Propagation speed of non-linear parabolic equations on Riemannian manifolds

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Contents

1 Introduction

We are concerned with an evolution equation

$$\partial_t u = \Delta_p u^q \tag{1.1}$$

where $p, q > 0$, $u(x,t)$ is an unknown non-negative function, and Δ_p is the p-Laplacian:

$$\Delta_p v = \operatorname{div}\left(|\nabla v|^{p-2} \nabla v\right).$$

Equation (1.1) was introduced by L. S. Leibenson in 1945 in order to describe filtration of turbulent compressible fluid through a porous medium. The physical meaning of u is the *volumetric moisture content,* i.e. the (infinitesimal) fraction of volume of the medium taken by the liquid. Parameter p characterizes the turbulence of a flow while $q-1$ is the index of *polytropy* of the liquid, that determines relation $PV^{q-1} = \text{const}$ between volume V and pressure P.

The physically interesting values of p and q are as follows: $\frac{3}{2} \le p \le 2$ and $q \ge 1$. The case $p = 2$ corresponds to laminar flow (=absence of turbulence). In this case (1.1) becomes a *porous medium* equation $\partial_t u = \Delta u^q$, if $q > 1$, and the classical heat equation $\partial_t u = \Delta u$ if $q = 1$.

From mathematical point of view, the entire range $p > 1$, $q > 0$ is interesting. G.I.Barenblatt constructed in 1952 spherically symmetric self-similar solutions of (1.1) in $\mathbb{R}^n$ that are nowadays called *Barenblatt solutions.* Let us assume that

$$\boxed{q(p-1) > 1}.$$

In this case the Barenblatt solution is as follows:

$$u(x,t) = \frac{1}{t^{n/\beta}} \left(C - \kappa \left(\frac{|x|}{t^{1/\beta}} \right)^{\frac{p}{p-1}} \right)_+^{\gamma}$$

where $C > 0$ is any constant, and

$$\beta = p + n\left[q(p-1) - 1\right], \quad \gamma = \tfrac{p-1}{q(p-1)-1}, \quad \kappa = \tfrac{q(p-1)-1}{pq} \beta^{-\frac{1}{p-1}}. \tag{1.2}$$

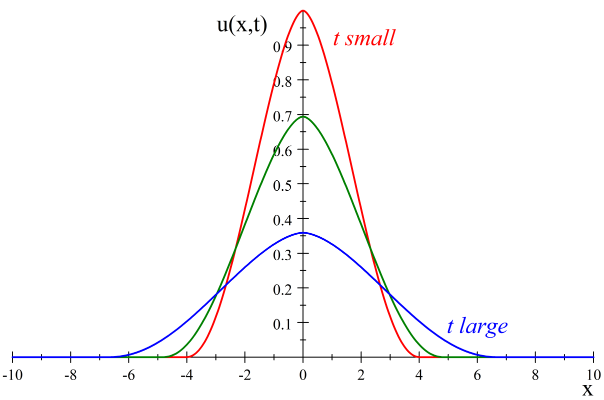
Parameter β determines the space/time scaling and is analogous to the *walk dimension.*

It is obvious that, for the Barenblatt solution,

$$u(x,t) = 0 \quad \text{for } |x| > ct^{1/\beta}$$

so that $u(\cdot, t)$ has a *compact support* for any t. One says that u has a *finite propagation speed.*

Here are the graphs of function $x \mapsto u(x,t)$ for different values of t in the case $n = 1$.



Assume now that

$$q\,(p-1)<1.$$

Then we have $\gamma,\kappa<0$, and the Barenblatt solution is

$$u\,(x,t)=\frac{1}{t^{n/\beta}}\left(C+|\kappa|\left(\frac{|x|}{t^{1/\beta}}\right)^{\frac{p}{p-1}}\right)^{-|\gamma|},$$

that is, it is positive for all x,t.

In the borderline case

$$q\,(p-1)=1,$$

the Barenblatt solution is

$$u\,(x,t)=\frac{1}{t^{n/p}}\exp\left(-c\left(\frac{|x|}{t^{1/p}}\right)^{\frac{p}{p-1}}\right),$$

where $c=(p-1)^2\,p^{-\frac{p}{p-1}}$. Hence, if $q\,(p-1)\leq 1$ then u has infinite propagation speed.

Of course, if here $p=2$ then $q=1$, and we obtain the fundamental solution of the heat equation $\partial_t u=\Delta u$:

$$u\,(x,t)=\frac{1}{t^{n/2}}\exp\left(-\frac{1}{4}\left(\frac{|x|}{t^{1/2}}\right)^{2}\right),$$

2 Leibenson's equation on manifolds

Consider on an arbitrary Riemannian manifold the operator

$$Lv=\Delta_p\,(v^q)$$

where

$$p>1\quad\text{and}\quad q>0,$$

and

$$\Delta_p u=\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right).$$

We will be concerned with the associated evolution equation

$$\boxed{\partial_t v=\Delta_p(v^q)}$$

that is called the Leibenson equation. Our aim is to prove the following theorem.

Theorem 1. *If $q(p-1)>1$ the any bounded non-negative solution to the Leibenson equation has a finite propagation speed.*

The exact meaning of "finite propagation speed" will be explained later on. The proof will also be given later on.

Now we show how to obtain the Barenblatt solutions in $\mathbb{R}^n$. We start with deriving a chain rule for the p-Laplacian. Consider on an arbitrary manifold the p-Laplacian

$$\Delta_p v = \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right), \tag{2.1}$$

where $p > 1$. Let us compute $\Delta_p f(u)$ assuming that f is smooth enough and

$$f \geq 0 \ \text{ and } f' \leq 0.$$

We have $\nabla f(u) = f'(u)\nabla u$ and

$$\begin{aligned}
\Delta_p f(u) &= \operatorname{div}(|f'(u)\nabla u|^{p-2} f'(u)\nabla u) \\
&= \operatorname{div}\left(|f'(u)|^{p-2} f'(u)\,|\nabla u|^{p-2}\nabla u\right) \\
&= -\operatorname{div}\left(|f'(u)|^{p-1}|\nabla u|^{p-2}\nabla u\right) \\
&= -|f'(u)|^{p-1}\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - \nabla\left(|f'(u)|^{p-1}\right)|\nabla u|^{p-2}\nabla u \\
&= -|f'(u)|^{p-1}\Delta_p u - \nabla\left((-f'(u))^{p-1}\right)|\nabla u|^{p-2}\nabla u \\
&= -|f'(u)|^{p-1}\Delta_p u + (p-1)(-f'(u))^{p-2} f''(u)\nabla u\,|\nabla u|^{p-2}\nabla u \\
&= -|f'(u)|^{p-1}\Delta_p u + (p-1)|f'(u)|^{p-2} f''(u)\,|\nabla u|^p.
\end{aligned}$$

Hence, we obtain

$$\Delta_p f(u) = -(-f'(u))^{p-1}\Delta_p u + (p-1)(-f'(u))^{p-2} f''(u)\,|\nabla u|^p. \tag{2.2}$$

3 Solutions on models

3.1 Model manifolds

Let M be a *model manifold* $\mathbb{R}_+ \times \mathbb{S}^{n-1}$ with the polar coordinates (r,θ) (where $r \in \mathbb{R}_+$ and $\theta \in \mathbb{S}^{n-1}$) and with the Riemannian metric

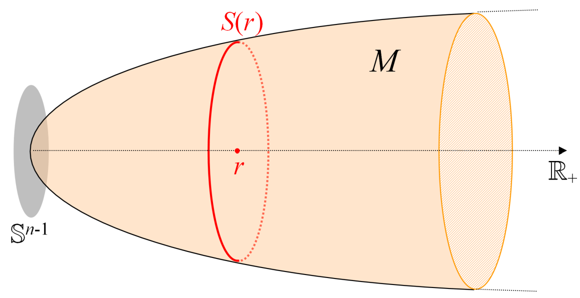
$$ds^2 = dr^2 + \psi^2(r)\,d\theta^2.$$

Here $d\theta^2$ is the standard Riemannian metric on $\mathbb{S}^{n-1}$ and ψ is a smooth positive function on $\mathbb{R}_+$. For example, $\mathbb{R}^n \setminus \{0\}$ can be considered as a model manifold with $\psi(r) = r$.

Denote by $S(r)$ the boundary area function

$$S(r) = \omega_n \psi(r)^{n-1}.$$

For example, in $\mathbb{R}^n$ we have $S(r) = \omega_n r^{n-1}$.



It is known that the Laplace-Beltrami operator Δ on M admits the following representation in the polar coordinates:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{S'}{S}\frac{\partial}{\partial r} + \frac{1}{\psi^2(r)}\Delta_\theta,$$

where Δ_θ is the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$. In particular, considering the polar radius r as a function in M, we obtain that

$$\Delta r = \frac{S'}{S}. \tag{3.1}$$

For example, in $\mathbb{R}^n$ we have $\Delta r = \frac{n-1}{r}$. Using (3.1) and $|\nabla r| = 1$, we obtain that

$$\Delta_p r = \operatorname{div}\left(|\nabla r|^{p-2}\nabla r\right) = \operatorname{div}(\nabla r) = \Delta r = \frac{S'}{S}.$$

Setting in (2.2) $u = r$, we obtain

$$\begin{aligned} \Delta_p f(r) &= -\left(-f'(r)\right)^{p-1}\Delta_p r + (p-1)\left(-f'(r)\right)^{p-2} f''(r) \\ &= -\left(-f'(r)\right)^{p-1}\frac{S'}{S} + (p-1)\left(-f'(r)\right)^{p-2} f''(r). \end{aligned} \tag{3.2}$$

Note that

$$\left(\left(-f'(r)\right)^{p-1} S\right)' = \left(-f'(r)\right)^{p-1} S' - (p-1)\left(-f'(r)\right)^{p-2} f''(r)\, S.$$

Hence, (3.2) can be rewritten in the form

$$\Delta_p f(r) = -\frac{1}{S}\left(S\left(-f'(r)\right)^{p-1}\right)'.$$

The parabolic equation $\partial_t u = \Delta_p u$ for a function $u = u(r,t)$ (such that $u \geq 0$ and $\partial_r u \leq 0$) becomes therefore

$$\partial_t u = -\frac{1}{S}\partial_r\left(S\left(-\partial_r u\right)^{p-1}\right), \tag{3.3}$$

and the Leibenson equation $\partial_t u = \Delta_p(u^q)$ becomes

$$\partial_t u = -\frac{1}{S}\partial_r\left(S\left(-\partial_r u^q\right)^{p-1}\right). \tag{3.4}$$

3.2 Barenblatt solutions

We solve here (3.4) assuming $p > 1$, $q > 0$ and

$$S(r) = r^{\alpha-1},$$

where α is a positive real. In particular, for $\alpha = n \in \mathbb{N}$, this will give us the Barenblatt solution in $\mathbb{R}^n$.

The equation (3.4) becomes with this $S(r)$

$$\partial_t u = -\frac{1}{r^{\alpha-1}} \partial_r \left(r^{\alpha-1} \left(-\partial_r u^q \right)^{p-1} \right), \tag{3.5}$$

and we look for a solution in the form

$$u(x,t) = t^a f\left(rt^b\right),$$

where f is a decreasing non-negative function and a, b are (negative) reals, yet to be determined.

Let us require in addition that the solution $u(x,t)$ has a constant L^1-norm, that is,

$$\int_M t^a f(rt^b) d\mu = \text{const} < \infty,$$

where μ is the Riemannian measure. Computing the integral in the polar coordinates and using

$$d\mu = \psi(r)^{n-1} dr d\theta = \frac{1}{\omega_n} S(r) dr d\theta,$$

we obtain that

$$\int_0^\infty t^a f(rt^b) r^{\alpha-1} dr = \text{const} < \infty.$$

A change $s = rt^b$ in the integral gives

$$\int_0^\infty t^a f(rt^b) r^{\alpha-1} dr = \int_0^\infty t^a f(s) \left(st^{-b}\right)^{\alpha-1} t^{-b} ds = t^{a-b\alpha} \int_0^\infty f(s) s^{\alpha-1} ds.$$

Hence, we must have

$$\int_0^\infty f(s) s^{\alpha-1} ds < \infty \tag{3.6}$$

and

$$\boxed{a = \alpha b.}$$

Using again the variable $s = rt^b$, we obtain

$$\begin{aligned} \partial_t u &= \partial_t \left(t^a f\left(rt^b\right) \right) \\ &= a t^{a-1} f(rt^b) + t^a f'(rt^b) r b t^{b-1} \\ &= b\alpha t^{a-1} f(rt^b) + b t^{a-1} r t^b f'(rt^b) \\ &= b t^{a-1} \left(\alpha f(s) + s f'(s) \right) \end{aligned}$$

$$= \frac{bt^{a-1}}{s^{\alpha-1}} \left(s^{\alpha} f\left(s\right)\right)'$$

and

$$\begin{aligned}
\partial_r u^q &= q u^{q-1} \partial_r u \\
&= q \left(t^a f\left(rt^b\right)\right)^{q-1} \partial_r \left(t^a f\left(rt^b\right)\right) \\
&= q t^{a(q-1)} f(rt^b)^{q-1} t^{a+b} f'(rt^b) \\
&= q t^{aq+b} f(s)^{q-1} f'(s).
\end{aligned}$$

Hence, (3.5) is equivalent to

$$\begin{aligned}
\frac{bt^{a-1}}{s^{\alpha-1}} \left(s^{\alpha} f\left(s\right)\right)' &= -\frac{1}{r^{\alpha-1}} \partial_r \left(r^{\alpha-1} \left(-q t^{aq+b} f(s)^{q-1} f'(s)\right)^{p-1}\right) \\
&= -\frac{q^{p-1} t^{(aq+b)(p-1)}}{\left(st^{-b}\right)^{\alpha-1}} \partial_r \left(\left(st^{-b}\right)^{\alpha-1} \left(-f(s)^{q-1} f'(s)\right)^{p-1}\right) \\
&= -\frac{q^{p-1} t^{(aq+b)(p-1)}}{s^{\alpha-1}} t^b \partial_s \left(s^{\alpha-1} \left(-f(s)^{q-1} f'(s)\right)^{p-1}\right). \qquad (3.7)
\end{aligned}$$

We require that the powers of t in the both sides to match, that is,

$$\left(aq+b\right)\left(p-1\right)+b = a-1,$$

which together with $a = b\alpha$ yields

$$\left[\left(\alpha q+1\right)\left(p-1\right)+1-\alpha\right] b = -1,$$

$$\left[\alpha\left(q\left(p-1\right)-1\right)+p\right] b = -1.$$

Setting

$$\boxed{\delta = q(p-1)-1}$$

we obtain

$$\left(\alpha\delta+p\right) b = -1$$

whence

$$\boxed{b = -\frac{1}{\alpha\delta+p}}.$$

In particular, we see that

$$\boxed{b<0 \Leftrightarrow \delta > -\frac{p}{\alpha} \Leftrightarrow q > \frac{1-p/\alpha}{p-1}}. \qquad (3.8)$$

In what follows, we always assume that (3.8) is satisfied.

With this choice of b and $a = \alpha b$, the powers of t and s in (3.7) cancel out, and we obtain an ODE for f:

$$b\left(s^{\alpha} f\left(s\right)\right)' = -q^{p-1} \left(s^{\alpha-1} \left(-f(s)^{q-1} f'(s)\right)^{p-1}\right)'.$$

Hence, we have

$$bs^{\alpha} f(s) = -q^{p-1} s^{\alpha-1} \left(-f(s)^{q-1} f'(s)\right)^{p-1} \tag{3.9}$$

(ignoring a constant of integration). Since $b<0$, we obtain

$$|b|\, sf = q^{p-1} \left(-f^{q-1} f'\right)^{p-1},$$

$$|b|\, sf = q^{p-1} \left(-f'\right)^{p-1} f^{(q-1)(p-1)},$$

$$f^{(q-1)(p-1)-1} \left(-f'\right)^{p-1} = \frac{|b|\, s}{q^{p-1}},$$

$$f^{(q-1)-\frac{1}{p-1}} f' = -\frac{\left(|b|\, s\right)^{\frac{1}{p-1}}}{q}.$$

Set

$$\gamma := q - \frac{1}{p-1} = \frac{q(p-1)-1}{p-1} = \frac{\delta}{p-1}$$

and rewrite the above ODE in the form

$$f^{\gamma-1} f' = -\frac{|b|^{\frac{1}{p-1}}}{q} s^{\frac{1}{p-1}} \tag{3.10}$$

Assume first that $\delta \neq 0$, that is, $\gamma \neq 0$. Then (3.10) is equivalent to

$$\left(f^{\gamma}\right)' = \gamma f^{\gamma-1} f' = -\frac{\gamma\, |b|^{\frac{1}{p-1}}}{q} s^{\frac{1}{p-1}},$$

and integrating it, we obtain

$$f^{\gamma} = C - \kappa s^{\frac{p}{p-1}}$$

where

$$\kappa = \frac{p-1}{p} \frac{\gamma\, |b|^{\frac{1}{p-1}}}{q} = \frac{\delta}{p} \frac{|b|^{\frac{1}{p-1}}}{q}.$$

Hence,

$$f(s) = \left(C - \kappa s^{\frac{p}{P-1}}\right)^{1/\gamma},$$

with a positive constant C.

Case 1. Let $\delta > 0$ that is,

$$q(p-1) > 1,$$

(which implies also that $b<0$).

Then $\kappa > 0$ and we see that $f(s)$ is well defined for $s \in [0, s_0]$ where s_0 is determined by

$$C = \kappa s_0^{\frac{p}{P-1}}.$$

Let us extend $f(s)$ for all $s \in [0,\infty)$ by setting $f(s) = 0$ for $s > s_0$, that is,

$$f(s) = \left(C - \kappa s^{\frac{p}{P-1}}\right)_+^{1/\gamma}.$$

Then this function f is a *weak* solution of the ODE (3.10) in $[0,\infty]$ because f is continuous in $[0,\infty]$ and solves (3.10) in the both intervals $[0,s_0]$ and $[s_0,\infty)$. Consequently, we obtain in this case a (weak) solution of (3.5)

$$u(x,t) = t^{\alpha b} f\left(rt^b\right) = \frac{1}{t^{\alpha/\beta}}\left(C-\kappa\left(\frac{r}{t^{1/\beta}}\right)^{\frac{p}{p-1}}\right)_+^{1/\gamma}$$

where

$$\boxed{\beta = -\frac{1}{b} = \alpha\delta + p} > 0.$$

Clearly, this solution has a finite propagation speed. Note that in this case $\beta > p$.

Case 2. Let $\delta < 0$ that is,

$$q(p-1) < 1.$$

Since $\kappa < 0$, the solution

$$f(s) = \left(C + |\kappa|\, s^{\frac{p}{p-1}}\right)^{-1/|\gamma|}$$

is defined and positive for all $s \geq 0$. Note that by (3.8)

$$\frac{p}{p-1}\frac{1}{|\gamma|} = \frac{p}{(p-1)\left(\frac{1}{p-1}-q\right)} = \frac{p}{1-q(p-1)} = \frac{p}{-\delta} > \alpha,$$

that is,

$$f(s) \simeq s^{-(\alpha+\varepsilon)} \quad \text{as } s \to \infty$$

where $\varepsilon > 0$. Since also

$$f(s) \simeq \text{const} \quad \text{as } s \to 0$$

we obtain the finiteness of the integral (3.6).

We obtain in this case a solution

$$u(x,t) = \frac{1}{t^{\alpha/\beta}}\left(C + |\kappa|\left(\frac{r}{t^{1/\beta}}\right)^{\frac{p}{p-1}}\right)^{-1/|\gamma|}$$

that is defined for all x and $t > 0$ and belongs to $L^1(M)$ for any $t > 0$. Hence, this solution has infinite propagation speed. Note that in this case $\beta < p$.

Case 3. Let $\delta = 0$ that is,

$$q(p-1) = 1,$$

In this case $\gamma = 0$ and

$$b = -\frac{1}{\alpha\delta + p} = -\frac{1}{p}.$$

Then (3.10) becomes

$$\frac{f'}{f} = -\frac{s^{\frac{1}{p-1}}}{p^{\frac{1}{p-1}} q}$$

whence

$$\ln f = -\frac{1}{p^{\frac{1}{p-1}} q} \frac{s^{\frac{p}{p-1}}}{\frac{p}{p-1}} = -\kappa s^{\frac{p}{p-1}},$$

where

$$\kappa = \frac{1}{p^{\frac{1}{p-1}} q \frac{p}{p-1}} = \frac{(p-1)^2}{p^{\frac{p}{p-1}}} > 0.$$

It follows that

$$f(s) = \exp\left(-\kappa s^{\frac{p}{p-1}}\right)$$

whence

$$u(x,t) = \frac{1}{t^{\alpha/p}} \exp\left(-\kappa \left(\frac{r}{t^{1/p}}\right)^{\frac{p}{p-1}}\right).$$

For example, in the case $p = 2$ and, hence, $q = 1$ we obtain $\kappa = \frac{1}{4}$ and

$$u(x,t) = \frac{1}{t^{\alpha/2}} \exp\left(-\frac{1}{4}\frac{r^2}{t}\right).$$

Hence, the finite propagation speed for the above solutions occurs if and only if $\delta > 0$, that is, $q(p-1) > 1$.

4 Weak solutions

Let Ω be an open subset of M and I be an interval in $[0, \infty)$. By a *subsolution* of the equation

$$\partial_t v = \Delta_p (v^q) \tag{4.1}$$

in the cylinder $\Omega \times I$ we mean a non-negative function v of an appropriate class satisfying

$$\partial_t v \leq \Delta_p (v^q). \tag{4.2}$$

In fact, this equation is understood in a certain weak sense, and a function v is taken from the following class:

$$v \in C\left(I; L^2(\Omega)\right) \quad \text{and} \quad v^q \in L^p_{loc}\left(I; W^{1,p}(\Omega)\right).$$

That is, for any $t \in I$,

$$v(\cdot, t) \in L^2(\Omega), \quad v^q(\cdot, t) \in W^{1,p}(\Omega),$$

the function $t \mapsto v(\cdot, t)$ is continuous in $L^2(\Omega)$, and, for any compact subinterval $J \subset I$,

$$\int_J \|v^q(\cdot, t)\|^p_{W^{1,p}(\Omega)} \, dt < \infty,$$

that is,

$$\int_J \int_\Omega (v^{qp} + |\nabla v^q|^p) \, d\mu dt < \infty. \tag{4.3}$$

Let us first show that the Leibenson operator $Lv = \Delta_p(v^q)$ can be rewritten in the form

$$Lv = c\,\mathrm{div}\left(v^m |\nabla v|^{p-2} \nabla v\right) \tag{4.4}$$

for some c, m, that is,

$$\mathrm{div}\left(|\nabla v^q|^{p-2} \nabla v^q\right) = c\,\mathrm{div}\left(v^m |\nabla v|^{p-2} \nabla v\right). \tag{4.5}$$

Indeed, we have

$$\nabla v^q = q v^{q-1} \nabla v$$

and

$$\mathrm{div}\left(|\nabla v^q|^{p-2} \nabla v^q\right) = q^{p-1}\,\mathrm{div}\left(v^{(q-1)(p-1)} |\nabla v|^{p-2} \nabla v\right).$$

Hence, (4.5) holds provided

$$\boxed{m = (q-1)(p-1)} \tag{4.6}$$

and $c = q^{p-1}$. The Leibenson equation becomes

$$\partial_t v = q^{p-1}\,\mathrm{div}\left(v^m |\nabla v|^{p-2} \nabla v\right),$$

and (4.2) becomes

$$\partial_t v \le q^{p-1}\,\mathrm{div}\left(v^m |\nabla v|^{p-2} \nabla v\right). \tag{4.7}$$

5 Caccioppoli type inequality

We start here the proof of Theorem 1. The first step is obtaining a Caccioppoli type inequality.

For simplicity of notation, we omit in all integrations the notation of measure. All integration in M is done with respect to $d\mu$, and in $M \times \mathbb{R}$ – with respect to $d\mu dt$. We assume that

$$p > 1, \quad q > 0$$

and use the notation

$$\delta = q(p-1) - 1.$$

Let Ω be an open subset of M and I be an interval in $\mathbb{R}$.

Lemma 2. *Let $v = v(x,t)$ be a bounded non-negative subsolution to (4.1) in a cylinder $\Omega \times I$. Let $\eta(x,t)$ be a Lipschitz non-negative bounded function in $\Omega \times (0,T)$ such that $\eta(\cdot,t)$ has compact support in Ω for all $t \in I$. Fix some real σ such that*

$$\sigma \ge \max(p, pq). \tag{5.1}$$

Set

$$\boxed{\lambda = \sigma - \delta \quad \text{and} \quad \alpha = \frac{\sigma}{p}}. \tag{5.2}$$

Then, for all $t_1, t_2 \in I$ such that $t_1 < t_2$, we have

$$\left[\int_\Omega v^\lambda \eta^p\right]_{t_1}^{t_2} + c_1 \int_{t_1}^{t_2} \int_\Omega |\nabla(v^\alpha \eta)|^p \le \int_{t_1}^{t_2} \int_\Omega \left[p v^\lambda \eta^{p-1} \eta_t + c_2 v^\sigma |\nabla \eta|^p\right], \tag{5.3}$$

where c_1, c_2 are positive constants depending on p, q, λ (see below (5.13) and (5.14)).

In particular, if η does not depend on t then

$$\left[\int_\Omega v^\lambda \eta^p\right]_{t_1}^{t_2} + c_1\int_{t_1}^{t_2}\int_\Omega |\nabla(v^\alpha\eta)|^p \le c_2\int_{t_1}^{t_2}\int_\Omega v^\sigma |\nabla\eta|^p . \tag{5.4}$$

Let us explain why all the integrals in (5.3) are well defined. Observe that always $\lambda \ge 2$. Indeed, if $q \ge 1$ then, using $\sigma \ge pq$, we obtain

$$\lambda = \sigma - \delta \ge pq - (q(p-1)-1) = q+1 \ge 2, \tag{5.5}$$

and if $q < 1$ then, using $\sigma \ge p$, we obtain

$$\lambda = \sigma - \delta \ge p - \delta = p - (q(p-1)-1) = (p+1) - (p-1)\,q > (p+1) - (p-1) = 2.$$

Since $v(\cdot,t) \in L^2(\Omega)$ and v is bounded, it follows that, for any $t \in I$,

$$\int_\Omega v^\lambda(\cdot,t) \le \|v\|_{L^\infty}^{\lambda-2}\int_\Omega v^2(\cdot,t) < \infty.$$

Consequently, the expression

$$\left[\int_\Omega v^\lambda\eta^p\right]_{t_1}^{t_2}$$

is well-defined. It also follows that

$$\int_{t_1}^{t_2}\int_\Omega v^\lambda \eta^{p-1}|\eta_t| \le \text{const}\int_{t_1}^{t_2}\int_\Omega v^2(\cdot,t) < \infty.$$

Since $\nabla\eta(\cdot,t)$ and v are bounded and $\sigma \ge pq$, we obtain

$$\int_{t_1}^{t_2}\int_\Omega v^\sigma|\nabla\eta|^p \le \text{const}\int_{t_1}^{t_2}\int_\Omega v^\sigma \le \text{const}\,\|v\|_{L^\infty}^{\sigma-pq}\int_{t_1}^{t_2}\int_\Omega v^{pq} < \infty, \tag{5.6}$$

where we have used (4.3). The hypothesis $\sigma \ge pq$ implies that $\alpha \ge q$. Hence, the function $\Phi(s) = s^{\frac{a}{q}}$ is Lipschitz on any bounded interval in $[0,\infty)$. Since

$$\nabla v^\alpha = \nabla\Phi(v^q) = \Phi'(v^q)\nabla v^q$$

and v^q is bounded, it follows that

$$|\nabla v^\alpha| \le C|\nabla v^q|.$$

We obtain that

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega|\nabla(v^\alpha\eta)|^p &\le C\int_{t_1}^{t_2}\int_\Omega|\nabla v^\alpha|^p\eta^p + v^{\alpha p}|\nabla\eta|^p\\ &\le C\int_{t_1}^{t_2}\int_\Omega|\nabla v^q|^p + v^\sigma|\nabla\eta|^p\\ &< \infty,\end{aligned}$$

where we have used (4.3) and (5.6). Hence, all the integrals in (5.3) are well-defined. Let us record for a later usage that

$$v^\alpha\eta \in L^p_{loc}\left(I; W_0^{1,p}(\Omega)\right) \tag{5.7}$$

because

$$\int_{t_1}^{t_2}\int_\Omega (v^\alpha\eta)^p + |\nabla(v^\alpha\eta)|^p \le \text{const}\int_{t_1}^{t_2}\int_\Omega v^\sigma + \int_{t_1}^{t_2}\int_\Omega |\nabla(v^\alpha\eta)|^p < \infty.$$

Proof of Lemma 2. Let us rewrite (4.7) in the form

$$q^{1-p}\partial_t v \le \operatorname{div}\left(v^m |\nabla v|^{p-2}\nabla v\right) \tag{5.8}$$

where

$$m = (q-1)(p-1). \tag{5.9}$$

Multiplying (5.8) by $v^{\lambda-1}\eta^p$ and integrating it over the cylinder $Q = \Omega\times[t_1,t_2]$, we obtain

$$\begin{aligned} q^{1-p}\int_Q v_t v^{\lambda-1}\eta^p &\le \int_Q \operatorname{div}\left(v^m|\nabla v|^{p-2}\nabla v\right)v^{\lambda-1}\eta^p \\ &= -\int_Q v^m|\nabla v|^{p-2}\nabla v\,\nabla\left(v^{\lambda-1}\eta^p\right) \\ &= -\int_Q v^m|\nabla v|^{p-2}\nabla v\,\left[(\lambda-1)v^{\lambda-2}\eta^p\nabla v + pv^{\lambda-1}\eta^{p-1}\nabla\eta\right] \\ &= -(\lambda-1)\int_Q v^{\lambda+m-2}|\nabla v|^p\eta^p - p\int_Q v^{\lambda+m-1}|\nabla v|^{p-2}\eta^{p-1}(\nabla v,\nabla\eta) \\ &\le -(\lambda-1)\int_Q v^{\lambda+m-2}|\nabla v|^p\eta^p + p\int_Q v^{\lambda+m-1}|\nabla v|^{p-1}\eta^{p-1}|\nabla\eta|. \end{aligned} \tag{5.10}$$

Observe that, for any fixed t, the function $v^{\lambda-1}\eta^p$ belongs to $W_0^{1,p}(\Omega)$ which allows to use the integration-by-part formula without the boundary term. Indeed, we have $v^q \in W^{1,p}(\Omega)$ by the definition of a weak solution, which implies $v^{\lambda-1}\in W^{1,p}(\Omega)$ because v is bounded and $\lambda-1\ge q$ by (5.5), whence the inclusion $v^{\lambda-1}\eta^p\in W_0^{1,p}(\Omega)$ follows because η is compactly supported in Ω.

Since

$$\lambda+m-2 = \lambda+(q-1)(p-1)-2 = \lambda+(p-1)q-1-p = \lambda+\delta-p = \sigma-p,$$

we rewrite (5.10) as follows:

$$q^{1-p}\int_Q v_t v^{\lambda-1}\eta^p \le -(\lambda-1)\int_Q v^{\sigma-p}|\nabla v|^p\eta^p + p\int_Q v^{\sigma-p+1}|\nabla v|^{p-1}\eta^{p-1}|\nabla\eta|. \tag{5.11}$$

Since $\sigma\ge p$, the function v enters all the integrals in (5.11) in non-negative powers; hence, the integrals are finite.

Next, let us use the following inequality for all $X,Y\ge 0$ and $\varepsilon>0$:

$$XY \le \varepsilon^{p'}X^{p'} + \frac{1}{\varepsilon^p}Y^p$$

where $p' = \frac{p}{p-1}$ is the Hölder conjugate of p (here we use that $p > 1$). Applying this inequality with

$$X = v^{\xi} |\nabla v|^{p-1} \eta^{p-1} \quad \text{and} \quad Y = v^{(\sigma-p+1-\xi)} |\nabla \eta|$$

(where ε and ξ are yet to be determined) we obtain

$$\begin{aligned} v^{\sigma-p+1} |\nabla v|^{p-1} \eta^{p-1} |\nabla \eta| &= XY \le \varepsilon^{p'} \left(v^{\xi} |\nabla v|^{p-1} \eta^{p-1} \right)^{p'} + \frac{1}{\varepsilon^p} \left(v^{\sigma-p+1-\xi} |\nabla \eta| \right)^p \\ &= \varepsilon^{p'} v^{\xi p'} |\nabla v|^p \eta^p + \frac{1}{\varepsilon^p} v^{(\sigma-p+1-\xi)p} |\nabla \eta|^p . \end{aligned}$$

We would like to have

$$\xi p' = \sigma - p$$

whence

$$\xi := \frac{\sigma - p}{p'}.$$

With this ξ we have

$$(\sigma - p + 1 - \xi)\, p = \left(\sigma - p + 1 - \frac{(\sigma - p)}{p'} \right) p = \left(\frac{\sigma - p}{p} + 1 \right) p = \sigma$$

and

$$v^{\sigma-p+1} |\nabla v|^{p-1} \eta^{p-1} |\nabla \eta| \le \varepsilon^{p'} v^{\sigma-p} |\nabla v|^p \eta^p + \frac{1}{\varepsilon^p} v^{\sigma} |\nabla \eta|^p .$$

It follows that

$$\begin{aligned} q^{1-p} \int_Q v_t v^{\lambda-1} \eta^p &\le -(\lambda - 1) \int_Q v^{\sigma-p} |\nabla v|^p \eta^p + p \int_Q \left[\varepsilon^{p'} v^{\sigma-p} |\nabla v|^p \eta^p + \frac{1}{\varepsilon^p} v^{\sigma} |\nabla \eta|^p \right] \\ &= -\left(\lambda - 1 - p\varepsilon^{p'} \right) \int_Q v^{\sigma-p} |\nabla v|^p \eta^p + \frac{p}{\varepsilon^p} \int_Q v^{\sigma} |\nabla \eta|^p . \end{aligned} \tag{5.12}$$

On the other hand, we have

$$\begin{aligned} |\nabla (v^{\alpha} \eta)|^p &= \left| \alpha v^{\alpha-1} \eta \nabla v + v^{\alpha} \nabla \eta \right|^p \\ &\le 2^{p-1} \alpha^p v^{p(\alpha-1)} |\nabla v|^p \eta^p + 2^{p-1} v^{\alpha p} |\nabla \eta|^p \\ &= 2^{p-1} \alpha^p v^{\sigma-p} |\nabla v|^p \eta^p + 2^{p-1} v^{\sigma} |\nabla \eta|^p , \end{aligned}$$

where we have used that, by (5.2),

$$p(\alpha - 1) = \sigma - p.$$

It follows that

$$v^{\sigma-p} |\nabla v|^p \eta^p \ge 2^{1-p} \alpha^{-p} |\nabla (v^{\alpha} \eta)|^p - \alpha^{-p} v^{\sigma} |\nabla \eta|^p .$$

Substituting into (5.12) yields

$$q^{1-p} \int_Q v_t v^{\lambda-1} \eta^p \le -\left(\lambda - 1 - p\varepsilon^{p'} \right) 2^{1-p} \alpha^{-p} \int_Q |\nabla (v^{\alpha} \eta)|^p$$

$$+\left(\left(\lambda-1-p\varepsilon^{p'}\right)\alpha^{-p}+\frac{p}{\varepsilon^{p}}\right)\int_Q v^{\sigma}\left|\nabla\eta\right|^{p}.$$

Hence,

$$\lambda\int_Q v_t v^{\lambda-1}\eta^{p}\leq -c_1\int_Q\left|\nabla\left(v^{\alpha}\eta\right)\right|^{p}+c_2\int_Q v^{\sigma}\left|\nabla\eta\right|^{p}$$

where

$$\boxed{c_1=\lambda q^{p-1}\left(\lambda-1-p\varepsilon^{p'}\right)2^{1-p}\alpha^{-p}}$$

and

$$\boxed{c_2=\lambda q^{p-1}\left(\left(\lambda-1-p\varepsilon^{p'}\right)\alpha^{-p}+\frac{p}{\varepsilon^{p}}\right)}.$$

We choose ε so small that $c_1>0$, that is,

$$p\varepsilon^{p'}<\lambda-1.$$

Since

$$\lambda\int_Q v_t v^{\lambda-1}\eta^{p}=\int_Q\partial_t v^{\lambda}\eta^{p}=\left[\int_\Omega v^{\lambda}\eta^{p}\right]_{t_1}^{t_2}-p\int_Q v^{\lambda}\eta^{p-1}\eta_t$$

we obtain

$$\begin{aligned}\left[\int_\Omega v^{\lambda}\eta^{p}\right]_{t_1}^{t_2}&=\lambda\int_Q v_t v^{\lambda-1}\eta^{p}+p\int_Q v^{\lambda}\eta^{p-1}\eta_t\\&\leq -c_1\int_Q\left|\nabla\left(v^{\alpha}\eta\right)\right|^{p}+c_2\int_Q v^{\sigma}\left|\nabla\eta\right|^{p}+p\int_Q v^{\lambda}\eta^{p-1}\eta_t\end{aligned}$$

and, hence,

$$\left[\int_\Omega v^{\lambda}\eta^{p}\right]_{t_1}^{t_2}+c_1\int_Q\left|\nabla\left(v^{\alpha}\eta\right)\right|^{p}\leq\int_Q\left[pv^{\lambda}\eta^{p-1}\eta_t+c_2v^{\sigma}\left|\nabla\eta\right|^{p}\right].$$

Finally, let us specify c_1 and c_2. Let us choose ε so that

$$p\varepsilon^{p'}=\frac{1}{2}\left(\lambda-1\right)$$

that is

$$\boxed{c_1=\lambda\left(\lambda-1\right)2^{-p}q^{p-1}\alpha^{-p}}.\tag{5.13}$$

It follows that

$$\begin{aligned}c_2&=\lambda q^{p-1}\left(\frac{1}{2}\left(\lambda-1\right)\alpha^{-p}+\frac{p}{\varepsilon^{p}}\right)\\&=\lambda q^{p-1}\left(\frac{1}{2}\left(\lambda-1\right)\alpha^{-p}+\frac{p}{\left(\frac{1}{2}\left(\lambda-1\right)/p\right)^{p/p'}}\right)\\&=\lambda q^{p-1}\left(\frac{1}{2}\left(\lambda-1\right)\alpha^{-p}+\frac{2^{p/p'}p^{1+p/p'}}{\left(\lambda-1\right)^{p/p'}}\right).\end{aligned}$$

Since

$$\frac{p}{p'} + 1 = \frac{p}{p/(p-1)} + 1 = p$$

we have

$$\boxed{c_2 = \tfrac{1}{2}\lambda(\lambda-1)\,q^{p-1}\alpha^{-p} + \frac{\lambda 2^{p-1}p^p q^{p-1}}{(\lambda-1)^{p-1}}}. \tag{5.14}$$

■

Remark. For the future we need the ratio $\frac{c_2}{c_1}$. It follows from (5.13) and (5.14) that

$$\begin{aligned}\frac{c_2}{c_1} &= 2^{p-1} + \lambda\frac{2^{p-1}p^p}{(\lambda-1)^{p-1}\lambda(\lambda-1)\,2^{-p}\alpha^{-p}} \\ &= 2^{p-1} + \frac{2^{2p-1}\sigma^p}{(\lambda-1)^p},\end{aligned}$$

where we have used that $\alpha p = \sigma$. Since $\sigma = \lambda + \delta$, we obtain

$$\boxed{\frac{c_2}{c_1} = 2^{p-1} + \frac{2^{2p-1}(\lambda+\delta)^p}{(\lambda-1)^p}}.$$

It follows that, for all $\lambda \geq 2$,

$$\boxed{\frac{c_2}{c_1} \leq C_{p,q}}, \tag{5.15}$$

where $C_{p,q}$ depend only on p and q but does not depend on λ.

Remark. Let obtain an upper bound of c_2. Using

$$\alpha = \frac{\sigma}{p} = \frac{\lambda+\delta}{p}$$

we obtain

$$c_2 = \frac{1}{2}\frac{\lambda(\lambda-1)}{(\lambda+\delta)^p}q^{p-1}p^p + \frac{\lambda 2^{p-1}p^p q^{p-1}}{(\lambda-1)^{p-1}}.$$

As $\lambda \geq 2$ and $\lambda + \delta \geq p > 1$, it follows that

$$\boxed{c_2 \leq C_{p,q}\lambda^{2-p}}. \tag{5.16}$$

Of course, if $p \geq 2$ then c_2 is uniformly bounded by a constant $C_{p,q}$ independently of λ, but if $p < 2$ then c_2 may grow with λ as λ^{2-p}.

Lemma 3. *Let M be geodesically complete. Let $v = v(x,t)$ be a bounded non-negative subsolution to (4.7) in $M \times I$. Then, for all large enough λ, including $\lambda = \infty$, the function*

$$t \mapsto \|v(\cdot,t)\|_{L^\lambda(M)}$$

is monotone decreasing. Consequently, if $I = [a,b]$ then

$$\|v\|_{L^\infty(M\times I)} \leq \|v(\cdot,a)\|_{L^\infty(M)}.$$

Proof. If M is geodesically complete, then $W_0^{1,p}(M) = W^{1,p}(M)$. Hence, $v^{\lambda-1}(\cdot,t) \in W_0^{1,p}(M)$ for any $t \in I$, and we can use the argument in the proof of Lemma 2 with $\eta \equiv 1$ (without assumption that $\eta(\cdot,t)$ is compactly supported). Assuming that λ is large enough so that $\sigma := \lambda + \delta$ satisfies (5.1), we obtain from (5.3) that, for all $t_1, t_2 \in I$, $t_1 < t_2$,

$$\left[\int_M v^\lambda\right]_{t_1}^{t_2} \le 0,$$

which proved the claim for a finite λ. The case of an infinite λ is obtained then by letting $\lambda \to \infty$. ■

6 Sobolev and Moser inequalities

Let B be a precompact ball in a manifold M of dimension n. The Sobolev inequality in B of order p says the following|: for any non-negative function $w \in W_0^{1,p}(B)$

$$\left(\int_B w^{p\kappa}\right)^{1/\kappa} \le S_B \int_B |\nabla w|^p \,, \tag{6.1}$$

where $\kappa > 1$ is some constant and S_B is called the *Sobolev constant* in B. The value of κ is independent of B and can be chosen as follows:

$$\kappa = \frac{n}{n-p} \quad \text{if } n > p,$$

and κ is an arbitrary real number > 1 if $n \le p$.

We always assume that S_B is chosen to be minimal possible. In this case the function

$$B \mapsto S_B$$

is clearly monotone increasing with respect to inclusion of balls.

Fix a precompact ball $B \subset M$ and set $Q = B \times I$, where $I \subset \mathbb{R}$ is an interval Assume that the Sobolev inequality (10.5) holds in B with exponent $\kappa > 1$. Let κ' be its Hölder conjugate. Set

$$\boxed{\nu = \frac{1}{\kappa'} = \frac{\kappa - 1}{\kappa}}.$$

Lemma 4. *Let $w \in L^p(I; W_0^{1,p}(B))$ be a non-negative function. Then*

$$\boxed{\int_Q w^{p(1+\nu)} \le S_B \left(\int_Q |\nabla w|^p\right) \sup_{t \in I} \left(\int_B w^p\right)^\nu} \tag{6.2}$$

Proof. By the Hölder inequality, we have, for any fixed $t \in I$,

$$\int_B w^{p(1+\nu)} = \int_B w^p w^{p\nu} \le \left(\int_B w^{p\kappa}\right)^{1/\kappa} \left(\int_B w^{p\nu\kappa'}\right)^{1/\kappa'}$$

$$= \left(\int_B w^{p\kappa}\right)^{1/\kappa}\left(\int_B w^p\right)^{\nu}$$
$$\leq \left(\int_B w^{p\kappa}\right)^{1/\kappa}\sup_{t\in I}\left(\int_B w^p\right)^{\nu},$$

where we have used that $\nu\kappa' = 1$.

By the Sobolev inequality (6.1) we have, for any $t \in I$,

$$\left(\int_B w^{p\kappa}\right)^{1/\kappa} \leq S_B \int_B |\nabla w|^p .$$

It follows that

$$\int_B w^{p(1+\nu)} \leq S_B \left(\int_B |\nabla w|^p\right)\sup_{t\in I}\left(\int_B w^p\right)^{\nu}.$$

Integrating this inequality in $t \in I$ gives (6.2). ■

7 Comparison in two cylinders

Here we assume that

$$\delta := q(p-1) - 1 \geq 0.$$

Lemma 5. *Consider two balls $B = B(x,r)$ and $B' = B(x,r')$ with $0 < r' < r$, and two cylinders*

$$Q = B \times [0,T], \quad Q' = B' \times [0,T].$$

Assume that B is precompact. Let σ be any real such that

$$\sigma \geq \max(p, pq). \tag{7.1}$$

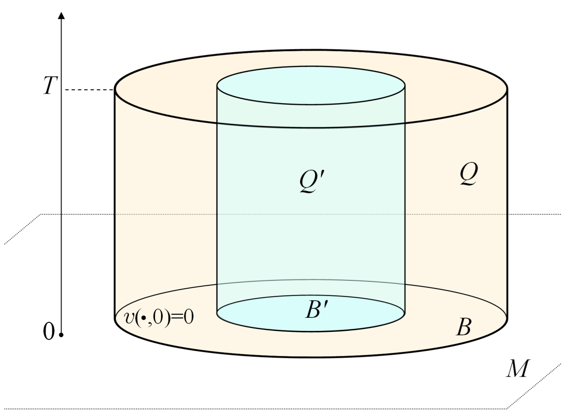
Let v be non-negative bounded subsolution of (4.1) in Q such that

$$v(\cdot,0) = 0 \text{ in } B.$$

Then

$$\int_{Q'} v^{\sigma(1+\nu)} \leq \frac{C S_B \sigma^{(2-p)\nu}}{(r-r')^{p(1+\nu)}}\left(\int_Q v^{\sigma}\right)\left(\int_Q v^{\sigma+\delta}\right)^{\nu}, \tag{7.2}$$

where C depends only on p, q and ν, while it is independent of σ.



Proof. As in Lemma 2, set $\lambda = \sigma - \delta$ and $\alpha = \frac{\sigma}{p}$ and recall that $\alpha \geq 1$ and $\lambda \geq 2$. Let $\eta = \eta(x)$ be a bump function of B' in B. By (5.7), we have

$$w := v^{\alpha}\eta \in L^p\left([0,T]\,;W_0^{1,p}(B)\right)$$

Applying (6.2) with this function w and using

$$w^p = v^{\sigma}\eta^p,$$

we obtain that

$$\int_Q v^{\sigma(1+\nu)}\eta^{p(1+\nu)} \leq S_B\left(\int_Q |\nabla(v^{\alpha}\eta)|^p\right)\sup_{t\in[0,T]}\left(\int_B v^{\sigma}\eta^p\right)^{\nu}.$$

By (5.4) we have

$$\int_Q |\nabla(v^{\alpha}\eta)|^p \leq \frac{c_2}{c_1}\int_Q v^{\sigma}|\nabla\eta|^p.$$

and

$$\sup_{t\in[0,T]}\left(\int_B v^{\lambda}\eta^p\right) \leq c_2\int_Q v^{\sigma}|\nabla\eta|^p.$$

Let us use the latter inequality also for other values of the parameters as follows:

$$\sup_{t\in[0,T]}\left(\int_B v^{\lambda'}\eta^p\right) \leq c_2'\int_Q v^{\sigma'}|\nabla\eta|^p,$$

where

$$\sigma' = \sigma + \delta \quad \text{and } \lambda' = \sigma' - \delta = \sigma.$$

Then we have

$$\sup_{t\in[0,T]}\left(\int_B v^{\sigma}\eta^p\right) \leq c_2'\int_Q v^{\sigma'}|\nabla\eta|^p.$$

It follows that

$$\int_Q v^{\sigma(1+\nu)}\eta^{p(1+\nu)} \leq S_B\frac{c_2}{c_1}\int_Q v^{\sigma}|\nabla\eta|^p\left(c_2'\int_Q v^{\sigma'}|\nabla\eta|^p\right)^{\nu}.$$

Using that $\eta = 1$ in B' and $|\nabla\eta| \leq \frac{1}{r-r'}$ we obtain

$$\int_{Q'} v^{\sigma(1+\nu)} \leq S_B\frac{c_2}{c_1}\frac{(c_2')^{\nu}}{(r_1-r_2)^{p(1+\nu)}}\left(\int_Q v^{\sigma}\right)\left(\int_Q v^{\sigma'}\right)^{\nu}.$$

By (5.15) we have

$$\frac{c_2}{c_1} \leq C_{p,q},$$

and (5.16)

$$c_2' \leq C_{p,q}(\lambda')^{2-p} = C_{p,q}\sigma^{2-p}.$$

Hence, (7.2) follows. ■

Corollary 6. *Under the hypotheses of* Lemma 5, *we have*

$$\int_{Q'} v^{\sigma(1+\nu)} \leq \frac{CS_B\sigma^{(2-p)\nu}\|v\|_{L^{\infty}(Q)}^{\delta\nu}}{(r-r')^{p(1+\nu)}}\left(\int_Q v^{\sigma}\right)^{1+\nu}, \tag{7.3}$$

where $C = C(p,q,\nu)$.

8 Auxiliary lemma about sequences

Lemma 7. *Let a sequence* $\{J_k\}_{k=0}^{\infty}$ *of non-negative reals satisfies*

$$J_{k+1} \leq \frac{A^k}{D} J_k^{1+\omega} \quad \text{for all } k \geq 0. \tag{8.1}$$

where $A, D, \omega > 0$*. Then, for all* $k \geq 0$*,*

$$J_k \leq \left(\left(A^{1/\omega} D^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^k} \left(A^{-k-1/\omega} D\right)^{1/\omega}. \tag{8.2}$$

In particular, if

$$D \geq A^{1/\omega} J_0^{\omega}, \tag{8.3}$$

then, for all $k \geq 0$*,*

$$J_k \leq A^{-k/\omega} J_0. \tag{8.4}$$

Proof. Consider the sequence

$$X_k = \left(\left(A^{1/\omega} D^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^k} \left(A^{-k-1/\omega} D\right)^{1/\omega}.$$

Then we have

$$X_0 = \left(A^{1/\omega} D^{-1}\right)^{1/\omega} J_0 \left(A^{-1/\omega} D\right)^{1/\omega} = J_0$$

and

$$\begin{aligned}
\frac{A^k}{D} X_k^{1+\omega} &= \frac{A^k}{D} \left(\left(A^{1/\omega} D^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^{k+1}} \left(A^{-k-1/\omega} D\right)^{\frac{1+\omega}{\omega}} \\
&= \left(\left(A^{1/\omega} D^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^{k+1}} A^k D^{-1} \left(A^{-k-1/\omega} D\right) \left(A^{-k-1/\omega} D\right)^{\frac{1}{\omega}} \\
&= \left(\left(A^{1/\omega} D^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^{k+1}} A^{-1/\omega} \left(A^{-k-1/\omega} D\right)^{1/\omega} \\
&= \left(\left(A^{1/\omega} D^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^{k+1}} \left(A^{-(k+1)-1/\omega} D\right)^{1/\omega} = X_{k+1}.
\end{aligned}$$

Hence, by comparison we obtain $J_k \leq X_k$, which was to be proved.

For the second statement, if (8.3) holds then we can assume without loss of generality that

$$D = A^{1/\omega} J_0^{\omega}.$$

Substituting this value of D into (8.2) we obtain

$$J_k \leq \left(A^{-k} J_0^{\omega}\right)^{1/\omega}$$

which is equivalent to (8.4). ∎

9 Mean value inequality

We assume here that

$$\delta = q\,(p-1) - 1 \geq 0.$$

Lemma 8. *Let $B = B\,(x_0, R)$ be a precompact ball. Let u be a non-negative bounded subsolution of (4.1) in*

$$Q = B \times [0,t]$$

such that

$$u\,(\cdot, 0) = 0 \text{ in } B.$$

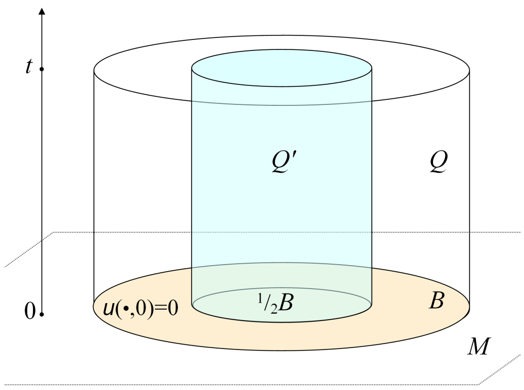
Let σ be as in (7.1). Then, for the cylinder

$$Q' = \frac{1}{2}B \times [0,t]\,,$$

we have

$$\|u\|_{L^\infty(Q')} \leq \left(\frac{CS_B}{R^{p(1+\nu)}}\right)^{\frac{1}{\sigma\nu}} \|u\|_{L^\infty(Q)}^{\frac{\delta}{\sigma}} \|u\|_{L^\sigma(Q)}\,, \tag{9.1}$$

where $C = C\,(p, q, \nu, \sigma)$.



Remark. Since

$$\|u\|_{L^\sigma(Q)} = \left(\int_0^t \int_\Omega u^\sigma\right)^{1/\sigma} \leq (t\mu(\Omega))^{\frac{1}{\sigma}} \|u\|_{L^\infty(Q)}\,,$$

we obtain from (9.1) that

$$\|u\|_{L^\infty(Q')} \leq \left(\left(\frac{CS_B}{R^{p(1+\nu)}}\right)^{\frac{1}{\nu}} t\mu(B)\right)^{\frac{1}{\sigma}} \|u\|_{L^\infty(Q)}^{1+\frac{\delta}{\sigma}}\,. \tag{9.2}$$

Remark. Unlike Lemma 5 where we have explicitly traced the dependence of the constants on σ, in (9.1) and (9.2) the dependence of C on σ is unimportant because these inequalities will be applied only with a fixed σ.

Proof. Consider a sequence of radii

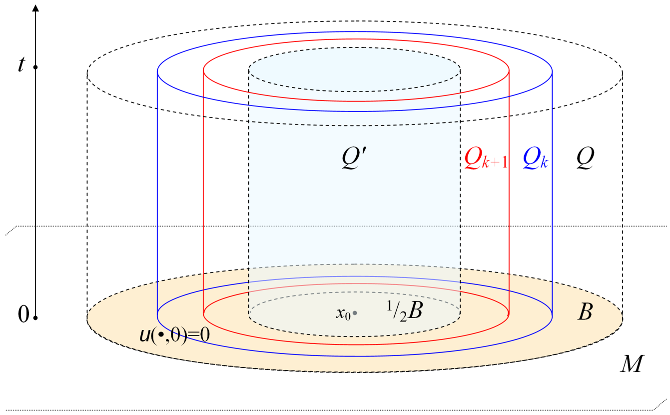
$$r_k = \left(\frac{1}{2} + 2^{-k-1}\right) R$$

so that $r_0 = R$ and $r_k \searrow \frac{1}{2}R$ as $k \to \infty$. Set

$$B_k = B\left(x_0, r_k\right), \qquad Q_k = B_k \times [0, t]$$

so that

$$B_0 = B, \qquad Q_0 = Q \qquad \text{and} \qquad Q_\infty := \lim_{k\to\infty} Q_k = Q'.$$



Set also

$$\sigma_k = \sigma\left(1+\nu\right)^k$$

and

$$J_k = \int_{Q_k} u^{\sigma_k}.$$

Applying (7.3) to the cylinders Q_k and Q_{k+1}, we obtain

$$\begin{aligned} J_{k+1} = \int_{Q_{k+1}} u^{\sigma_k(1+\nu)} &\le \frac{CS_{B_k}\sigma_k^{(2-p)\nu}\left\|u\right\|_{L^\infty(Q_k)}^{\delta\nu}}{\left(r_k - r_{k+1}\right)^{p(1+\nu)}} \int_{Q_k} u^{\sigma_k} \\ &= \frac{CS_{B_k}\sigma_k^{(2-p)\nu}\left\|u\right\|_{L^\infty(Q_k)}^{\delta\nu}}{\left(r_k - r_{k+1}\right)^{p(1+\nu)}} J_k^{1+\nu} \\ &\le \frac{C2^{kp(1+\nu)}\left(1+\nu\right)^{k(2-p)\nu}\sigma^{(2-p)\nu}S_B\left\|u\right\|_{L^\infty(Q)}^{\delta\nu}}{R^{p(1+\nu)}} J_k^{1+\nu} \\ &\le A^k D^{-1} J_k^{1+\nu}, \end{aligned}$$

where

$$A = 2^{p(1+\nu)}\left(1+\nu\right)^{(2-p)_+\nu} \ge 1$$

and

$$D^{-1} = \frac{CS_B\left\|u\right\|_{L^\infty(Q)}^{\delta\nu}}{R^{p(1+\nu)}},$$

where we have absorbed $\sigma^{(2-p)\nu}$ into C. By Lemma 7 we conclude that

$$J_k \le \left(\left(A^{1/\nu} D^{-1} \right)^{1/\nu} J_0 \right)^{(1+\nu)^k} \left(A^{-1/\nu} D \right)^{1/\nu}$$
$$= A^{\frac{(1+\nu)^k - 1}{\nu^2}} D^{-\frac{(1+\nu)^k - 1}{\nu}} J_0^{(1+\nu)^k}.$$

It follows that

$$\left(\int_{Q_k} u^{\sigma_k} \right)^{1/\sigma_k} = J_k^{\frac{1}{\sigma(1+\nu)^k}} \le A^{\frac{1-(1+\nu)^{-k}}{\sigma\nu^2}} D^{-\frac{1-(1+\nu)^{-k}}{\sigma\nu}} \left(\int_Q u^{\sigma} \right)^{1/\sigma}.$$

As $k \to \infty$, we obtain

$$\begin{aligned} \|u\|_{L^\infty(Q')} &\le A^{\frac{1}{\sigma\nu^2}} D^{-\frac{1}{\sigma\nu}} \|u\|_{L^\sigma(Q)} = \left(A^{\frac{1}{\nu}} D^{-1} \right)^{\frac{1}{\sigma\nu}} \|u\|_{L^\sigma(Q)} \\ &\le \left(A^{\frac{1}{\nu}} \frac{C S_B \|u\|_{L^\infty(Q)}^{\delta\nu}}{R^{p(1+\nu)}} \right)^{\frac{1}{\sigma\nu}} \|u\|_{L^\sigma(Q)} \\ &= \left(\frac{C S_B}{R^{p(1+\nu)}} \right)^{\frac{1}{\sigma\nu}} \|u\|_{L^\infty(Q)}^{\frac{\delta}{\sigma}} \|u\|_{L^\sigma(Q)}, \end{aligned}$$

where $A^{\frac{1}{\nu}}$ was absorbed into C. ∎

10 Normalized Sobolev constant

Let B be a precompact ball in M and $w \in W_0^{1,p}(B)$. Dividing the Sobolev inequality (6.1) by $\mu(B)^{1/\kappa}$, we obtain

$$\left(⨍_B w^{p\kappa} \right)^{1/\kappa} \le \mu(B)^\nu S_B ⨍_B |\nabla w|^p$$

where

$$\nu = \frac{\kappa - 1}{\kappa} = \frac{1}{\kappa'},$$

and

$$\left(⨍_B w^{p\kappa} \right)^{1/(p\kappa)} \le \left(\mu(B)^\nu S_B \right)^{1/p} \left(⨍_B |\nabla w|^p \right)^{1/p}, \tag{10.3}$$

Denoting by $r(B)$ the radius of B, let us define a new quantity

$$\boxed{\iota(B) = \frac{1}{\mu(B)} \left(\frac{r(B)^p}{S_B} \right)^{1/\nu}}$$

so that

$$S_B = \frac{r(B)^p}{\left(\iota(B) \mu(B) \right)^\nu} \tag{10.4}$$

and

$$(\mu(B)^\nu S_B)^{1/p} = \frac{r(B)}{\iota(B)^{\frac{\nu}{p}}}.$$

Hence, (10.3) can be rewritten in the form

$$\boxed{\left(\fint_B |\nabla w|^p\right)^{1/p} \geq \frac{\iota(B)^{\frac{\nu}{p}}}{r(B)} \left(\fint_B w^{p\kappa}\right)^{1/p\kappa}}. \tag{10.5}$$

It is clear from (10.5) that the value of κ can be always reduced (by modifying the value of $\iota(B)$). It is only important that $\kappa > 1$. In fact, the exact value of κ does not affect the results, although various constants depend on κ.

The constant $\iota(B)$ is called the *normalized Sobolev* constant in B. It is known that if M is complete and $Ricci_M \geq 0$ then, for all balls B, the normalized Sobolev constant $\iota(B)$ is bounded below by a positive constant.

11 Propagation speed inside a ball

We assume here that M is geodesically complete and

$$\delta = q(p-1) - 1 > 0.$$

Theorem 9. *Let u be a bounded non-negative subsolution of (4.1) in $M \times [0,T]$ with the initial condition $u(\cdot,0) = u_0$. Let $B_0 = B(z_0, R)$ be a ball in M such that*

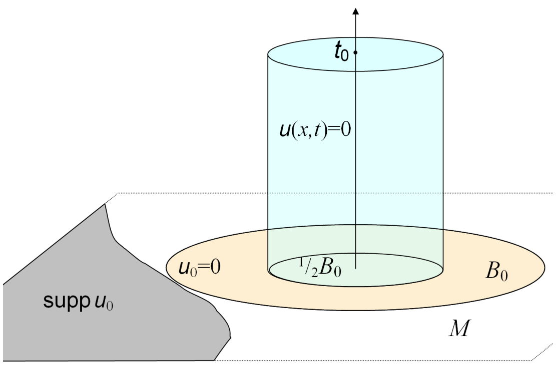
$$u_0 = 0 \text{ in } B_0.$$

Set

$$t_0 = \eta\, \iota(B_0) R^p \, \|u_0\|_{L^\infty(M)}^{-\delta} \wedge T, \tag{11.1}$$

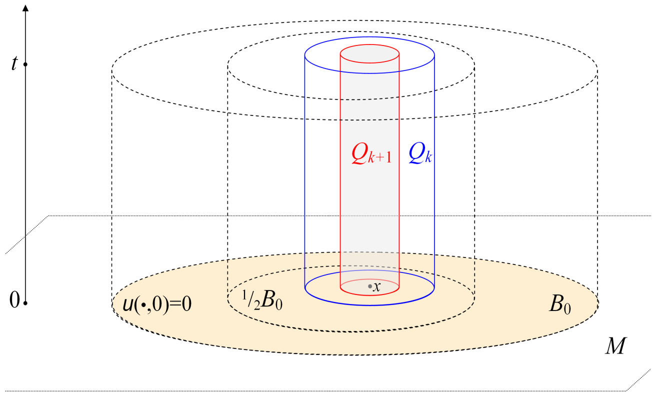
where η is a sufficiently small positive constant depending only on p, q, ν. Then

$$u = 0 \text{ in } \frac{1}{2}B_0 \times [0, t_0].$$



Proof. Set $r = \frac{1}{2}R$ and fix for a while a point $x \in \frac{1}{2}B_0$ so that $B := B(x, r) \subset B_0$. Fix also some $t \in (0, T]$ and set

$$Q_k = 2^{-k}B \times [0, t] \quad \text{and} \quad J_k = \|u\|_{L^\infty(Q_k)}.$$



Our purpose is to obtain an upper bound for $J_k(x) = \|u\|_{L^\infty(Q_k)}$ that ensures that $J_k(x) \to 0$ as $k \to \infty$ uniformly in $x \in \frac{1}{2}B_0$.

Let us fix σ satisfying (7.1), for example, $\sigma = \max(p, pq)$. Applying inequality (9.2) of Lemma 8 in the cylinders Q_k, Q_{k+1}, we obtain

$$\begin{aligned}
J_{k+1} = \|u\|_{L^\infty(Q_{k+1})} &\le \left(\left(\frac{CS_{2^{-k}B}}{(2^{-k}r)^{p(1+\nu)}} \right)^{\frac{1}{\nu}} t\mu(2^{-k}B) \right)^{\frac{1}{\sigma}} \|u\|_{L^\infty(Q_k)}^{1+\frac{\delta}{\sigma}} \\
&\le 2^{(k+1)\frac{p(1+\nu)}{\sigma\nu}} \left(\left(\frac{CS_{B_0}}{R^{p(1+\nu)}} \right)^{\frac{1}{\nu}} t\mu(B_0) \right)^{\frac{1}{\sigma}} J_k^{1+\frac{\delta}{\sigma}},
\end{aligned}$$

where we have used that $S_{2^{-k}B} \le S_{B_0}$ and $\mu(2^{-k}B) \le \mu(B_0)$. By (10.4) we have

$$(S_{B_0})^{1/\nu} = \left(\frac{R^p}{(\iota(B_0)\mu(B_0))^\nu} \right)^{1/\nu} = \frac{R^{p/\nu}}{\iota(B_0)\mu(B_0)},$$

whence

$$\left(\frac{S_{B_0}}{R^{p(1+\nu)}} \right)^{\frac{1}{\nu}} \mu(B_0) = \frac{R^{p/\nu}}{R^{p\frac{(1+\nu)}{\nu}} \iota(B_0)\mu(B_0)} \mu(B_0) = \frac{1}{\iota(B_0)R^p}.$$

It follows that

$$\begin{aligned}
J_{k+1} &\le 2^{(k+1)\frac{p(1+\nu)}{\sigma\nu}} \left(\frac{Ct}{\iota(B_0)R^p} \right)^{\frac{1}{\sigma}} J_k^{1+\frac{\delta}{\sigma}} \\
&= A^k D^{-1} J_k^{1+\omega},
\end{aligned}$$

where

$$\omega = \frac{\delta}{\sigma}, \qquad A = 2^{\frac{p(1+\nu)}{\sigma\nu}}$$

and

$$D^{-1} = A\left(\frac{Ct}{\iota(B_0)R^p}\right)^{\frac{1}{\sigma}}.$$

By Lemma 7, if

$$D^{-1} \le A^{-1/\omega} J_0^{-\omega} \tag{11.2}$$

then, for all $k \ge 0$,

$$J_k \le A^{-k/\omega} J_0. \tag{11.3}$$

The condition (11.2) is equivalent to

$$A\left(\frac{Ct}{\iota(B_0)R^p}\right)^{\frac{1}{\sigma}} \le A^{-1/\omega} J_0^{-\omega}$$

that is, to

$$t \le C^{-1}\iota(B_0)R^p J_0^{-\delta}, \tag{11.4}$$

where A is absorbed to C. Since by Lemma 3

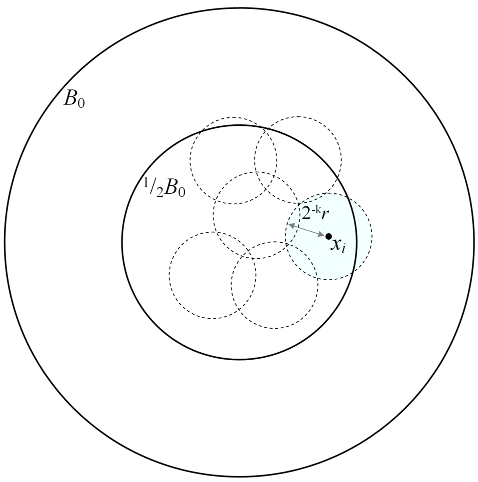
$$J_0 = \|u\|_{L^\infty(Q)} \le \|u_0\|_{L^\infty(M)},$$

the condition (11.4) is satisfied for $t = t_0$, where t_0 is determined by (11.1) with $\eta = C^{-1}$.

Hence, for $t = t_0$ we obtain from (11.3) that, for any k,

$$\|u\|_{L^\infty\left(2^{-k}B\times[0,t]\right)} \le A^{-k/\omega}\|u_0\|_{L^\infty}.$$

For any k, we cover the ball $\frac{1}{2}B_0$ by a finite sequence of balls $B\left(x_i, 2^{-k}r\right)$ with $x_i \in \frac{1}{2}B_0$.



Since for all i

$$\|u\|_{L^\infty\left(B(x_i,2^{-k}r)\times[0,t]\right)} \le A^{-k/\omega}\|u_0\|_{L^\infty},$$

we obtain that

$$\|u\|_{L^\infty\left(\frac{1}{2}B_0\times[0,t]\right)} \le A^{-k/\omega}\|u_0\|_{L^\infty}.$$

Finally, letting $k \to \infty$, we obtain that $u = 0$ in $\frac{1}{2}B_0 \times [0,t]$, which was to be proved. ■

12 Propagation speed of support

In this section we assume M is geodesically complete, that is, all geodesic balls are precompact. Let also

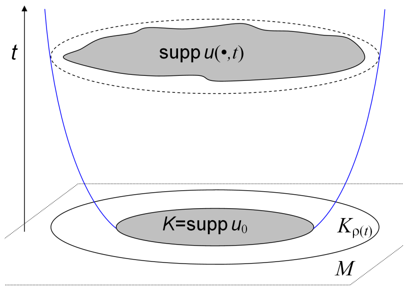
$$\delta = q(p-1) - 1 > 0.$$

For any set $K \subset M$ and any $r > 0$, denote by K_r a closed r-neighborhood of K.

Theorem 10. *Let $u(x,t)$ be a non-negative bounded subsolution of (4.1) in $M \times \mathbb{R}_+$ with the initial function $u_0 = u(\cdot,0)$. Assume that the support $K = \operatorname{supp} u_0$ is compact. Then there exists $T > 0$ and an increasing continuous function $\rho : (0,T) \to \mathbb{R}_+$ such that*

$$\operatorname{supp} u(\cdot,t) \subset K_{\rho(t)}$$

for all $t \in (0,T)$.



Here both T and $\rho(t)$ may depend on u. The function $\rho(t)$ is called a *propagation rate* of u.

Proof. Let us fix a reference point $x_0 \in K$ and define the following function for all $r > 0$:

$$\varphi(r) = \frac{\eta}{4^{p+p/\nu}} \iota(B(x_0,r)) r^p \|u_0\|_{L^\infty(M)}^{-\delta}. \tag{12.1}$$

Denote

$$r_0 = \operatorname{diam} K.$$

Let us prove that that, for any $r \ge r_0$,

$$t \le \varphi(3r + r_0) \Rightarrow \operatorname{supp} u(\cdot,t) \subset K_r,$$

that is,

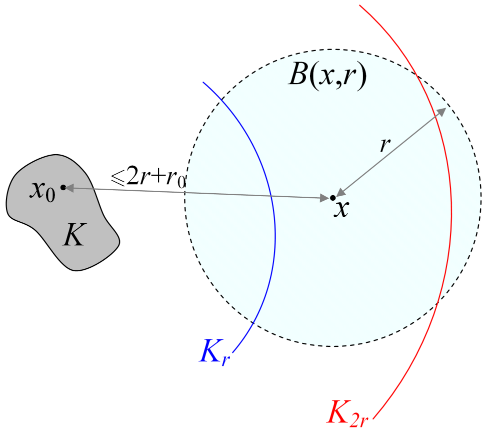
$$u(\cdot,t) = 0 \text{ in } M \setminus K_r.$$

Let us first prove that

$$u(\cdot,t) = 0 \text{ in } K_{2r} \setminus K_r.$$

Fix a point $x \in K_{2r} \setminus K_r$.We have

$$d(x,K) \le 2r \Rightarrow d(x,x_0) \le 2r + r_0.$$



It follows that

$$B\left(x,r\right)\subset B\left(x_0,3r+r_0\right)=B(x_0,R)$$

where

$$R:=3r+r_0.$$

The condition $r\geq r_0$ implies $R\leq 4r$. Since $B(x,r)\subset B(x_0,R)$, we have by the monotonicity of function (10.4) that

$$\frac{\iota(B(x,r))\mu(B(x,r))}{r^{p/\nu}}\geq\frac{\iota(B(x_0,R))\mu(B(x_0,R))}{R^{p/\nu}}.$$

It follows that

$$\begin{aligned}\frac{\iota(B(x,r))r^p}{\iota(B(x_0,R))R^p}&\geq\left(\frac{r}{R}\right)^{p+p/\nu}\frac{\mu(B(x_0,R))}{\mu(B(x,r))}\\&\geq\frac{1}{4^{p+p/\nu}}\iota(B(x_0,R))R^p.\end{aligned}$$

Therefore, the hypothesis

$$t\leq\varphi\left(R\right)=\frac{\eta}{4^{p+p/\nu}}\iota(B(x_0,R))R^p\left\|u_0\right\|_{L^\infty(M)}^{-\delta}$$

implies that

$$t\leq\eta\iota(B(x,r)))r^p\left\|u_0\right\|_{L^\infty(M)}^{-\delta}.$$

Since $u(\cdot,0)=0$ in $B(x,r)$, we conclude by Theorem 9 that

$$u(\cdot,t)=0\ \text{ in }\ B(x,r/2).$$

Since this is true for any $x\in K_{2r}\setminus K_r$, we obtain that

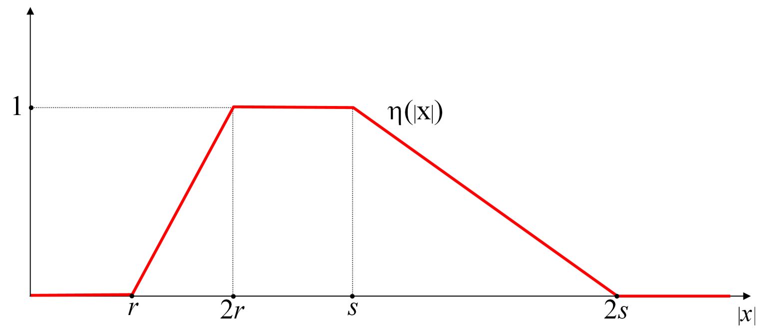
$$u(\cdot,t)=0\ \text{ in } K_{2r}\setminus K_r. \tag{12.2}$$

Let us show that also

$$u(\cdot,t)=0\ \text{ in } M\setminus K_{2r}. \tag{12.3}$$

Fix some $s >> 2r$ and let $\eta(x)$ be a bump function of $K_s \setminus K_{2r}$ in $K_{2s} \setminus K_r$; that is, η is the following function of $|x| := d(x, K)$:

$$\eta(x) = \begin{cases} \left(\frac{|x|}{r} - 1\right)_+, & |x| \le 2r, \\ 1, \quad |x| \in [2r, s], \\ 2\left(1 - \frac{|x|}{2s}\right)_+, & |x| \ge s. \end{cases}$$



Function η

Applying the inequality (5.4) of Lemma 2 with some fixed λ, we obtain

$$\left[\int_M u^\lambda \eta^p\right]_0^t \le c_2 \int_0^t \int_M u^\sigma |\nabla \eta|^p. \tag{12.4}$$

Since $u(\cdot, 0) = 0$ on $\operatorname{supp} \eta$ and $\eta = 1$ on $K_s \setminus K_{2r}$, the left hand side here is bounded below by

$$\int_{K_s \setminus K_{2r}} u^\lambda(\cdot, t).$$

Since $\eta = 0$ in K_r, $u(\cdot, \tau) = 0$ in $K_{2r} \setminus K_r$ for all $\tau \le t$ (by (12.2)), and $\nabla \eta = 0$ in $K_s \setminus K_{2r}$, the right hand side in (12.4) is equal to

$$c_2 \int_0^t \int_{M \setminus K_s} u^\sigma |\nabla \eta|^p.$$

Since

$$|\nabla \eta| \le \frac{1}{s} \text{ in } M \setminus K_s,$$

we obtain that

$$\int_{K_s \setminus K_{2r}} u^\lambda(\cdot, t) \le c_2 \int_0^t \int_{M \setminus K_s} u^\sigma |\nabla \eta|^p \le \frac{c_2}{s^p} \int_0^t \int_{M \setminus K_s} u^\sigma.$$

The right hand side goes to 0 as $s \to \infty$, which implies that $u(\cdot, t) = 0$ in $M \setminus K_{2r}$, thus proving (12.3).

Now let us define in $[r_0, \infty)$ a function

$$\psi(r) = \frac{1}{2} \sup_{r_0 \le s \le r} \varphi(3s + r_0)$$

so that $\psi(r)$ is monotone increasing. If $t \leq \psi(r)$ then $t \leq \varphi(3s + r_0)$ for some $s \in [r_0, r]$, which implies by the first part of the proof that

$$u(\cdot, t) = 0 \text{ in } M \setminus K_s$$

and, hence,

$$u(\cdot, t) = 0 \text{ in } M \setminus K_r.$$

It is unclear whether ψ is continuous or not. As a monotone function, ψ may have only jump discontinuities. By subtracting all these jumps, we obtain a continuous monotone function $\widetilde{\psi} \leq \psi$ with the same property|:

$$t \leq \widetilde{\psi}(r) \Rightarrow u(\cdot, t) = 0 \text{ in } M \setminus K_r. \tag{12.5}$$

As a continuous monotone increasing function, $\widetilde{\psi}$ has an inverse $\rho = \widetilde{\psi}^{-1}$ on $[t_0, T)$ where

$$t_0 = \widetilde{\psi}(r_0) \text{ and } T = \sup \widetilde{\psi}.$$

Let us extend $\rho(t)$ to $t < t_0$ by setting $\rho(t) = \rho(t_0)$. Then $r = \rho(t)$ implies $t \leq \widetilde{\psi}(r)$, and by (12.5)

$$u(\cdot, t) = 0 \text{ in } M \setminus K_r,$$

which was to be proved. ■

13 Curvature and propagation rate

In this section we assume again that M is geodesically complete and

$$\delta = q(p-1) - 1 > 0.$$

Theorem 11. *Let M be geodesically complete, non-compact, and let $Ricci_M \geq 0$. Let u be a bounded non-negative subsolution of (4.1) in $M \times \mathbb{R}_+$ with the initial condition $u(\cdot, 0) = u_0$. Set $K = \operatorname{supp} u_0$. Then, for any $t \geq 0$,*

$$\operatorname{supp} u(\cdot, t) \subset K_{Ct^{1/p}},$$

where C depends on $\|u_0\|_{L^\infty}$, p, q, n.

Proof. It is known that on such manifolds $\iota(B) \geq \text{const} > 0$ for all balls $B \subset M$.

Let $B = B(x, r)$ be any ball that is disjoint with K. It follows from Theorem 9, that is

$$t \leq c r^p \|u_0\|_{L^\infty(M)}^{-\delta},$$

where $c > 0$ is a small enough constant, then

$$u(\cdot, t) = 0 \text{ in } \frac{1}{2}B.$$

Hence, if

$$r \geq C t^{1/p}$$

where $C = c^{-1/p} \|u_0\|_{L^\infty(M)}^{\delta/p}$, then

$$\operatorname{supp} u(\cdot, t) \cap \frac{1}{2}B = \emptyset.$$

It follows that

$$\operatorname{supp} u(\cdot, t) \subset K_{\frac{1}{2}r},$$

whence the claim follows. ■