = \#_{i\in I} M_i$ is a manifold with (COE). Assume first that $I_{super} \neq \emptyset$. Then the index of the largest end m is in I_{super} and $\min_\eta h_\eta(r) \simeq 1$. Taking $i = m$, we obtain

$$\Lambda(B,\kappa B) \lesssim \max_{j\neq m}\{V_j(r)h_j(r)\}.$$

By the second condition in (5.4) and the definition of (COE), we conclude

$$\max_{j\neq m}\{V_j(r)h_j(r)\} \simeq V_n(r)h_n(r),$$

which gives the estimate in (2.18) for some $\kappa \ge 1$.

If $I_{super} = \emptyset$ and $I_{middle} \neq \emptyset$ then, by Lemma 5.2 there exists a dominating volume function V_l satisfying (5.6) with $l \in I_{middle}$ and $\min_i h_i(r) \simeq h_l(r)$. Estimate (5.24) with $i = l$ implies that

$$\Lambda(B,\kappa B) \lesssim \max_{j\neq l}\{V_j(r)h_j(r)\} \simeq V_n(r)h_n(r),$$

which gives the estimate in (2.18) for some $\kappa \ge 1$.

Finally, assume that $I_{super} = I_{middle} = \emptyset$, that is, all ends are subcritical. By the definition of subcriticality given in Definition 2.8 (b),

$$\min_\eta h_\eta(r) \lesssim \min_\eta \frac{r^2}{V_\eta(r)} = \frac{r^2}{V_m(r)}.$$

Taking $i = m$ in the estimate in (5.24), we obtain

$$\Lambda(B,\kappa B) \lesssim \max_{j\neq m}\{V_j(r)h_j(r)\} \simeq r^2,$$

which completes the proof of the estimate in(2.18) with $\kappa \ge 1$.

By using an additional argument in Section 6, we will show that we can reduce $\kappa > 1$ to $\kappa = 1$. ■

5.4 Lower bound on the Poincaré constant

In this section we prove the matching lower bound for the Poincaré constant under a different hypothesis.

Theorem 5.6. *Let $M = M_1\#\cdots\#M_k$ be a manifolds with ends such that each end M_i satisfies* (VD) *and*

$$CV_i(r) \geq rV_i'(r) \quad \text{for all } r \geq r_0. \tag{5.25}$$

Then for all large $r \gg 1$

$$\Lambda(B(o,r)) \geq c\max_{i\neq m}\{V_i(r)h_i(r)\}, \tag{5.26}$$

where $m = m(r)$ is the index of the largest end at scale r, namely for all $i \in I$

$$V_m(r) \geq V_i(r). \tag{5.27}$$

Let $n = n(r)$ be the index of the second largest end. Then the estimate in (5.26) implies that

$$\Lambda(B(o,r)) \geq cV_n(r)h_n(r). \tag{5.28}$$

In view of the upper estimate of the Poincaré constant in Theorems 2.7 and 2.10, the lower bound in (5.28) is optimal if either all ends are non-parabolic, or M is a manifold with (COE).

Proof. Set $B = B(o,r)$. First we show that for any $\varepsilon \in (0,1)$,

$$\varepsilon \sup_{\substack{f\in C^1(B)\\ \mu(B\cap\{f=0\})\geq\varepsilon\mu(B)}} \frac{\int_B |f|^2 d\mu}{\int_B |\nabla f|^2 d\mu} \leq \Lambda(B,B). \tag{5.29}$$

Indeed, for any $f \in C^1(B)$ with $\mu(B\cap\{f=0\}) \geq \varepsilon\mu(B)$,

$$\begin{aligned}\int_B |f-f_B|^2 d\mu &= \int_B |f|^2 d\mu - \mu(B)f_B^2 \\ &= \int_B |f|^2 d\mu - \frac{1}{\mu(B)}\left(\int_{B\cap\{f\neq 0\}} f d\mu\right)^2.\end{aligned}$$

Cauchy-Schwarz inequality implies that

$$\begin{aligned}\int_B |f-f_B|^2 d\mu &\geq \int_B |f|^2 d\mu - \frac{\mu(B\cap\{f\neq 0\})}{\mu(B)}\int_B |f|^2 d\mu \\ &= \frac{\mu(B\cap\{f=0\})}{\mu(B)}\int_B |f|^2 d\mu \geq \varepsilon\int_B |f|^2 d\mu.\end{aligned}$$

By the expression of the Poincaré constant (2.6), we obtain the estimate in (5.29).

Now we prove the estimate in (5.26) by choosing a test function. For $i \neq m$, take a C^1-function f_i defined by

$$f_i(x) = \begin{cases} h_i(|x|) & x \in B\cap E_i, \\ 0 & x \in B\cap E_j,\ j\neq i. \end{cases}$$

We note that the assumption (5.27) implies that

$$\mu(B\cap E_m) \geq \frac{1}{2}\mu(B\cap E_j)$$

for large $r \gg 1$ and all $j \in \{1, \ldots, k\}$. Because $f_i = 0$ in $B \cap E_m$, we observe that

$$
\begin{aligned}
\mu(B \cap \{f_i = 0\}) &\geq \mu(B \cap E_m) \\
&= \frac{1}{2k}\left(2k\mu(B \cap E_m) + \mu(K)\right) - \frac{1}{2k}\mu(K) \\
&\geq \frac{1}{2k}\left(\mu(B) - \mu(K)\right).
\end{aligned}
$$

Then for large $r \gg 1$ so that $\mu(B) \geq 2\mu(K)$, we obtain

$$
\mu(B \cap \{f = 0\}) \geq \frac{1}{4k}\mu(B).
$$

Hence, (5.29) implies that

$$
\frac{1}{4k}\frac{\int_B |f_i|^2 d\mu}{\int_B |\nabla f_i|^2 d\mu} \leq \Lambda(B, B).
$$

By using the assumptions (VD) and (5.25), we obtain for large $r \gg 1$

$$
\begin{aligned}
\int_B |f_i|^2 d\mu &\geq \int_{(B \setminus \frac{1}{2}B) \cap E_i} |f_i|^2 d\mu \\
&\geq \left(\mu(B \cap E_i) - \mu(\frac{1}{2}B \cap E_i)\right) \inf_{(B \setminus \frac{1}{2}B) \cap E_i} |f_i|^2 \\
&\geq cV_i(r)h_i^2(r), \\
\int_B |\nabla f_i|^2 d\mu &= \int_{B(o,r_0)} |\nabla f_i|^2 d\mu + \int_{r_0}^r |\partial_s h_i(s)|^2 V'(s) ds \\
&= C + \int_{r_0}^r \frac{s}{V_i(s)} \frac{sV'(s)}{V(s)} ds \leq C + Ch_i(r) \\
&\leq C' h_i(r).
\end{aligned}
$$

Then we obtain for any $i \neq m$,

$$
cV_i(r)h_i(r) \leq \Lambda(B)
$$

which concludes (5.26). ■

6 Appendix: Spectral gap on central balls

Recall that M is a connected sum of manifolds $M_1, \ldots, M_k$ with a central part K. The purpose of this section is to obtain a general upper bound of the Poincaré constant $\Lambda(B(o, r)) = \Lambda(B(o, r), B(o, r))$ at the central reference point $o \in K$ for any large $r > 0$ by using a collection of $\Lambda(B(x, s), B(x, \kappa s))$, where $B(x, s) \subset B(o, r)$ and $\kappa \geq 1$. This is an important procedure to obtain a lower bound of the spectral gap $\lambda(B(o, r))$ of $-\Delta$ on $B(o, r)$ defined in (2.7). First of all, we only assume the volume doubling condition (VD) on each M_i.

The main tool to obtain a desired bound is a *Whitney covering* $\mathcal{W}$ of $B(o, r)$ which is a collection of balls defined as the following.

Definition 6.1. For any $0 < \eta < 1$, a collection of balls $\mathcal{W} = \mathcal{W}(\eta)$ in $B(o, r)$ is called a Whitney covering of $B(o, r)$ with parameter η if

(W1) All $F \in \mathcal{W}$ are disjoint,

(W2) $\cup_{F \in \mathcal{W}} 3F = B(o, r)$, (W3) $r(F) = \eta d(F, B(o,r)^c)$. Here $r(F)$ is the radius of F and $B(o,r)^c$ is the complement of $B(o,r)$.

(W4) For any $\alpha \geq 1$, there exists $N = N(\eta, \alpha)$ independent of r such that for any $x \in B$

$$\#\{F \in \mathcal{W} \; : \; x \in \alpha F\} \leq N.$$

It is a well-known fact that there exists such a covering. For $F \in \mathcal{W}$, we denote by γ_F a distance-minimizing curve joining the center of F and o. Let $\mathcal{F}(F) = (F_0, F_1, \ldots, F_{l(F)})$ be a string of F, that is, a sequence of balls in $\mathcal{W}$ so that $3F_j \cap 3F_{j+1} \neq \emptyset$ $(j = 0, 1, \ldots, l(F)-1)$, $o \in 3F_0$ and $F_{l(F)} = F$. It is known that there exists such a string $\mathcal{F}(F)$ for any $F \in \mathcal{W}$. See [26, Section 5.3.3] for the detail.

Proposition 6.2 (c.f. [26, Section 5.3.3])**.** *Let $\mathcal{W}$ be a Whitney covering of $B(o,r)$ with parameter $0 < \eta < 1/4$. Then $\mathcal{W}$ satisfies the following.*

(PW1) *([26, Lemma 5.3.6]) For any $F, F' \in \mathcal{W}$ so that $3F' \cap \gamma_F \neq \emptyset$,*

$$r(F') \geq \frac{1}{4\eta + 1} r(F).$$

(PW2) *([26, Lemma 5.3.7]) For $F \in \mathcal{W}$, let $\mathcal{F}(F) = (F_0, F_1, \ldots, F_{l(F)})$ be a string of F. Then for any $j = 0, 1, \ldots, l(F)-1$,*

$$\begin{aligned} (\eta^{-1} - 4) r(F_j) &\leq (4 + \eta^{-1}) r(F_{j+1}) \\ (\eta^{-1} - 4) r(F_{j+1}) &\leq (4 + \eta^{-1}) r(F_j), \end{aligned}$$

and

$$3F_{j+1} \subset \left(6\frac{\eta^{-1}+4}{\eta^{-1}-4} + 3\right) F_j. \tag{6.1}$$

We note that $6\frac{\eta^{-1}+4}{\eta^{-1}-4} + 3 < 12$ if $\eta < 1/20$. Thus, we always assume that $\eta < 1/20$ in the sequel.

Lemma 6.3. *For any $\kappa \geq 1$ and any $0 < \eta < 1/20$, let $B_0 = B(o, 12\kappa\eta r)$. Then for any r large enough so that $\mathrm{diam} K \leq \eta r$ and for any $F \in \mathcal{W}$ so that*

$$F \cap B_0 = \emptyset, \tag{6.2}$$

we have

$$12\kappa F \cap K = \emptyset.$$

Proof. Since $r(F) \leq \eta r$ by (W3), the condition (6.2) implies that

$$\begin{aligned} d(o, o(F)) &\geq r(F) + r(B_0) = r(F) + 12\kappa\eta r \\ &= r(F) + (12\kappa - 1)\eta r + \eta r \\ &\geq 12\kappa r(F) + \mathrm{diam} K, \end{aligned}$$

which concludes the lemma. ■

In the sequel, we always assume that r is large enough so that

$$\mathrm{diam} K \leq \eta r. \tag{6.3}$$

Lemma 6.4. *For $F \in \mathcal{W}$, if $3F \not\subset 3B_0$, then $F \cap B_0 = \emptyset$.*

Proof. Suppose that $F \cap B_0 \neq \emptyset$. Then for any $x \in 3F$ and $z \in F \cap B_0$,

$$\begin{aligned} d(o,x) &\leq d(o,z) + d(z,x) \\ &\leq r(B_0) + 4r(F) \\ &\leq 12\kappa\eta r + 4\eta r \leq 3r(B_0). \end{aligned}$$

This implies that $3F \subset 3B_0$. By contraposition, we conclude the lemma. ■

Lemma 6.5. *For $F \in \mathcal{W}$, if $3F \cap 3B_0 \neq \emptyset$, then*

$$3F \subset B_1 = B(o, (36\kappa + 6)\eta r).$$

Proof. For $x \in 3F$ and $z \in 3F \cap 3B_0$, we obtain

$$\begin{aligned} d(o,x) &\leq d(o,z) + d(z,x) \\ &\leq 36\kappa\eta r + 6r(F) \\ &\leq (36\kappa + 6)\eta r, \end{aligned}$$

which concludes the lemma. ■

Now we modify the Whitney covering $\mathcal{W} = \mathcal{W}(\eta)$ to fit the manifolds with ends in concern. Set

$$\mathcal{W}' = \{F \in \mathcal{W} \ : \ 3F \not\subset 3B_0\}.$$

Here we note that

$$B(o,r) = \cup_{F \in \mathcal{W}'} 3F \cup 3B_0. \tag{6.4}$$

Moreover, for $i = 1, \ldots, k$, set $\mathcal{W}'_i = \{F \in \mathcal{W}' \ : \ F \subset E_i\}$. We can easily see that they are disjoint each other by the definition, and, Lemmas 6.3 and 6.4 imply that

$$\mathcal{W}' = \cup_{i=1}^k \mathcal{W}'_i.$$

For $F \in \mathcal{W}'_i$, we retake a string $\mathcal{F}(F) = (F_0, F_1, \ldots, F_{l'(F)})$ given in (PW2) so that $F_j \subset E_i$ $(j \neq 0)$ and $F_0 = B_0$.

The following inclusion conditions play a key role in the proof of Theorem 6.7.

Lemma 6.6. *For a constant $\kappa \geq 1$, let $\eta > 0$ be a parameter satisfying*

$$\eta < \frac{1}{\kappa(36\kappa + 6)}. \tag{6.5}$$

For such $\eta > 0$, let $r > 0$ be a number so that (6.3) holds. Then we have

$$3\kappa B_0 \subset \kappa B_1 \subset B(o,r), \tag{6.6}$$

$$12\kappa F \subset B(o,r) \cap E_i \quad (F \in \mathcal{W}'_i). \tag{6.7}$$

The main result of this section is the following.

Theorem 6.7. *Let M be a connected sum of manifolds $M_1, \ldots, M_k$ with a compact central part K, where each M_i admits* (VD). *Let $o \in K$ be a central reference point. Fix $\kappa \geq 1$. Fix $\eta > 0$ satisfying (6.5). For any $r > 0$ satisfying* $\mathrm{diam} K \leq \eta r$*, there exists a constant $C > 0$ such that*

$$\begin{aligned} &\Lambda(B(o,r)) \\ &\leq C \left(\Lambda(3B_0, 3\kappa B_0) + \Lambda(B_1, \kappa B_1) + \max_{F \in \mathcal{W}'(\eta)} \{\Lambda(3F, 3\kappa F), \Lambda(12F, 12\kappa F)\} \right), \end{aligned}$$

where $B_0 = B(o, 12\kappa\eta r)$ and $B_1 = B(o, (36\kappa + 6)\eta r)$.

We note that by (6.7), if $F \in \mathcal{W}_i'$, then $12\kappa F \subset E_i$. Hence, the Poincaré constants $\Lambda(3F, 3\kappa F)$ and $\Lambda(12F, 12\kappa F)$ for $F \in \mathcal{W}_i'$ can be computed by using a local information of E_i.

Proof. By using (6.4), we have

$$\begin{aligned}
\int_{B(o,r)} |f - f_{B(o,r)}|^2 d\mu \leq \int_{B(o,r)} |f - f_{3B_0}|^2 d\mu \\
&\leq \int_{3B_0} |f - f_{3B_0}|^2 d\mu + \sum_{F \in \mathcal{W}'} \int_{3F} |f - f_{3B_0}|^2 d\mu \\
&\leq \int_{3B_0} |f - f_{3B_0}|^2 d\mu + 4 \sum_{F \in \mathcal{W}'} \int_{3F} |f - f_{3F}|^2 d\mu \\
&\quad + 4 \sum_{i=1}^{k} \sum_{F \in \mathcal{W}_i'} \int_{3F} |f_{3F} - f_{3F_1}|^2 d\mu + 4 \sum_{i=1}^{k} \sum_{F \in \mathcal{W}_i'} \int_{3F} |f_{3F_1} - f_{3B_0}|^2 d\mu \\
&= I + 4II + 4 \sum_{i=1}^{k} III_i + 4 \sum_{i=1}^{k} IV_i, \qquad (6.8)
\end{aligned}$$

where F_1 is an element of the string $\mathcal{F}(F)$. We estimate terms I, II, III_i and IV_i in (6.8) separately.

Estimate of I

By using (6.6), we obtain

$$\begin{aligned}
I = \int_{3B_0} |f - f_{3B_0}|^2 d\mu &\leq \Lambda(3B_0, 3\kappa B_0) \int_{3\kappa B_0} |\nabla f|^2 d\mu \\
&\leq \Lambda(3B_0, 3\kappa B_0) \int_{B(o,r)} |\nabla f|^2 d\mu,
\end{aligned}$$

which gives a desired bound.

Estimate of II

By using (W2) and (W4), we obtain

$$\begin{aligned}
II = \sum_{F \in \mathcal{W}'} \int_{3F} |f - f_{3F}|^2 d\mu &\leq \sum_{F \in \mathcal{W}'} \Lambda(3F, 3\kappa F) \int_{3\kappa F} |\nabla f|^2 d\mu \\
&\leq \max_{F \in \mathcal{W}'} \Lambda(3F, 3\kappa F) \sum_{F \in \mathcal{W}'} \int_{3\kappa F} |\nabla f|^2 d\mu \\
&\leq N(\eta, 3\kappa) \max_{F \in \mathcal{W}'} \Lambda(3F, 3\kappa F) \int_{B(o,r)} |\nabla f|^2 d\mu,
\end{aligned}$$

which gives a desired bound.

Estimate of III_i

For $F \in \mathcal{W}_i'$, let $F_j, F_{j+1} \in \mathcal{F}(F)$ $(j \neq 0)$. Then the Cauchy-Schwartz inequality implies that

$$|f_{3F_{j+1}} - f_{3F_j}| \leq \frac{1}{\mu(3F_{j+1})^{1/2} \mu(3F_j)^{1/2}} \left(\int_{3F_{j+1} \times 3F_j} |f(x) - f(y)|^2 d\mu(x) d\mu(y) \right)^{1/2}.$$

We note that for any open set $D \subset M$,

$$\int_{D\times D} |f(x)-f(y)|^2 d\mu(x)d\mu(y) = 2\mu(D)\int_D |f-f_D|^2 d\mu. \tag{6.9}$$

Moreover, by using (6.1), (6.9), and the volume doubling property of M_i, we have

$$|f_{3F_{j+1}} - f_{3F_j}| \le \left(\frac{C}{\mu(F_j)}\int_{12F_j} |f-f_{12F_j}|^2 d\mu\right)^{1/2}.$$

Then we obtain

$$\begin{aligned} |f_{3F}-f_{3F_1}| &\le \sum_{j=1}^{l'(F)-1} |f_{3F_{j+1}}-f_{3F_j}| \\ &\le \sum_{j=1}^{l'(F)-1}\left(\frac{C}{\mu(F_j)}\int_{12F_j}|f-f_{12F_j}|^2 d\mu\right)^{1/2} \\ &\le \sum_{j=1}^{l'(F)-1}\left(\frac{C\Lambda(12F_j,12\kappa F_j)}{\mu(F_j)}\int_{12\kappa F_j}|\nabla f|^2 d\mu\right)^{1/2}. \end{aligned}$$

By using [26, Lemma 5.3.8], for any $F_j \in \mathcal{F}(F)$ with $j \neq 0$,

$$F \subset AF_j \cap E_i,$$

where $A = 8 + 4\eta + \eta^{-1}$. Then we obtain

$$\begin{aligned} III_i =& \sum_{F\in\mathcal{W}_i'}\int_{3F}|f_{3F}-f_{3F_1}|^2 = \int_{B\cap E_i}\sum_{F\in\mathcal{W}_i'}\chi_{3F}(x)|f_{3F}-f_{3F_1}|^2 d\mu(x) \\ \le& C\int_{B\cap E_i}\sum_{F\in\mathcal{W}_i'}\chi_{3F}(x)\left(\sum_{j=1}^{l'(F)-1}\Big(\frac{\Lambda(12F_j,12\kappa F_j)}{\mu(F_j)}\int_{12\kappa F_j}|\nabla f|^2 d\mu\Big)^{1/2}\right)^2 d\mu(x) \\ =& C\int_{B\cap E_i}\sum_{F\in\mathcal{W}_i'}\chi_{3F}(x)\Big(\sum_{j=1}^{l'(F)-1}\Big(\frac{\Lambda(12F_j,12\kappa F_j)}{\mu(F_j)}\int_{12\kappa F_j}|\nabla f|^2 d\mu\Big)^{1/2}\chi_{AF_j\cap E_i}(x)\Big)^2 d\mu(x) \\ \le& C\int_{B\cap E_i}\sum_{F\in\mathcal{W}_i'}\chi_{3F}(x)\Big(\sum_{G\in\mathcal{W}_i'}\Big(\frac{\Lambda(12G,12\kappa G)}{\mu(G)}\int_{12\kappa G}|\nabla f|^2 d\mu\Big)^{1/2}\chi_{AG\cap E_i}(x)\Big)^2 d\mu(x) \\ \le& CN(\eta,3)\max_{G\in\mathcal{W}_i'}\Lambda(12G,12\kappa G)\int_{B\cap E_i}\Big(\sum_{G\in\mathcal{W}_i'}\Big(\frac{1}{\mu(G)}\int_{12\kappa G}|\nabla f|^2 d\mu\Big)^{1/2}\chi_{AG\cap E_i}(x)\Big)^2 d\mu(x) \\ =& CN(\eta,3)\max_{G\in\mathcal{W}_i'}\Lambda(12G,12\kappa G)\left\|\sum_{G\in\mathcal{W}_i'}a_G\chi_{AG\cap E_i}\right\|_{L^2(E_i)}^2, \end{aligned}$$

where χ_D is the characteristic function of a set D and

$$a_G = \left(\frac{1}{\mu(G)}\int_{12\kappa G}|\nabla f|^2 d\mu\right)^{1/2}.$$

Since E_i satisfies (VD), applying Lemma [26, 5.3.12], there exists a constant $C = C(A) > 0$ such that

$$\left\|\sum_{G\in\mathcal{W}_i'}a_G\chi_{AG\cap E_i}\right\|_{L^2(E_i)} \le C\left\|\sum_{G\in\mathcal{W}_i'}a_G\chi_G\right\|_{L^2(E_i)}.$$

By using (W1), (W4) and (6.7), we obtain

$$
\begin{aligned}
III_i &\leq C'N(\eta,3)\max_{F\in\mathcal{W}_i'}\Lambda(12F,12\kappa F)\int_{B\cap E_i}\left(\sum_{F\in\mathcal{W}_i'}a_F\chi_F(x)\right)^2 d\mu(x)\\
&= C'N(\eta,3)\max_{F\in\mathcal{W}_i'}\Lambda(12F,12\kappa F)\int_{B\cap E_i}\sum_{F\in\mathcal{W}_i'}\frac{1}{\mu(F)}\int_{12\kappa F}|\nabla f|^2 d\mu\chi_F(x)d\mu(x)\\
&= C'N(\eta,3)\max_{F\in\mathcal{W}_i'}\Lambda(12F,12\kappa F)\sum_{F\in\mathcal{W}_i'}\int_{12\kappa F}|\nabla f|^2 d\mu\\
&\leq C'N(\eta,3)N(\eta,12\kappa)\max_{F\in\mathcal{W}_i'}\Lambda(12F,12\kappa F)\int_B|\nabla f|^2 d\mu,
\end{aligned}
$$

which gives a desired bound.

Estimate of IV_i

Because $3F_1, 3B_0 \subset B_1$, by using the same argument as in the estimate of III_i, we obtain

$$
\begin{aligned}
IV_i &= \sum_{F\in\mathcal{W}_i'}\int_{3F}|f_{3F_1}-f_{3B_0}|^2 d\mu\\
&\leq \sum_{F\in\mathcal{W}_i'}\frac{\mu(3F)}{\mu(3F_1)\mu(3B_0)}\int_{B_1\times B_1}|f(x)-f(y)|^2 d\mu(x)d\mu(y)\\
&= 2\left(\sum_{F\in\mathcal{W}_i'}\frac{\mu(3F)}{\mu(3F_1)}\right)\frac{\mu(B_1)}{\mu(3B_0)}\int_{B_1}|f-f_{B_1}|^2 d\mu.
\end{aligned}
$$

Here we recall that the central ball satisfies (VD). Indeed,

$$
\begin{aligned}
\mu(B(o,s)) &\simeq \mu(B_1(o_1,s))+\mu(B_2(o_2,s))+\cdots\mu(B_k(o_k,s))\\
&\simeq \max_i \mu(B_i(o_i,s)),
\end{aligned}
\tag{6.10}
$$

where $B_i(o_i,s)$ is the geodesic ball in M_i centered at $o_i \in K_i$. Since each $B_i(o_i,s)$ satisfies (VD), so does $B(o,s)$ by (6.10). Hence $\mu(B_1)/\mu(3B_0)$ is bounded in r. Now we estimate $\mu(3F_1)$ from below. By using (W3), Lemma 6.5 and (6.6), for any $F \in \mathcal{W}_i'$

$$
\begin{aligned}
r(F_1) &= \eta d(F_1, B(o,r)^c)\\
&\geq \eta d(B_1, B(o,r)^c) = \eta(1-(36\kappa+6)\eta)r,
\end{aligned}
$$

which implies that

$$A'r(F_1) \geq 2r,$$

where $A' = \frac{2}{\eta(1-(36\kappa+6)\eta)}$. Then we obtain

$$A'F_1 \supset B(o,r).$$

Since each end E_i is doubling, we obtain

$$\mu(3F_1) \geq C^{-1}\mu(A'F_1\cap E_i) \geq C^{-1}\mu(B(o,r)\cap E_i).$$

By using (W2), we obtain for all $i = 1,\ldots,k$

$$\sum_{F\in\mathcal{W}_i'}\frac{\mu(3F)}{\mu(3F_1)} \leq \frac{C}{\mu(B(o,r)\cap E_i)}\sum_{F\in\mathcal{W}_i'}\mu(3F) \leq C.$$

Combining the above estimates, we obtain

$$\begin{aligned} IV_i &\le C' \int_{B_1} |f - f_{B_1}|^2 d\mu \\ &\le C'\Lambda(B_1, \kappa B_1) \int_{\kappa B_1} |\nabla f|^2 d\mu. \end{aligned}$$

Since $\kappa B_1 \subset B(o, r)$ by (6.6), we conclude a desired estimate of IV_i. ■

We use the following corollary to show the upper bound of the Poincaré constant (i.e., lower bound of the spectral gap) of central balls in Theorems 2.7 and 2.10.

Corollary 6.8. *Let M be a connected sum of $M_1, \ldots, M_k$, where each end M_i satisfies* (VD) *and* (PI)*. If there is at least one parabolic end, assume that M satisfies* (COE)*. Then for any large $r > 0$*

$$\Lambda(B(o, r)) \lesssim V_n(r)h_n(r),$$

that is,

$$\lambda(B(o, r)) \gtrsim \frac{1}{V_n(r)h_n(r)},$$

where n is the index of the second largest end defined in (2.9).

Proof. Because $3\kappa F, 12\kappa F \subset E_i$ for all $F \in \mathcal{W}_i'$ by (6.7), the assumption of (VD) and (PI) on each M_i implies that for all $F \in \mathcal{W}'$

$$\Lambda(3F, 3\kappa F), \Lambda(12F, 12\kappa F) \lesssim r^2.$$

Using the estimates in section 5.3, we obtain

$$\Lambda(3B_0, 3\kappa B_0) + \Lambda(B_1, \kappa B_1) \lesssim V_n(r)h_n(r).$$

Since the inequality $V_i(r)h_i(r) \gtrsim r^2$ is always true by (VD), we conclude the corollary from Theorem 6.7 immediately. ■

For the estimates of Poincaré constants of central balls on some examples of manifolds with ends, we refer to Section 3.

Acknowledgments

The first and second authors would like to express their gratitude for the hospitality and support by the Institute of Mathematical Sciences of the Chinese University of Hong Kong and School of Mathematical Sciences, Nankai University.

References

[1] D. Bate and S. Li, *Differentiability and Poincar-type inequalities in metric measure spaces.* Adv. Math. 333 (2018), 868-930.

[2] A. Bendikov and L. Saloff-Coste, *Central Gaussian semigroups of measures with continuous density.* J. Func. Anal., 186 (2001), 206–286.

[3] I. Benjamini, I. Chavel and E. A. Feldman, *Heat kernel lower bounds on Riemannian manifolds using the old ideas of Nash.* Proc. London Math. Soc., 72 (1996) 215–240.

[4] G. Besson, G. Courtois and S. Hersonsky, *Poincaré inequality on complete Riemannian manifolds with Ricci curvature bounded below*. Math. Res. Lett. 25(2018), no. 6, 1741–1769.

[5] G. Carron and D. Tewodrose, *A rigidity result for metric measure spaces with Euclidean heat kernel*. J. Éc. polytech. Math. 9 (2022), 101-154.

[6] T. H. Colding and W. Minicozzi II, *Harmonic functions on manifolds*. Ann. of Math. (2) 146 (1997), no. 3, 725-747.

[7] T. Coulhon and A. Grigor'yan, *On-diagonal lower bounds for heat kernels on no-compact manifolds and Markov chains*. Duke Math. J., 89 (1997) no. 1, 133–199.

[8] T. Coulhon, R. Jiang, P. Koskela and A. Sikora, *Gradient estimates for heat kernels and harmonic functions*. J. Func. Anal., 278 (2020), no. 8, 108398.

[9] T. Coulhon and L. Saloff-Coste, *Isopérimétrie pour les groupes et les variétés, Revista Matemática Iberoamericana*, 9 (1993) 293–314.

[10] A. Grigor'yan, *The heat equation on noncompact Riemannian manifolds* (in Russian). Mat. Sb., 182 (1991) no. 1, 55-87; English translation in Math. USSR-Sb., 72 (1992) no. 1, 47–77.

[11] A. Grigor'yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*. Bull. AMS, 36 (1999) 135–249.

[12] A. Grigor'yan, Heat kernel and analysis on manifolds. AMS/IP Studies in Advanced Mathematics 47, AMS, 2009.

[13] A. Grigor'yan, S. Ishiwata and L. Saloff-Coste, *Heat kernel estimates on connected sums of parabolic manifolds*, J. Math. Pures Appl. 113 (2018), 155–194.

[14] A. Grigor'yan, S. Ishiwata and L. Saloff-Coste, *Geometric analysis on manifolds with ends*, Advances in Analysis and Geometry 3 (2020), 325–344.

[15] A. Grigoryan and L. Saloff-Coste, *Heat kernel on connected sums of Riemannian manifolds*. Math. Research Letters, 6 (1999) no. 3-4, 307–321.

[16] A.Grigor'yan and L. Saloff-Coste, *Dirichlet heat kernel in the exteior of a compact set*. Comm. Pure Appl. Math, 55 (2002), 93–133.

[17] A. Grigor'yan and L. Saloff-Coste, *Hitting probabilities for Brownian motion on Riemannian manifolds*. J. Math. Pures Appl., 81 (2002) no. 2, 115–142.

[18] A. Grigor'yan and L. Saloff-Coste, *Stability results for Harnack inequalities*. Ann. Inst. Fourier, Grenoble, 55 (2005) no.3, 825–890.

[19] A. Grigor'yan and L. Saloff-Coste, *Heat kernel on manifolds with ends*. Ann. Inst. Fourier, Grenoble, 59 (2009) no. 5, 1917–1997.

[20] A. Grigor'yan and L. Saloff-Coste, *Surgery of the Faber-Krahn inequality and applications to heat kernel bounds*, J. Nonlinear Analysis, 131 (2016) 243–272.

[21] S. Kusuoka and D. Stroock, *Application of Malliavin calculus, Part III*, J. Fac. Sci. Tokyo Univ., Sect. 1A, Math., 34 (1987) 391–442.

[22] Yu. T. Kuz'menko and S. A. Molchanov, *Counterexamples of Liouville type theorems*, (in Russian) Vestnik Moscov. Univ. Ser. I Mat. Mekh., (1976) no. 6, 39–43; Engl. transl. Moscow Univ. Math. Bull., 34 (1979) 35–39.

[23] P. Li and S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math., 156 (1986) no. 3-4, 153–201.

[24] J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math., 17 (1964) 101–134. Correction: Comm. Pure Appl. Math. 20 (1967) 231-236.

[25] L. Saloff-Coste, *A note on Poincaré, Sobolev, and Harnack inequalities*. Internat. Math. Res. Notices (1992), no. 2, 27–38.

[26] L. Saloff-Coste, Aspects of Sobolev type inequalities. London Math. Soc. Lecture Notes Series 289, Cambridge Univ. Press, 2002.