Propagation speed of non-linear parabolic equations on Riemannian manifolds

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Contents

1   Introduction

We are concerned with an evolution equation

$$\partial_t u = \Delta_p u^q \tag{1.1}$$

where $p, q > 0$, $u(x,t)$ is an unknown non-negative function, and Δ_p is the p-Laplacian:

$$\Delta_p v = \mathrm{div}\left(|\nabla v|^{p-2}\nabla v\right).$$

Equation (1.1) was introduced by L. S. Leibenson in 1945 in order to describe filtration of turbulent compressible fluid through a porous medium. The physical meaning of u is the *volumetric moisture content,* i.e. the (infinitesimal) fraction of volume of the medium taken by the liquid. Parameter p characterizes the turbulence of a flow while $q - 1$ is the index of *polytropy* of the liquid, that determines relation $PV^{q-1} = \mathrm{const}$ between volume V and pressure P.

The physically interesting values of p and q are as follows: $\frac{3}{2} \le p \le 2$ and $q \ge 1$. The case $p = 2$ corresponds to laminar flow (=absence of turbulence). In this case (1.1) becomes a *porous medium* equation $\partial_t u = \Delta u^q$, if $q > 1$, and the classical heat equation $\partial_t u = \Delta u$ if $q = 1$.

From mathematical point of view, the entire range $p > 1$, $q > 0$ is interesting. G.I.Barenblatt constructed in 1952 spherically symmetric self-similar solutions of (1.1) in $\mathbb{R}^n$ that are nowadays called *Barenblatt solutions.* Let us assume that

$$\boxed{q\,(p-1) > 1}.$$

In this case the Barenblatt solution is as follows:

$$u(x,t) = \frac{1}{t^{n/\beta}}\left(C - \kappa\left(\frac{|x|}{t^{1/\beta}}\right)^{\frac{p}{p-1}}\right)_+^{\gamma}$$

where $C > 0$ is any constant, and

$$\beta = p + n\left[q\,(p-1) - 1\right]\ ,\quad \gamma = \frac{p-1}{q(p-1)-1},\quad \kappa = \frac{q(p-1)-1}{pq}\beta^{-\frac{1}{p-1}}. \tag{1.2}$$

Parameter β determines the space/time scaling and is analogous to the *walk dimension.*
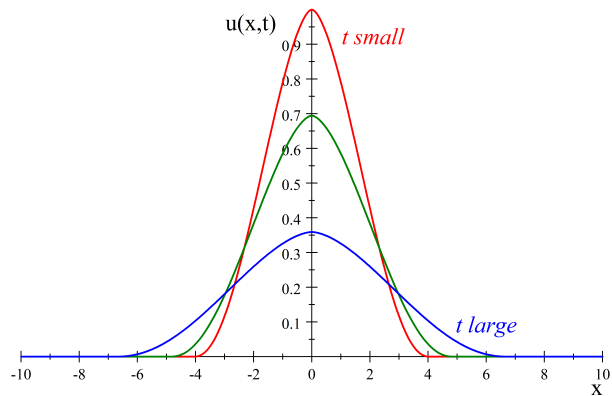
It is obvious that, for the above Barenblatt solution,

$$u(x,t) = 0 \quad \text{for} \quad |x| > ct^{1/\beta}$$

so that $u(\cdot, t)$ has a *compact support* for any t. One says that u has a *finite propagation speed.* Here are the graphs of function $x \mapsto u(x,t)$ for different values of t in the case $n = 1$.
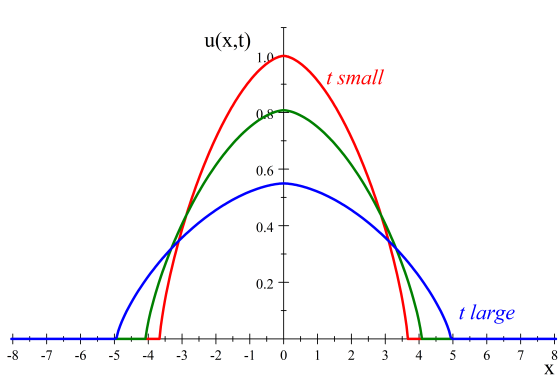
Case $p = 2.5$ and $q = 2$

Case $p = 2.5$ and $q = 1.1$

Assume now that
$$q(p-1) < 1.$$
Then we have $\gamma, \kappa < 0$, and the Barenblatt solution is

$$u(x,t) = \frac{1}{t^{n/\beta}} \left(C + |\kappa| \left(\frac{|x|}{t^{1/\beta}} \right)^{\frac{p}{p-1}} \right)^{-|\gamma|},$$

that is, it is positive for all x, t.

In the borderline case
$$q(p-1) = 1,$$
the Barenblatt solution is

$$u(x,t) = \frac{1}{t^{n/p}} \exp \left(-c \left(\frac{|x|}{t^{1/p}} \right)^{\frac{p}{p-1}} \right),$$

where $c = (p-1)^2 p^{-\frac{p}{p-1}}$. Hence, if $q(p-1) \le 1$ then u has infinite propagation speed.

Of course, if here $p = 2$ then $q = 1$, and we obtain the fundamental solution of the heat equation $\partial_t u = \Delta u$:

$$u(x,t) = \frac{1}{t^{n/2}} \exp \left(-\frac{1}{4} \left(\frac{|x|}{t^{1/2}} \right)^2 \right),$$

2 Leibenson's equation on manifolds

Consider on an arbitrary Riemannian manifold the operator

$$Lv = \Delta_p (v^q)$$

where
$$p > 1 \quad \text{and} \quad q > 0,$$
and
$$\Delta_p u = \text{div} \left(|\nabla u|^{p-2} \nabla u \right).$$

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We will be concerned with the associated evolution equation

$$\boxed{\partial_t v = \Delta_p(v^q)}$$

that is called the Leibenson equation. Our aim is to prove the following theorem.

Theorem 1. *If $q(p-1) > 1$ the any bounded non-negative solution to the Leibenson equation has a finite propagation speed.*

The exact meaning of "finite propagation speed" will be explained later on. The proof will also be given later on.

Now we show how to obtain the Barenblatt solutions in $\mathbb{R}^n$. We start with deriving a chain rule for the p-Laplacian. Consider on an arbitrary manifold the p-Laplacian

$$\Delta_p v = \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right), \tag{2.1}$$

where $p > 1$. Let us compute $\Delta_p f(u)$ assuming that f is smooth enough and

$$f \geq 0 \quad \text{and} \quad f' \leq 0.$$

We have $\nabla f(u) = f'(u)\nabla u$ and

$$\begin{aligned}
\Delta_p f(u) &= \operatorname{div}(|f'(u)\nabla u|^{p-2}f'(u)\nabla u) \\
&= \operatorname{div}\left(|f'(u)|^{p-2}f'(u)|\nabla u|^{p-2}\nabla u\right) \\
&= -\operatorname{div}\left(|f'(u)|^{p-1}|\nabla u|^{p-2}\nabla u\right) \\
&= -|f'(u)|^{p-1}\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - \nabla\left(|f'(u)|^{p-1}\right)|\nabla u|^{p-2}\nabla u \\
&= -|f'(u)|^{p-1}\Delta_p u - \nabla\left((-f'(u))^{p-1}\right)|\nabla u|^{p-2}\nabla u \\
&= -|f'(u)|^{p-1}\Delta_p u + (p-1)(-f'(u))^{p-2}f''(u)\nabla u|\nabla u|^{p-2}\nabla u \\
&= -|f'(u)|^{p-1}\Delta_p u + (p-1)|f'(u)|^{p-2}f''(u)|\nabla u|^p.
\end{aligned}$$

Hence, we obtain

$$\Delta_p f(u) = -(-f'(u))^{p-1}\Delta_p u + (p-1)(-f'(u))^{p-2}f''(u)|\nabla u|^p. \tag{2.2}$$

3 Solutions on models

3.1 Model manifolds

Let M be a *model manifold* $\mathbb{R}_+ \times \mathbb{S}^{n-1}$ with the polar coordinates (r, θ) (where $r \in \mathbb{R}_+$ and $\theta \in \mathbb{S}^{n-1}$) and with the Riemannian metric
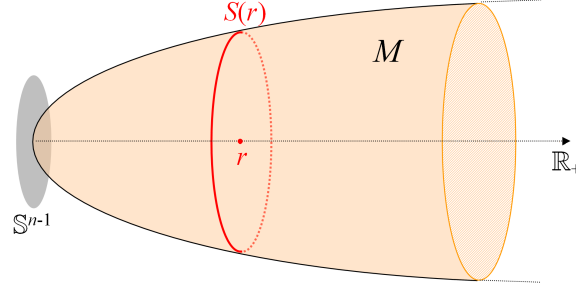
$$ds^2 = dr^2 + \psi^2(r)\,d\theta^2.$$

Here $d\theta^2$ is the standard Riemannian metric on $\mathbb{S}^{n-1}$ and ψ is a smooth positive function on $\mathbb{R}_+$. For example, $\mathbb{R}^n \setminus \{0\}$ can be considered as a model manifold with $\psi(r) = r$.

Denote by $S(r)$ the boundary area function

$$S(r) = \omega_n \psi(r)^{n-1}.$$

For example, in $\mathbb{R}^n$ we have $S(r) = \omega_n r^{n-1}$.

It is known that the Laplace-Beltrami operator Δ on M admits the following representation in the polar coordinates:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{S'}{S}\frac{\partial}{\partial r} + \frac{1}{\psi^2(r)}\Delta_\theta,$$

where Δ_θ is the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$. In particular, considering the polar radius r as a function in M, we obtain that

$$\Delta r = \frac{S'}{S}. \tag{3.1}$$

For example, in $\mathbb{R}^n$ we have $\Delta r = \frac{n-1}{r}$. Using (3.1) and $|\nabla r| = 1$, we obtain that

$$\Delta_p r = \operatorname{div}\left(|\nabla r|^{p-2}\nabla r\right) = \operatorname{div}(\nabla r) = \Delta r = \frac{S'}{S}.$$

Setting in (2.2) $u = r$, we obtain

$$\begin{aligned}
\Delta_p f(r) &= -\left(-f'(r)\right)^{p-1}\Delta_p r + (p-1)\left(-f'(r)\right)^{p-2}f''(r)\\
&= -\left(-f'(r)\right)^{p-1}\frac{S'}{S} + (p-1)\left(-f'(r)\right)^{p-2}f''(r). \tag{3.2}
\end{aligned}$$

Note that

$$\left(\left(-f'(r)\right)^{p-1}S\right)' = \left(-f'(r)\right)^{p-1}S' - (p-1)\left(-f'(r)\right)^{p-2}f''(r)S.$$

Hence, (3.2) can be rewritten in the form

$$\Delta_p f(r) = -\frac{1}{S}\left(S\left(-f'(r)\right)^{p-1}\right)'.$$

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The parabolic equation $\partial_t u = \Delta_p u$ for a function $u = u(r,t)$ (such that $u \geq 0$ and $\partial_r u \leq 0$) becomes therefore

$$\partial_t u = -\frac{1}{S} \partial_r \left(S \left(-\partial_r u \right)^{p-1} \right), \tag{3.3}$$

and the Leibenson equation $\partial_t u = \Delta_p (u^q)$ becomes

$$\partial_t u = -\frac{1}{S} \partial_r \left(S \left(-\partial_r u^q \right)^{p-1} \right). \tag{3.4}$$

3.2   Barenblatt solutions

We solve here (3.4) assuming $p > 1$, $q > 0$ and

$$S(r) = r^{\alpha-1},$$

where α is a positive real. In particular, for $\alpha = n \in \mathbb{N}$, this will give us the Barenblatt solution in $\mathbb{R}^n$.

The equation (3.4) becomes with this $S(r)$

$$\partial_t u = -\frac{1}{r^{\alpha-1}} \partial_r \left(r^{\alpha-1} \left(-\partial_r u^q \right)^{p-1} \right), \tag{3.5}$$

and we look for a solution in the form

$$u(x,t) = t^a f \left(rt^b \right),$$

where f is a decreasing non-negative function and a, b are (negative) reals, yet to be determined.

Let us require in addition that the solution $u(x,t)$ has a constant L^1-norm, that is,

$$\int_M t^a f(rt^b) d\mu = \text{const} < \infty,$$

where μ is the Riemannian measure. Computing the integral in the polar coordinates and using

$$d\mu = \psi(r)^{n-1} dr d\theta = \frac{1}{\omega_n} S(r) dr d\theta,$$

we obtain that

$$\int_0^\infty t^a f(rt^b) r^{\alpha-1} dr = \text{const} < \infty.$$

A change $s = rt^b$ in the integral gives

$$\int_0^\infty t^a f(rt^b) r^{\alpha-1} dr = \int_0^\infty t^a f(s) \left(st^{-b} \right)^{\alpha-1} t^{-b} ds = t^{a-b\alpha} \int_0^\infty f(s) s^{\alpha-1} ds.$$

Hence, we must have

$$\int_0^\infty f(s) s^{\alpha-1} ds < \infty \tag{3.6}$$

and
$$a = \alpha b.$$

Using again the variable $s = rt^b$, we obtain

$$\begin{aligned}
\partial_t u &= \partial_t \left(t^a f \left(rt^b \right) \right) \\
&= at^{a-1} f(rt^b) + t^a f'(rt^b) rbt^{b-1} \\
&= b\alpha t^{a-1} f(rt^b) + bt^{a-1} rt^b f'(rt^b) \\
&= bt^{a-1} \left(\alpha f(s) + sf'(s) \right) \\
&= \frac{bt^{a-1}}{s^{\alpha-1}} \left(s^\alpha f(s) \right)'
\end{aligned}$$

and

$$\begin{aligned}
\partial_r u^q &= qu^{q-1} \partial_r u \\
&= q \left(t^a f \left(rt^b \right) \right)^{q-1} \partial_r \left(t^a f \left(rt^b \right) \right) \\
&= qt^{a(q-1)} f(rt^b)^{q-1} t^{a+b} f'(rt^b) \\
&= qt^{aq+b} f(s)^{q-1} f'(s).
\end{aligned}$$

Hence, (3.5) is equivalent to

$$\begin{aligned}
\frac{bt^{a-1}}{s^{\alpha-1}} \left(s^\alpha f(s) \right)' &= -\frac{1}{r^{\alpha-1}} \partial_r \left(r^{\alpha-1} \left(-qt^{aq+b} f(s)^{q-1} f'(s) \right)^{p-1} \right) \\
&= -\frac{q^{p-1} t^{(aq+b)(p-1)}}{(st^{-b})^{\alpha-1}} \partial_r \left((st^{-b})^{\alpha-1} \left(-f(s)^{q-1} f'(s) \right)^{p-1} \right) \\
&= -\frac{q^{p-1} t^{(aq+b)(p-1)}}{s^{\alpha-1}} t^b \partial_s \left(s^{\alpha-1} \left(-f(s)^{q-1} f'(s) \right)^{p-1} \right). \quad (3.7)
\end{aligned}$$

We require that the powers of t in the both sides to match, that is,

$$(aq + b)(p - 1) + b = a - 1,$$

which together with $a = b\alpha$ yields

$$[(\alpha q + 1)(p - 1) + 1 - \alpha] b = -1,$$
$$[\alpha (q(p - 1) - 1) + p] b = -1.$$

Setting

$$\boxed{\delta = q(p - 1) - 1}$$

we obtain

$$(\alpha\delta + p) b = -1$$

whence

$$\boxed{b = -\frac{1}{\alpha\delta + p}}.$$

In particular, we see that

$$\boxed{b < 0 \Leftrightarrow \delta > -\frac{p}{\alpha} \Leftrightarrow q > \frac{1 - p/\alpha}{p - 1}}. \quad (3.8)$$

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In what follows, we always assume that (3.8) is satisfied.

With this choice of b and $a = \alpha b$, the powers of t and s in (3.7) cancel out, and we obtain an ODE for f:

$$b\left(s^\alpha f\left(s\right)\right)' = -q^{p-1}\left(s^{\alpha-1}\left(-f(s)^{q-1}f'(s)\right)^{p-1}\right)'.$$

Hence, we have

$$bs^\alpha f\left(s\right) = -q^{p-1}s^{\alpha-1}\left(-f(s)^{q-1}f'(s)\right)^{p-1} \tag{3.9}$$

(ignoring a constant of integration). Since $b < 0$, we obtain

$$\left|b\right|sf = q^{p-1}\left(-f^{q-1}f'\right)^{p-1},$$

$$\left|b\right|sf = q^{p-1}\left(-f'\right)^{p-1}f^{(q-1)(p-1)},$$

$$f^{(q-1)(p-1)-1}\left(-f'\right)^{p-1} = \frac{\left|b\right|s}{q^{p-1}},$$

$$f^{(q-1)-\frac{1}{p-1}}f' = -\frac{\left(\left|b\right|s\right)^{\frac{1}{p-1}}}{q}.$$

Set

$$\gamma := q - \frac{1}{p-1} = \frac{q\left(p-1\right)-1}{p-1} = \frac{\delta}{p-1}$$

and rewrite the above ODE in the form

$$f^{\gamma-1}f' = -\frac{\left|b\right|^{\frac{1}{p-1}}}{q}s^{\frac{1}{p-1}} \tag{3.10}$$

Assume first that $\delta \neq 0$, that is, $\gamma \neq 0$. Then (3.10) is equivalent to

$$\left(f^\gamma\right)' = \gamma f^{\gamma-1}f' = -\frac{\gamma\left|b\right|^{\frac{1}{p-1}}}{q}s^{\frac{1}{p-1}},$$

and integrating it, we obtain

$$f^\gamma = C - \kappa s^{\frac{p}{p-1}}$$

where

$$\kappa = \frac{p-1}{p}\frac{\gamma\left|b\right|^{\frac{1}{p-1}}}{q} = \frac{\delta}{p}\frac{\left|b\right|^{\frac{1}{p-1}}}{q}.$$

Hence,

$$f\left(s\right) = \left(C - \kappa s^{\frac{p}{p-1}}\right)^{1/\gamma},$$

with a positive constant C.

Case 1. Let $\delta > 0$ that is,

$$q\left(p-1\right) > 1,$$

(which implies also that $b < 0$).

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Then $\kappa > 0$ and we see that $f(s)$ is well defined for $s \in [0, s_0]$ where s_0 is determined by

$$C = \kappa s_0^{\frac{p}{P-1}}.$$

Let us extend $f(s)$ for all $s \in [0, \infty)$ by setting $f(s) = 0$ for $s > s_0$, that is,

$$f(s) = \left(C - \kappa s^{\frac{p}{P-1}} \right)_+^{1/\gamma}.$$

Then this function f is a *weak* solution of the ODE (3.10) in $[0, \infty]$ because f is continuous in $[0, \infty]$ and solves (3.10) in the both intervals $[0, s_0]$ and $[s_0, \infty)$. Consequently, we obtain in this case a (weak) solution of (3.5)

$$u(x, t) = t^{\alpha b} f\left(r t^b \right) = \frac{1}{t^{\alpha/\beta}} \left(C - \kappa \left(\frac{r}{t^{1/\beta}} \right)^{\frac{p}{p-1}} \right)_+^{1/\gamma}$$

where

$$\boxed{\beta = -\frac{1}{b} = \alpha\delta + p} > 0.$$

Clearly, this solution has a finite propagation speed. Note that in this case $\beta > p$.

Case 2. Let $\delta < 0$ that is,

$$q(p - 1) < 1.$$

Since $\kappa < 0$, the solution

$$f(s) = \left(C + |\kappa| s^{\frac{p}{P-1}} \right)^{-1/|\gamma|}$$

is defined and positive for all $s \geq 0$. Note that by (3.8)

$$\frac{p}{p-1} \frac{1}{|\gamma|} = \frac{p}{(p-1)\left(\frac{1}{p-1} - q\right)} = \frac{p}{1 - q(p-1)} = \frac{p}{-\delta} > \alpha,$$

that is,

$$f(s) \simeq s^{-(\alpha+\varepsilon)} \quad \text{as } s \to \infty$$

where $\varepsilon > 0$. Since also

$$f(s) \simeq \text{const} \quad \text{as } s \to 0$$

we obtain the finiteness of the integral (3.6).

We obtain in this case a solution

$$u(x, t) = \frac{1}{t^{\alpha/\beta}} \left(C + |\kappa| \left(\frac{r}{t^{1/\beta}} \right)^{\frac{p}{p-1}} \right)^{-1/|\gamma|}$$

that is defined for all x and $t > 0$ and belongs to $L^1(M)$ for any $t > 0$. Hence, this solution has infinite propagation speed. Note that in this case $\beta < p$.

Case 3. Let $\delta = 0$ that is,

$$q(p - 1) = 1,$$

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In this case $\gamma = 0$ and

$$b = -\frac{1}{\alpha\delta + p} = -\frac{1}{p}.$$

Then (3.10) becomes

$$\frac{f'}{f} = -\frac{s^{\frac{1}{p-1}}}{p^{\frac{1}{p-1}}q}$$

whence

$$\ln f = -\frac{1}{p^{\frac{1}{p-1}}q}\frac{s^{\frac{p}{p-1}}}{\frac{p}{p-1}} = -\kappa s^{\frac{p}{p-1}},$$

where

$$\kappa = \frac{1}{p^{\frac{1}{p-1}}q\frac{p}{p-1}} = \frac{(p-1)^2}{p^{\frac{p}{p-1}}} > 0.$$

It follows that

$$f(s) = \exp\left(-\kappa s^{\frac{p}{p-1}}\right)$$

whence

$$u(x,t) = \frac{1}{t^{\alpha/p}}\exp\left(-\kappa\left(\frac{r}{t^{1/p}}\right)^{\frac{p}{p-1}}\right).$$

For example, in the case $p = 2$ and, hence, $q = 1$ we obtain $\kappa = \frac{1}{4}$ and

$$u(x,t) = \frac{1}{t^{\alpha/2}}\exp\left(-\frac{1}{4}\frac{r^2}{t}\right).$$

Hence, the finite propagation speed for the above solutions occurs if and only if $\delta > 0$, that is, $q(p-1) > 1$.

4 Weak solutions

Let Ω be an open subset of M and I be an interval in $[0, \infty)$. By a *subsolution* of the equation

$$\partial_t v = \Delta_p(v^q) \tag{4.1}$$

in the cylinder $\Omega \times I$ we mean a non-negative function v of an appropriate class satisfying

$$\partial_t v \leq \Delta_p(v^q). \tag{4.2}$$

In fact, this equation is understood in a certain weak sense, and a function v is taken from the following class:

$$v \in C\left(I; L^2(\Omega)\right) \quad \text{and} \quad v^q \in L^p_{loc}\left(I; W^{1,p}(\Omega)\right).$$

That is, for any $t \in I$,

$$v(\cdot, t) \in L^2(\Omega), \quad v^q(\cdot, t) \in W^{1,p}(\Omega),$$

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the function $t \mapsto v(\cdot, t)$ is continuous in $L^2(\Omega)$, and, for any compact subinterval $J \subset I$,

$$\int_J \|v^q(\cdot, t)\|^p_{W^{1,p}(\Omega)} \, dt < \infty,$$

that is,

$$\int_J \int_\Omega \left(v^{qp} + |\nabla v^q|^p\right) d\mu dt < \infty. \tag{4.3}$$

Let us first show that the Leibenson operator $Lv = \Delta_p(v^q)$ can be rewritten in the form

$$Lv = c \operatorname{div} \left(v^m |\nabla v|^{p-2} \nabla v\right) \tag{4.4}$$

for some c, m, that is,

$$\operatorname{div} \left(|\nabla v^q|^{p-2} \nabla v^q\right) = c \operatorname{div} \left(v^m |\nabla v|^{p-2} \nabla v\right). \tag{4.5}$$

Indeed, we have

$$\nabla v^q = q v^{q-1} \nabla v$$

and

$$\operatorname{div} \left(|\nabla v^q|^{p-2} \nabla v^q\right) = q^{p-1} \operatorname{div} \left(v^{(q-1)(p-1)} |\nabla v|^{p-2} \nabla v\right).$$

Hence, (4.5) holds provided

$$\boxed{m = (q - 1)(p - 1)} \tag{4.6}$$

and $c = q^{p-1}$. The Leibenson equation becomes

$$\partial_t v = q^{p-1} \operatorname{div} \left(v^m |\nabla v|^{p-2} \nabla v\right),$$

and (4.2) becomes

$$\partial_t v \leq q^{p-1} \operatorname{div} \left(v^m |\nabla v|^{p-2} \nabla v\right). \tag{4.7}$$

5 Caccioppoli type inequality

We start here the proof of Theorem 1. The first step is obtaining a Caccioppoli type inequality.

For simplicity of notation, we omit in all integrations the notation of measure. All integration in M is done with respect to $d\mu$, and in $M \times \mathbb{R}$ – with respect to $d\mu dt$. We assume that

$$p > 1, \quad q > 0$$

and use the notation

$$\delta = q(p - 1) - 1.$$

Let Ω be an open subset of M and I be an interval in $\mathbb{R}$.

Lemma 2. *Let $v = v(x, t)$ be a bounded non-negative subsolution to (4.1) in a cylinder $\Omega \times I$. Let $\eta(x, t)$ be a Lipschitz non-negative bounded function in $\Omega \times (0, T)$ such that $\eta(\cdot, t)$ has compact support in Ω for all $t \in I$. Fix some real σ such that*

$$\sigma \geq \max(p, pq). \tag{5.1}$$

Set

$$\boxed{\lambda = \sigma - \delta \quad and \quad \alpha = \frac{\sigma}{p}}. \tag{5.2}$$

Then, for all $t_1, t_2 \in I$ such that $t_1 < t_2$, we have

$$\left[\int_\Omega v^\lambda \eta^p \right]_{t_1}^{t_2} + c_1 \int_{t_1}^{t_2} \int_\Omega |\nabla(v^\alpha \eta)|^p \leq \int_{t_1}^{t_2} \int_\Omega \left[p v^\lambda \eta^{p-1} \eta_t + c_2 v^\sigma |\nabla \eta|^p \right], \tag{5.3}$$

where c_1, c_2 are positive constants depending on p, q, λ (see below (5.13) and (5.14)).

In particular, if η does not depend on t then

$$\left[\int_\Omega v^\lambda \eta^p \right]_{t_1}^{t_2} + c_1 \int_{t_1}^{t_2} \int_\Omega |\nabla(v^\alpha \eta)|^p \leq c_2 \int_{t_1}^{t_2} \int_\Omega v^\sigma |\nabla \eta|^p. \tag{5.4}$$

Let us explain why all the integrals in (5.3) are well defined. Observe that always $\lambda \geq 2$. Indeed, if $q \geq 1$ then, using $\sigma \geq pq$, we obtain

$$\lambda = \sigma - \delta \geq pq - (q(p-1) - 1) = q + 1 \geq 2, \tag{5.5}$$

and if $q < 1$ then, using $\sigma \geq p$, we obtain

$$\lambda = \sigma - \delta \geq p - \delta = p - (q(p-1) - 1) = (p+1) - (p-1)q > (p+1) - (p-1) = 2.$$

Since $v(\cdot, t) \in L^2(\Omega)$ and v is bounded, it follows that, for any $t \in I$,

$$\int_\Omega v^\lambda(\cdot, t) \leq \|v\|_{L^\infty}^{\lambda-2} \int_\Omega v^2(\cdot, t) < \infty.$$

Consequently, the expression

$$\left[\int_\Omega v^\lambda \eta^p \right]_{t_1}^{t_2}$$

is well-defined. It also follows that

$$\int_{t_1}^{t_2} \int_\Omega v^\lambda \eta^{p-1} |\eta_t| \leq \text{const} \int_{t_1}^{t_2} \int_\Omega v^2(\cdot, t) < \infty.$$

Since $\nabla \eta(\cdot, t)$ and v are bounded and $\sigma \geq pq$, we obtain

$$\int_{t_1}^{t_2} \int_\Omega v^\sigma |\nabla \eta|^p \leq \text{const} \int_{t_1}^{t_2} \int_\Omega v^\sigma \leq \text{const} \|v\|_{L^\infty}^{\sigma-pq} \int_{t_1}^{t_2} \int_\Omega v^{pq} < \infty, \tag{5.6}$$

where we have used (4.3). The hypothesis $\sigma \geq pq$ implies that $\alpha \geq q$. Hence, the function $\Phi(s) = s^{\frac{\alpha}{q}}$ is Lipschitz on any bounded interval in $[0, \infty)$. Since

$$\nabla v^\alpha = \nabla \Phi(v^q) = \Phi'(v^q) \nabla v^q$$

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and v^q is bounded, it follows that

$$|\nabla v^\alpha| \le C \, |\nabla v^q| \, .$$

We obtain that

$$
\begin{aligned}
\int_{t_1}^{t_2} \int_\Omega |\nabla \, (v^\alpha \eta)|^p \;\; &\le \;\; C \int_{t_1}^{t_2} \int_\Omega |\nabla v^\alpha|^p \, \eta^p + v^{\alpha p} \, |\nabla \eta|^p \\
&\le \;\; C \int_{t_1}^{t_2} \int_\Omega |\nabla v^q|^p + v^\sigma \, |\nabla \eta|^p \\
&< \;\; \infty \, ,
\end{aligned}
$$

where we have used (4.3) and (5.6). Hence, all the integrals in (5.3) are well-defined. Let us record for a later usage that

$$v^\alpha \eta \in L^p_{loc} \left(I; W^{1,p}_0(\Omega) \right) \tag{5.7}$$

because

$$\int_{t_1}^{t_2} \int_\Omega (v^\alpha \eta)^p + |\nabla \, (v^\alpha \eta)|^p \le \mathrm{const} \int_{t_1}^{t_2} \int_\Omega v^\sigma + \int_{t_1}^{t_2} \int_\Omega |\nabla \, (v^\alpha \eta)|^p < \infty.$$

Proof of Lemma 2. Let us rewrite (4.7) in the form

$$q^{1-p} \partial_t v \le \mathrm{div} \left(v^m \, |\nabla v|^{p-2} \nabla v \right) \tag{5.8}$$

where

$$m = (q-1)(p-1) \, . \tag{5.9}$$

Multiplying (5.8) by $v^{\lambda-1} \eta^p$ and integrating it over the cylinder $Q = \Omega \times [t_1, t_2]$, we obtain

$$
\begin{aligned}
q^{1-p} \int_Q v_t v^{\lambda-1} \eta^p &\le \int_Q \mathrm{div} \left(v^m \, |\nabla v|^{p-2} \nabla v \right) v^{\lambda-1} \eta^p \\
&= - \int_Q v^m \, |\nabla v|^{p-2} \nabla v \, \nabla \left(v^{\lambda-1} \eta^p \right) \\
&= - \int_Q v^m \, |\nabla v|^{p-2} \nabla v \, \left[(\lambda-1) \, v^{\lambda-2} \eta^p \nabla v + p v^{\lambda-1} \eta^{p-1} \nabla \eta \right] \\
&= - (\lambda - 1) \int_Q v^{\lambda+m-2} \, |\nabla v|^p \, \eta^p - p \int_Q v^{\lambda+m-1} \, |\nabla v|^{p-2} \, \eta^{p-1} \, (\nabla v, \nabla \eta) \\
&\le - (\lambda - 1) \int_Q v^{\lambda+m-2} \, |\nabla v|^p \, \eta^p + p \int_Q v^{\lambda+m-1} \, |\nabla v|^{p-1} \, \eta^{p-1} \, |\nabla \eta| \, .
\end{aligned}
$$
$$\tag{5.10}$$

Observe that, for any fixed t,

$$v^{\lambda-1} \eta^p(\cdot, t) \in W^{1,p}_0(\Omega),$$

which allows to use the integration-by-part formula without the boundary term. Indeed, we have $v^q \in W^{1,p}(\Omega)$ by the definition of a weak solution, which implies

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$v^{\lambda-1} \in W^{1,p}(\Omega)$ because v is bounded and $\lambda - 1 \geq q$ by (5.5), whence the inclusion $v^{\lambda-1}\eta^p \in W_0^{1,p}(\Omega)$ follows because η is compactly supported in Ω.

Since

$$\lambda + m - 2 = \lambda + (q-1)(p-1) - 2 = \lambda + (p-1)q - 1 - p = \lambda + \delta - p = \sigma - p,$$

we rewrite (5.10) as follows:

$$q^{1-p} \int_Q v_t v^{\lambda-1} \eta^p \leq -(\lambda-1) \int_Q v^{\sigma-p} |\nabla v|^p \eta^p + p \int_Q v^{\sigma-p+1} |\nabla v|^{p-1} \eta^{p-1} |\nabla \eta|. \quad (5.11)$$

Since $\sigma \geq p$, the function v enters all the integrals in (5.11) in non-negative powers; hence, the integrals are finite.

Next, let us use the following inequality for all $X, Y \geq 0$ and $\varepsilon > 0$:

$$XY \leq \varepsilon^{p'} X^{p'} + \frac{1}{\varepsilon^p} Y^p$$

where $p' = \frac{p}{p-1}$ is the Hölder conjugate of p (here we use that $p > 1$). Applying this inequality with

$$X = v^\xi |\nabla v|^{p-1} \eta^{p-1} \quad \text{and} \quad Y = v^{(\sigma-p+1-\xi)} |\nabla \eta|$$

(where ε and ξ are yet to be determined) we obtain

$$
\begin{aligned}
v^{\sigma-p+1} |\nabla v|^{p-1} \eta^{p-1} |\nabla \eta| &= XY \leq \varepsilon^{p'} \left(v^\xi |\nabla v|^{p-1} \eta^{p-1} \right)^{p'} + \frac{1}{\varepsilon^p} \left(v^{\sigma-p+1-\xi} |\nabla \eta| \right)^p \\
&= \varepsilon^{p'} v^{\xi p'} |\nabla v|^p \eta^p + \frac{1}{\varepsilon^p} v^{(\sigma-p+1-\xi)p} |\nabla \eta|^p.
\end{aligned}
$$

We would like to have

$$\xi p' = \sigma - p$$

whence

$$\xi := \frac{\sigma - p}{p'}.$$

With this ξ we have

$$(\sigma - p + 1 - \xi)p = \left((\sigma - p) + 1 - \frac{(\sigma - p)}{p'} \right) p = \left(\frac{\sigma - p}{p} + 1 \right) p = \sigma$$

and

$$v^{\sigma-p+1} |\nabla v|^{p-1} \eta^{p-1} |\nabla \eta| \leq \varepsilon^{p'} v^{\sigma-p} |\nabla v|^p \eta^p + \frac{1}{\varepsilon^p} v^\sigma |\nabla \eta|^p.$$

It follows that

$$
\begin{aligned}
q^{1-p} \int_Q v_t v^{\lambda-1} \eta^p &\leq -(\lambda-1) \int_Q v^{\sigma-p} |\nabla v|^p \eta^p + p \int_Q \left[\varepsilon^{p'} v^{\sigma-p} |\nabla v|^p \eta^p + \frac{1}{\varepsilon^p} v^\sigma |\nabla \eta|^p \right] \\
&= -\left(\lambda - 1 - p\varepsilon^{p'} \right) \int_Q v^{\sigma-p} |\nabla v|^p \eta^p + \frac{p}{\varepsilon^p} \int_Q v^\sigma |\nabla \eta|^p. \quad (5.12)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
|\nabla \left(v^\alpha \eta\right)|^p &= \left|\alpha v^{\alpha-1}\eta\nabla v + v^\alpha\nabla\eta\right|^p \\
&\leq 2^{p-1}\alpha^p v^{p(\alpha-1)} |\nabla v|^p \, \eta^p + 2^{p-1}v^{\alpha p}|\nabla\eta|^p \\
&= 2^{p-1}\alpha^p v^{\sigma-p}|\nabla v|^p \, \eta^p + 2^{p-1}v^\sigma|\nabla\eta|^p,
\end{aligned}
$$

where we have used that, by (5.2),

$$
p\left(\alpha-1\right) = \sigma - p.
$$

It follows that

$$
v^{\sigma-p}|\nabla v|^p \, \eta^p \geq 2^{1-p}\alpha^{-p}|\nabla \left(v^\alpha\eta\right)|^p - \alpha^{-p}v^\sigma|\nabla\eta|^p.
$$

Substituting into (5.12) yields

$$
\begin{aligned}
q^{1-p}\int_Q v_t v^{\lambda-1}\eta^p \leq & -\left(\lambda-1-p\varepsilon^{p'}\right)2^{1-p}\alpha^{-p}\int_Q |\nabla \left(v^\alpha\eta\right)|^p \\
& + \left(\left(\lambda-1-p\varepsilon^{p'}\right)\alpha^{-p} + \frac{p}{\varepsilon^p}\right)\int_Q v^\sigma|\nabla\eta|^p.
\end{aligned}
$$

Hence,

$$
\lambda\int_Q v_t v^{\lambda-1}\eta^p \leq -c_1\int_Q |\nabla \left(v^\alpha\eta\right)|^p + c_2\int_Q v^\sigma|\nabla\eta|^p
$$

where

$$
\boxed{c_1 = \lambda q^{p-1}\left(\lambda-1-p\varepsilon^{p'}\right)2^{1-p}\alpha^{-p}}
$$

and

$$
\boxed{c_2 = \lambda q^{p-1}\left(\left(\lambda-1-p\varepsilon^{p'}\right)\alpha^{-p} + \frac{p}{\varepsilon^p}\right)}.
$$

We choose ε so small that $c_1 > 0$, that is,

$$
p\varepsilon^{p'} < \lambda - 1.
$$

Since

$$
\lambda\int_Q v_t v^{\lambda-1}\eta^p = \int_Q \partial_t v^\lambda\eta^p = \left[\int_\Omega v^\lambda\eta^p\right]_{t_1}^{t_2} - p\int_Q v^\lambda\eta^{p-1}\eta_t
$$

we obtain

$$
\begin{aligned}
\left[\int_\Omega v^\lambda\eta^p\right]_{t_1}^{t_2} &= \lambda\int_Q v_t v^{\lambda-1}\eta^p + p\int_Q v^\lambda\eta^{p-1}\eta_t \\
&\leq -c_1\int_Q |\nabla \left(v^\alpha\eta\right)|^p + c_2\int_Q v^\sigma|\nabla\eta|^p + p\int_Q v^\lambda\eta^{p-1}\eta_t
\end{aligned}
$$

and, hence,

$$
\left[\int_\Omega v^\lambda\eta^p\right]_{t_1}^{t_2} + c_1\int_Q |\nabla \left(v^\alpha\eta\right)|^p \leq \int_Q \left[pv^\lambda\eta^{p-1}\eta_t + c_2 v^\sigma|\nabla\eta|^p\right].
$$

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Finally, let us specify c_1 and c_2. Let us choose ε so that

$$p\varepsilon^{p'} = \frac{1}{2}(\lambda - 1)$$

that is

$$\boxed{c_1 = \lambda(\lambda - 1)\, 2^{-p} q^{p-1} \alpha^{-p}}. \tag{5.13}$$

It follows that

$$
\begin{aligned}
c_2 &= \lambda q^{p-1} \left(\frac{1}{2}(\lambda - 1)\alpha^{-p} + \frac{p}{\varepsilon^p} \right) \\
&= \lambda q^{p-1} \left(\frac{1}{2}(\lambda - 1)\alpha^{-p} + \frac{p}{\left(\frac{1}{2}(\lambda - 1)/p \right)^{p/p'}} \right) \\
&= \lambda q^{p-1} \left(\frac{1}{2}(\lambda - 1)\alpha^{-p} + \frac{2^{p/p'} p^{1+p/p'}}{(\lambda - 1)^{p/p'}} \right).
\end{aligned}
$$

Since

$$\frac{p}{p'} + 1 = \frac{p}{p'} + \frac{p}{p} = p$$

we have

$$\boxed{c_2 = \tfrac{1}{2}\lambda(\lambda - 1)\, q^{p-1}\alpha^{-p} + \frac{\lambda 2^{p-1} p^p q^{p-1}}{(\lambda - 1)^{p-1}}}. \tag{5.14}$$

∎

Remark. For the future we need the ratio $\frac{c_2}{c_1}$. It follows from (5.13) and (5.14) that

$$
\begin{aligned}
\frac{c_2}{c_1} &= 2^{p-1} + \lambda \frac{2^{p-1} p^p}{(\lambda - 1)^{p-1} \lambda(\lambda - 1)\, 2^{-p}\alpha^{-p}} \\
&= 2^{p-1} + \frac{2^{2p-1}\sigma^p}{(\lambda - 1)^p},
\end{aligned}
$$

where we have used that $\alpha p = \sigma$. Since $\sigma = \lambda + \delta$, we obtain

$$\boxed{\frac{c_2}{c_1} = 2^{p-1} + \frac{2^{2p-1}(\lambda + \delta)^p}{(\lambda - 1)^p}}.$$

It follows that, for all $\lambda \geq 2$,

$$\boxed{\frac{c_2}{c_1} \leq C_{p,q}}, \tag{5.15}$$

where $C_{p,q}$ depend only on p and q but does not depend on λ.

Remark. Let obtain an upper bound of c_2. Using

$$\alpha = \frac{\sigma}{p} = \frac{\lambda + \delta}{p}$$

we obtain

$$c_2 = \frac{1}{2}\frac{\lambda(\lambda - 1)}{(\lambda + \delta)^p} q^{p-1} p^p + \frac{\lambda 2^{p-1} p^p q^{p-1}}{(\lambda - 1)^{p-1}}.$$

As $\lambda \geq 2$ and $\lambda + \delta \geq p > 1$, it follows that

$$\boxed{c_2 \leq C_{p,q} \lambda^{2-p}}. \tag{5.16}$$

Of course, if $p \geq 2$ then c_2 is uniformly bounded by a constant $C_{p,q}$ independently of λ, but if $p < 2$ then c_2 may grow with λ as λ^{2-p}.

Lemma 3. *Let M be geodesically complete. Let $v = v(x,t)$ be a bounded non-negative subsolution to* (4.7) *in $M \times I$. Then, for all large enough λ, including $\lambda = \infty$, the function*

$$t \mapsto \|v(\cdot, t)\|_{L^\lambda(M)}$$

is monotone decreasing. Consequently, if $I = [a,b]$ then

$$\|v\|_{L^\infty(M \times I)} \leq \|v(\cdot, a)\|_{L^\infty(M)}.$$

Proof. If M is geodesically complete, then $W_0^{1,p}(M) = W^{1,p}(M)$. Hence, $v^{\lambda-1}(\cdot, t) \in W_0^{1,p}(M)$ for any $t \in I$, and we can use the argument in the proof of Lemma 2 with $\eta \equiv 1$ (without assumption that $\eta(\cdot, t)$ is compactly supported). Assuming that λ is large enough so that $\sigma := \lambda + \delta$ satisfies (5.1), we obtain from (5.3) that, for all $t_1, t_2 \in I$, $t_1 < t_2$,

$$\left[\int_M v^\lambda \right]_{t_1}^{t_2} \leq 0,$$

which proved the claim for a finite λ. The case of an infinite λ is obtained then by letting $\lambda \to \infty$. ∎

6 Sobolev and Moser inequalities

Let B be a precompact ball in a manifold M of dimension n. The Sobolev inequality in B of order p says the following: for any non-negative function $w \in W_0^{1,p}(B)$

$$\left(\int_B w^{p\kappa} \right)^{1/\kappa} \leq S_B \int_B |\nabla w|^p, \tag{6.1}$$

where $\kappa > 1$ is some constant and S_B is called the *Sobolev constant* in B. The value of κ is independent of B and can be chosen as follows:

$$\kappa = \frac{n}{n-p} \quad \text{if } n > p,$$

and κ is an arbitrary real number > 1 if $n \leq p$.

We always assume that S_B is chosen to be minimal possible. In this case the function

$$B \mapsto S_B$$

is clearly monotone increasing with respect to inclusion of balls.

Fix a precompact ball $B \subset M$ and set $Q = B \times I$, where $I \subset \mathbb{R}$ is an interval Assume that the Sobolev inequality (10.5) holds in B with exponent $\kappa > 1$. Let κ' be its Hölder conjugate. Set

$$\nu = \frac{1}{\kappa'} = \frac{\kappa - 1}{\kappa}.$$

Lemma 4. *Let $w \in L^p(I; W_0^{1,p}(B))$ be a non-negative function. Then*

$$\int_Q w^{p(1+\nu)} \leq S_B \left(\int_Q |\nabla w|^p \right) \sup_{t \in I} \left(\int_B w^p \right)^{\nu} \tag{6.2}$$

Proof. By the Hölder inequality, we have, for any fixed $t \in I$,

$$\int_B w^{p(1+\nu)} = \int_B w^p w^{p\nu} \leq \left(\int_B w^{p\kappa} \right)^{1/\kappa} \left(\int_B w^{p\nu\kappa'} \right)^{1/\kappa'}$$

$$= \left(\int_B w^{p\kappa} \right)^{1/\kappa} \left(\int_B w^p \right)^{\nu}$$

$$\leq \left(\int_B w^{p\kappa} \right)^{1/\kappa} \sup_{t \in I} \left(\int_B w^p \right)^{\nu},$$

where we have used that $\nu\kappa' = 1$.

By the Sobolev inequality (6.1) we have, for any $t \in I$,

$$\left(\int_B w^{p\kappa} \right)^{1/\kappa} \leq S_B \int_B |\nabla w|^p.$$

It follows that

$$\int_B w^{p(1+\nu)} \leq S_B \left(\int_B |\nabla w|^p \right) \sup_{t \in I} \left(\int_B w^p \right)^{\nu}.$$

Integrating this inequality in $t \in I$ gives (6.2). $\blacksquare$

7 Comparison in two cylinders

Here we assume that

$$\delta := q(p-1) - 1 \geq 0.$$

Lemma 5. *Consider two balls $B = B(x, r)$ and $B' = B(x, r')$ with $0 < r' < r$, and two cylinders*

$$Q = B \times [0, T], \quad Q' = B' \times [0, T].$$

Assume that B is precompact. Let σ be any real such that

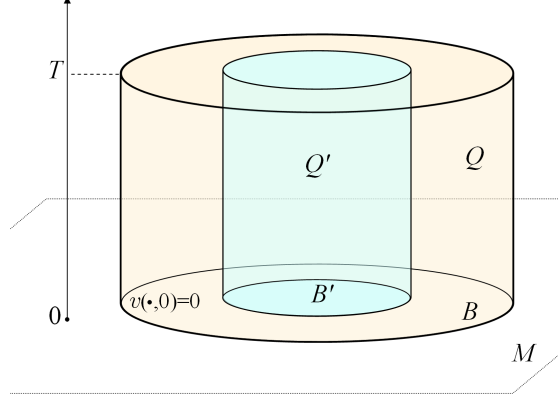$$\sigma \geq \max(p, pq). \tag{7.1}$$

Let v be non-negative bounded subsolution of (4.1) in Q such that

$$v(\cdot, 0) = 0 \text{ in } B.$$

Then

$$\int_{Q'} v^{\sigma(1+\nu)} \leq \frac{CS_B \sigma^{(2-p)\nu}}{(r-r')^{p(1+\nu)}} \left(\int_Q v^\sigma \right) \left(\int_Q v^{\sigma+\delta} \right)^\nu, \tag{7.2}$$

where C depends only on p, q and ν, while it is independent of σ.

Proof. As in Lemma 2, set $\lambda = \sigma - \delta$ and $\alpha = \frac{\sigma}{p}$ and recall that $\alpha \geq 1$ and $\lambda \geq 2$. Let $\eta = \eta(x)$ be a bump function of B' in B. By (5.7), we have

$$w := v^\alpha \eta \in L^p \left([0,T]; W_0^{1,p}(B) \right)$$

Applying (6.2) with this function w and using

$$w^p = v^\sigma \eta^p,$$

we obtain that

$$\int_Q v^{\sigma(1+\nu)} \eta^{p(1+\nu)} \leq S_B \left(\int_Q |\nabla (v^\alpha \eta)|^p \right) \sup_{t \in [0,T]} \left(\int_B v^\sigma \eta^p \right)^\nu.$$

By (5.4) we have

$$\int_Q |\nabla (v^\alpha \eta)|^p \leq \frac{c_2}{c_1} \int_Q v^\sigma |\nabla \eta|^p.$$

and

$$\sup_{t \in [0,T]} \left(\int_B v^\lambda \eta^p \right) \leq c_2 \int_Q v^\sigma |\nabla \eta|^p.$$

Let us use the latter inequality also for other values of the parameters as follows:

$$\sup_{t \in [0,T]} \left(\int_B v^{\lambda'} \eta^p \right) \leq c_2' \int_Q v^{\sigma'} |\nabla \eta|^p,$$

where

$$\sigma' = \sigma + \delta \quad \text{and} \quad \lambda' = \sigma' - \delta = \sigma.$$

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Then we have

$$\sup_{t\in[0,T]} \left(\int_B v^\sigma \eta^p \right) \le c_2' \int_Q v^{\sigma'} |\nabla\eta|^p.$$

It follows that

$$\int_Q v^{\sigma(1+\nu)} \eta^{p(1+\nu)} \le S_B \frac{c_2}{c_1} \int_Q v^\sigma |\nabla\eta|^p \left(c_2' \int_Q v^{\sigma'} |\nabla\eta|^p \right)^\nu.$$

Using that $\eta = 1$ in B' and $|\nabla\eta| \le \frac{1}{r-r'}$ we obtain

$$\int_{Q'} v^{\sigma(1+\nu)} \le S_B \frac{c_2}{c_1} \frac{(c_2')^\nu}{(r_1 - r_2)^{p(1+\nu)}} \left(\int_Q v^\sigma \right) \left(\int_Q v^{\sigma'} \right)^\nu.$$

By (5.15) we have

$$\frac{c_2}{c_1} \le C_{p,q},$$

and (5.16)

$$c_2' \le C_{p,q} (\lambda')^{2-p} = C_{p,q} \sigma^{2-p}.$$

Hence, (7.2) follows. ∎

Corollary 6. *Under the hypotheses of* Lemma 5, *we have*

$$\int_{Q'} v^{\sigma(1+\nu)} \le \frac{C S_B \sigma^{(2-p)\nu} \|v\|_{L^\infty(Q)}^{\delta\nu}}{(r - r')^{p(1+\nu)}} \left(\int_Q v^\sigma \right)^{1+\nu}, \tag{7.3}$$

where $C = C(p, q, \nu)$.

8 Auxiliary lemma about sequences

Lemma 7. *Let a sequence* $\{J_k\}_{k=0}^\infty$ *of non-negative reals satisfies*

$$J_{k+1} \le \frac{A^k}{D} J_k^{1+\omega} \quad \text{for all } k \ge 0. \tag{8.1}$$

where $A, D, \omega > 0$. *Then, for all* $k \ge 0$,

$$J_k \le \left(\left(A^{1/\omega} D^{-1} \right)^{1/\omega} J_0 \right)^{(1+\omega)^k} \left(A^{-k-1/\omega} D \right)^{1/\omega}. \tag{8.2}$$

In particular, if

$$D \ge A^{1/\omega} J_0^\omega, \tag{8.3}$$

then, for all $k \ge 0$,

$$J_k \le A^{-k/\omega} J_0. \tag{8.4}$$

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Proof. Consider the sequence

$$X_k = \left(\left(A^{1/\omega}D^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^k} \left(A^{-k-1/\omega}D\right)^{1/\omega}.$$

Then we have

$$X_0 = \left(A^{1/\omega}D^{-1}\right)^{1/\omega} J_0 \left(A^{-1/\omega}D\right)^{1/\omega} = J_0$$

and

$$\begin{aligned}
\frac{A^k}{D}X_k^{1+\omega} &= \frac{A^k}{D}\left(\left(A^{1/\omega}D^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^{k+1}} \left(A^{-k-1/\omega}D\right)^{\frac{1+\omega}{\omega}} \\
&= \left(\left(A^{1/\omega}D^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^{k+1}} A^k D^{-1} \left(A^{-k-1/\omega}D\right) \left(A^{-k-1/\omega}D\right)^{\frac{1}{\omega}} \\
&= \left(\left(A^{1/\omega}D^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^{k+1}} A^{-1/\omega} \left(A^{-k-1/\omega}D\right)^{1/\omega} \\
&= \left(\left(A^{1/\omega}D^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^{k+1}} \left(A^{-(k+1)-1/\omega}D\right)^{1/\omega} = X_{k+1}.
\end{aligned}$$

Hence, by comparison we obtain $J_k \leq X_k$, which was to be proved.

For the second statement, if (8.3) holds then we can assume without loss of generality that

$$D = A^{1/\omega}J_0^\omega.$$

Substituting this value of D into (8.2) we obtain

$$J_k \leq \left(A^{-k}J_0^\omega\right)^{1/\omega}$$

which is equivalent to (8.4). ∎

9   Mean value inequality

We assume here that

$$\delta = q(p-1) - 1 \geq 0.$$

Lemma 8. *Let $B = B(x_0, R)$ be a precompact ball. Let u be a non-negative bounded subsolution of (4.1) in*

$$Q = B \times [0, t]$$

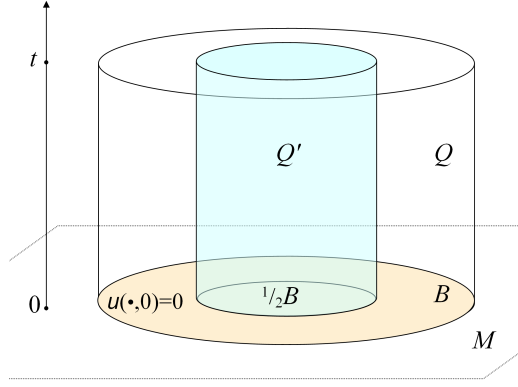*such that*

$$u(\cdot, 0) = 0 \ \text{in} \ B.$$

Let σ be as in (7.1). Then, for the cylinder

$$Q' = \frac{1}{2}B \times [0, t],$$

we have

$$\|u\|_{L^\infty(Q')} \leq \left(\frac{CS_B}{R^{p(1+\nu)}}\right)^{\frac{1}{\sigma\nu}} \|u\|_{L^\infty(Q)}^{\frac{\delta}{\sigma}} \|u\|_{L^\sigma(Q)}, \tag{9.1}$$

where $C = C(p, q, \nu, \sigma)$.

Remark. Since

$$\|u\|_{L^\sigma(Q)} = \left(\int_0^t \int_B u^\sigma\right)^{1/\sigma} \le (t\mu(B))^{\frac{1}{\sigma}} \|u\|_{L^\infty(Q)},$$

we obtain from (9.1) that

$$\|u\|_{L^\infty(Q')} \le \left(\left(\frac{CS_B}{R^{p(1+\nu)}}\right)^{\frac{1}{\nu}} t\mu(B)\right)^{\frac{1}{\sigma}} \|u\|_{L^\infty(Q)}^{1+\frac{\delta}{\sigma}}. \tag{9.2}$$

Remark. Unlike Lemma 5 where we have explicitly traced the dependence of the constants on σ, in (9.1) and (9.2) the dependence of C on σ is unimportant because these inequalities will be applied only with a fixed σ.

Proof. Consider a sequence of radii

$$r_k = \left(\frac{1}{2} + 2^{-k-1}\right) R$$

so that $r_0 = R$ and $r_k \searrow \frac{1}{2}R$ as $k \to \infty$. Set

$$B_k = B(x_0, r_k), \qquad Q_k = B_k \times [0, t]$$

so that

$$B_0 = B, \qquad Q_0 = Q \quad \text{and} \quad Q_\infty := \lim_{k\to\infty} Q_k = Q'.$$

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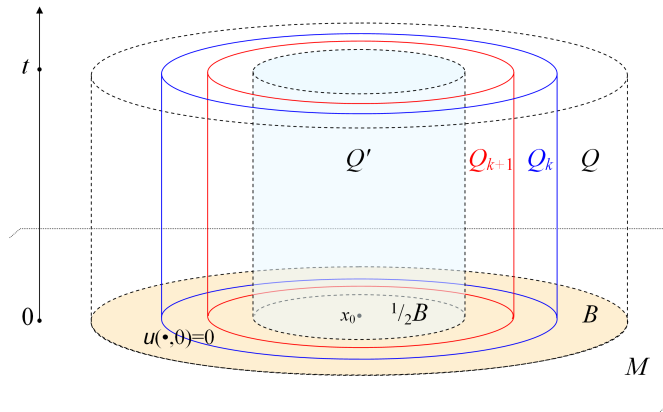
Set also
$$\sigma_k = \sigma \left(1 + \nu\right)^k$$
and
$$J_k = \int_{Q_k} u^{\sigma_k}.$$

Applying (7.3) to the cylinders Q_k and Q_{k+1}, we obtain

$$
\begin{aligned}
J_{k+1} &= \int_{Q_{k+1}} u^{\sigma_k(1+\nu)} \leq \frac{CS_{B_k}\sigma_k^{(2-p)\nu}\|u\|_{L^\infty(Q_k)}^{\delta\nu}}{(r_k - r_{k+1})^{p(1+\nu)}} \left(\int_{Q_k} u^{\sigma_k}\right)^{1+\nu}\\
&= \frac{CS_{B_k}\sigma_k^{(2-p)\nu}\|u\|_{L^\infty(Q_k)}^{\delta\nu}}{(r_k - r_{k+1})^{p(1+\nu)}} J_k^{1+\nu}\\
&\leq \frac{C2^{kp(1+\nu)}\left(1+\nu\right)^{k(2-p)\nu}\sigma^{(2-p)\nu}S_B\|u\|_{L^\infty(Q)}^{\delta\nu}}{R^{p(1+\nu)}} J_k^{1+\nu}\\
&\leq A^k D^{-1} J_k^{1+\nu},
\end{aligned}
$$

where
$$A = 2^{p(1+\nu)}\left(1+\nu\right)^{(2-p)_+\nu} \geq 1$$
and
$$D^{-1} = \frac{CS_B\|u\|_{L^\infty(Q)}^{\delta\nu}}{R^{p(1+\nu)}},$$

where we have absorbed $\sigma^{(2-p)\nu}$ into C. By Lemma 7 we conclude that

$$J_k \leq \left(\left(A^{1/\nu}D^{-1}\right)^{1/\nu} J_0\right)^{(1+\nu)^k} D^{1/\nu} = A^{\frac{(1+\nu)^k}{\nu^2}} D^{-\frac{(1+\nu)^k-1}{\nu}} J_0^{(1+\nu)^k}.$$

It follows that

$$\left(\int_{Q_k} u^{\sigma_k}\right)^{1/\sigma_k} = J_k^{\frac{1}{\sigma(1+\nu)^k}} \leq A^{\frac{1}{\sigma\nu^2}} D^{-\frac{1-(1+\nu)^{-k}}{\sigma\nu}} \left(\int_Q u^\sigma\right)^{1/\sigma}.$$

Leting $k \to \infty$, we obtain

$$
\begin{aligned}
\|u\|_{L^\infty(Q')} &\leq A^{\frac{1}{\sigma\nu^2}} D^{-\frac{1}{\sigma\nu}}\|u\|_{L^\sigma(Q)} = \left(A^{\frac{1}{\nu}}D^{-1}\right)^{\frac{1}{\sigma\nu}}\|u\|_{L^\sigma(Q)}\\
&\leq \left(A^{\frac{1}{\nu}}\frac{CS_B\|u\|_{L^\infty(Q)}^{\delta\nu}}{R^{p(1+\nu)}}\right)^{\frac{1}{\sigma\nu}}\|u\|_{L^\sigma(Q)}\\
&= \left(\frac{CS_B}{R^{p(1+\nu)}}\right)^{\frac{1}{\sigma\nu}}\|u\|_{L^\infty(Q)}^{\frac{\delta}{\sigma}}\|u\|_{L^\sigma(Q)},
\end{aligned}
$$

where $A^{\frac{1}{\nu}}$ was absorbed into C. ∎

10 Normalized Sobolev constant

Let B be a precompact ball in M and $w \in W_0^{1,p}(B)$. Dividing the Sobolev inequality (6.1) by $\mu(B)^{1/\kappa}$, we obtain

$$\left(\fint_B w^{p\kappa} \right)^{1/\kappa} \leq \mu(B)^{\nu} S_B \fint_B |\nabla w|^p$$

where

$$\nu = \frac{\kappa - 1}{\kappa} = \frac{1}{\kappa'},$$

and

$$\left(\fint_B w^{p\kappa} \right)^{1/(p\kappa)} \leq (\mu(B)^{\nu} S_B)^{1/p} \left(\fint_B |\nabla w|^p \right)^{1/p}, \tag{10.3}$$

Denoting by $r(B)$ the radius of B, let us define a new quantity

$$\boxed{\iota(B) = \frac{1}{\mu(B)} \left(\frac{r(B)^p}{S_B} \right)^{1/\nu}}$$

so that

$$S_B = \frac{r(B)^p}{(\iota(B)\mu(B))^{\nu}} \tag{10.4}$$

and

$$(\mu(B)^{\nu} S_B)^{1/p} = \frac{r(B)}{\iota(B)^{\frac{\nu}{p}}}.$$

Hence, (10.3) can be rewritten in the form

$$\boxed{\left(\fint_B |\nabla w|^p \right)^{1/p} \geq \frac{\iota(B)^{\frac{\nu}{p}}}{r(B)} \left(\fint_B w^{p\kappa} \right)^{1/p\kappa}}. \tag{10.5}$$

It is clear from (10.5) that the value of κ can be always reduced (by modifying the value of $\iota(B)$). It is only important that $\kappa > 1$. In fact, the exact value of κ does not affect the results, although various constants depend on κ.

The constant $\iota(B)$ is called the *normalized Sobolev* constant in B. It is known that if M is complete and $Ricci_M \geq 0$ then, for all balls B, the normalized Sobolev constant $\iota(B)$ is bounded below by a positive constant.

11 Propagation speed inside a ball

We assume here that M is geodesically complete and

$$\delta = q(p - 1) - 1 > 0.$$

Theorem 9. *Let u be a bounded non-negative subsolution of* (4.1) *in $M \times [0, T]$ with the initial condition $u(\cdot, 0) = u_0$. Let $B_0 = B(z_0, R)$ be a ball in M such that*
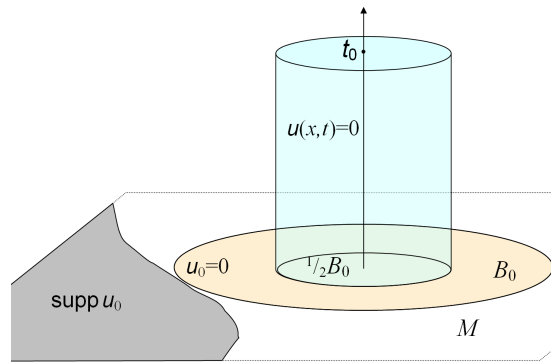
$$u_0 = 0 \ \text{in} \ B_0.$$

Set

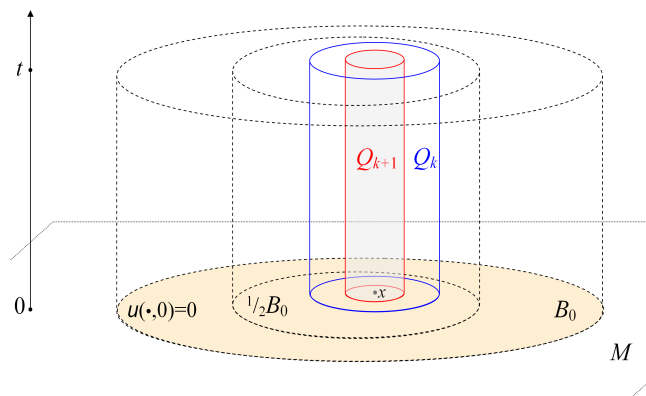$$t_0 = \eta \, \iota(B_0) R^p \, \|u_0\|_{L^\infty(M)}^{-\delta} \wedge T, \tag{11.1}$$

where η is a sufficiently small positive constant depending only on p, q, ν. Then

$$u = 0 \ \ \text{in} \ \ \frac{1}{2} B_0 \times [0, t_0].$$

Proof. Set $r = \frac{1}{2}R$ and fix for a while a point $x \in \frac{1}{2}B_0$ so that $B := B(x, r) \subset B_0$. Fix also some $t \in (0, T]$ and set

$$Q_k = 2^{-k} B \times [0, t] \quad \text{and} \quad J_k = \|u\|_{L^\infty(Q_k)}.$$

Our purpose is to obtain an upper bound for $J_k(x) = \|u\|_{L^\infty(Q_k)}$ that ensures that $J_k(x) \to 0$ as $k \to \infty$ uniformly in $x \in \frac{1}{2}B_0$.

Let us fix σ satisfying (7.1), for example, $\sigma = \max(p, pq)$. Applying inequality (9.2) of Lemma 8 in the cylinders Q_k, Q_{k+1}, we obtain

$$J_{k+1} = \|u\|_{L^\infty(Q_{k+1})} \leq \left(\left(\frac{CS_{2^{-k}B}}{(2^{-k}r)^{p(1+\nu)}} \right)^{\frac{1}{\nu}} t\mu(2^{-k}B) \right)^{\frac{1}{\sigma}} \|u\|_{L^\infty(Q_k)}^{1+\frac{\delta}{\sigma}}$$

$$\leq 2^{(k+1)\frac{p(1+\nu)}{\sigma\nu}} \left(\left(\frac{CS_{B_0}}{R^{p(1+\nu)}} \right)^{\frac{1}{\nu}} t\mu(B_0) \right)^{\frac{1}{\sigma}} J_k^{1+\frac{\delta}{\sigma}},$$

where we have used that $S_{2^{-k}B} \leq S_{B_0}$ and $\mu(2^{-k}B) \leq \mu(B_0)$. By (10.4) we have

$$(S_{B_0})^{1/\nu} = \left(\frac{R^p}{(\iota(B_0)\mu(B_0))^\nu} \right)^{1/\nu} = \frac{R^{p/\nu}}{\iota(B_0)\mu(B_0)},$$

whence

$$\left(\frac{S_{B_0}}{R^{p(1+\nu)}} \right)^{\frac{1}{\nu}} \mu(B_0) = \frac{R^{p/\nu}}{R^{p\frac{(1+\nu)}{\nu}}\iota(B_0)\mu(B_0)}\mu(B_0) = \frac{1}{\iota(B_0)R^p}.$$

It follows that

$$J_{k+1} \leq 2^{(k+1)\frac{p(1+\nu)}{\sigma\nu}} \left(\frac{Ct}{\iota(B_0)R^p} \right)^{\frac{1}{\sigma}} J_k^{1+\frac{\delta}{\sigma}}$$

$$= A^k D^{-1} J_k^{1+\omega},$$

where

$$\omega = \frac{\delta}{\sigma}, \qquad A = 2^{\frac{p(1+\nu)}{\sigma\nu}}$$

and

$$D^{-1} = A \left(\frac{Ct}{\iota(B_0)R^p} \right)^{\frac{1}{\sigma}}.$$

By Lemma 7, if

$$D^{-1} \leq A^{-1/\omega} J_0^{-\omega} \tag{11.2}$$

then, for all $k \geq 0$,

$$J_k \leq A^{-k/\omega} J_0. \tag{11.3}$$

The condition (11.2) is equivalent to

$$A \left(\frac{Ct}{\iota(B_0)R^p} \right)^{\frac{1}{\sigma}} \leq A^{-1/\omega} J_0^{-\omega}$$

that is, to

$$t \leq C^{-1}\iota(B_0)R^p J_0^{-\delta}, \tag{11.4}$$
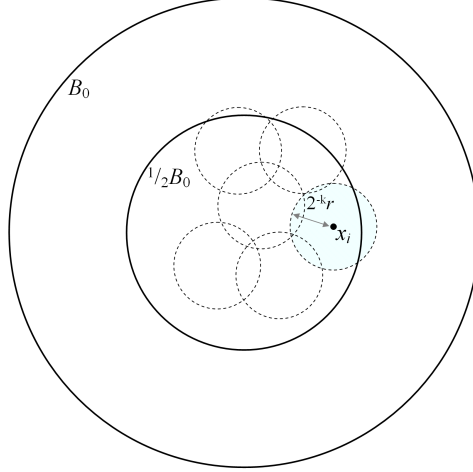
where A is absorbed to C. Since by Lemma 3

$$J_0 = \|u\|_{L^\infty(Q)} \leq \|u_0\|_{L^\infty(M)},$$

the condition (11.4) is satisfied for $t = t_0$, where t_0 is determined by (11.1) with $\eta = C^{-1}$.

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Hence, for $t = t_0$ we obtain from (11.3) that, for any k,

$$\|u\|_{L^\infty\left(2^{-k}B\times[0,t]\right)} \le A^{-k/\omega} \|u_0\|_{L^\infty} .$$

For any k, we cover the ball $\frac{1}{2}B_0$ by a finite sequence of balls $B\left(x_i, 2^{-k}r\right)$ with $x_i \in \frac{1}{2}B_0$.

Since for all i

$$\|u\|_{L^\infty\left(B(x_i,2^{-k}r)\times[0,t]\right)} \le A^{-k/\omega} \|u_0\|_{L^\infty} ,$$

we obtain that

$$\|u\|_{L^\infty\left(\frac{1}{2}B_0\times[0,t]\right)} \le A^{-k/\omega} \|u_0\|_{L^\infty} .$$

Finally, letting $k \to \infty$, we obtain that $u = 0$ in $\frac{1}{2}B_0 \times [0,t]$, which was to be proved.
∎

12   Propagation speed of support

In this section we assume M is geodesically complete, that is, all geodesic balls are precompact. Let also
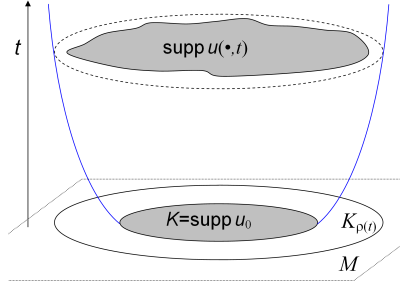
$$\delta = q(p-1) - 1 > 0.$$

For any set $K \subset M$ and any $r > 0$, denote by K_r a closed r-neighborhood of K.

Theorem 10. *Let $u(x,t)$ be a non-negative bounded subsolution of (4.1) in $M \times \mathbb{R}_+$ with the initial function $u_0 = u(\cdot, 0)$. Assume that the support $K = \operatorname{supp} u_0$ is compact. Then there exists $T > 0$ and an increasing continuous function $\rho : (0,T) \to \mathbb{R}_+$ such that*

$$\operatorname{supp} u(\cdot, t) \subset K_{\rho(t)}$$

for all $t \in (0,T)$.

Here both T and $\rho(t)$ may depend on u. The function $\rho(t)$ is called a *propagation rate* of u.

Proof. Let us fix a reference point $x_0 \in K$ and define the following function for all $r > 0$:

$$\varphi(r) = \frac{\eta}{4^{p+p/\nu}} \iota(B(x_0, r)) r^p \|u_0\|_{L^\infty(M)}^{-\delta}. \tag{12.1}$$

Denote

$$r_0 = \operatorname{diam} K.$$

Let us prove that that, for any $r \geq r_0$,

$$t \leq \varphi(3r + r_0) \Rightarrow \operatorname{supp} u(\cdot, t) \subset K_r,$$
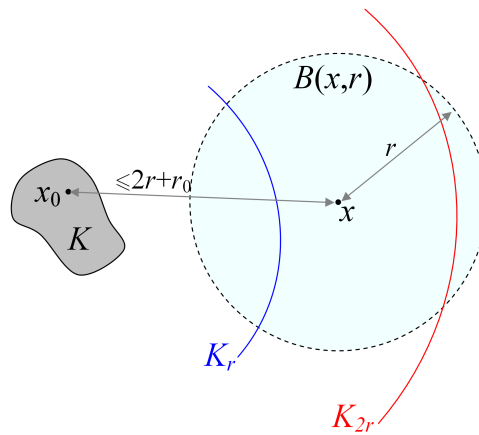
that is,

$$u(\cdot, t) = 0 \quad \text{in} \quad M \setminus K_r.$$

Let us first prove that

$$u(\cdot, t) = 0 \quad \text{in} \quad K_{2r} \setminus K_r.$$

Fix a point $x \in K_{2r} \setminus K_r$. We have

$$d(x, K) \leq 2r \Rightarrow d(x, x_0) \leq 2r + r_0.$$

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It follows that
$$B(x,r) \subset B(x_0, 3r + r_0) = B(x_0, R)$$
where
$$R := 3r + r_0.$$
The condition $r \geq r_0$ implies $R \leq 4r$. Since $B(x,r) \subset B(x_0, R)$, we have by the monotonicity of function (10.4) that
$$\frac{\iota(B(x,r))\mu(B(x,r))}{r^{p/\nu}} \geq \frac{\iota(B(x_0,R))\mu(B(x_0,R))}{R^{p/\nu}}.$$

It follows that
$$\frac{\iota(B(x,r))r^p}{\iota(B(x_0,R))R^p} \geq \left(\frac{r}{R}\right)^{p+p/\nu} \frac{\mu(B(x_0,R))}{\mu(B(x,r))}$$
$$\geq \frac{1}{4^{p+p/\nu}}\iota(B(x_0,R))R^p.$$

Therefore, the hypothesis
$$t \leq \varphi(R) = \frac{\eta}{4^{p+p/\nu}}\iota(B(x_0,R))R^p \|u_0\|_{L^\infty(M)}^{-\delta}$$
implies that
$$t \leq \eta\iota(B(x,r)))r^p \|u_0\|_{L^\infty(M)}^{-\delta}.$$
Since $u(\cdot, 0) = 0$ in $B(x,r)$, we conclude by Theorem 9 that
$$u(\cdot, t) = 0 \quad \text{in} \quad B(x, r/2).$$
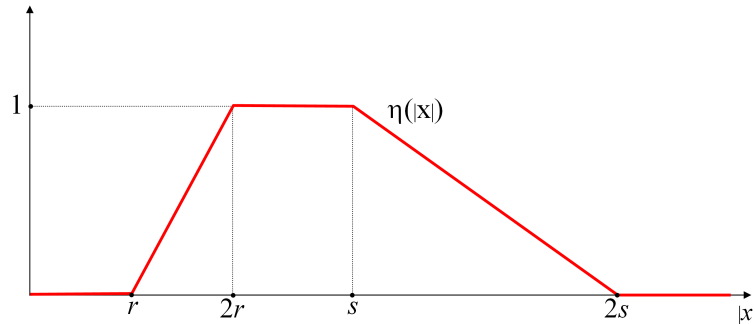Since this is true for any $x \in K_{2r} \setminus K_r$, we obtain that
$$u(\cdot, t) = 0 \quad \text{in } K_{2r} \setminus K_r. \tag{12.2}$$

Let us show that also
$$u(\cdot, t) = 0 \quad \text{in } M \setminus K_{2r}. \tag{12.3}$$
Fix some $s >> 2r$ and let $\eta(x)$ be a bump function of $K_s \setminus K_{2r}$ in $K_{2s} \setminus K_r$; that is, η is the following function of $|x| := d(x, K)$:

$$\eta(x) = \begin{cases} \left(\frac{|x|}{r} - 1\right)_+, & |x| \leq 2r, \\ 1, & |x| \in [2r, s], \\ 2\left(1 - \frac{|x|}{2s}\right)_+, & |x| \geq s. \end{cases}$$

Function η

Applying the inequality (5.4) of Lemma 2 with some fixed λ, we obtain

$$\left[\int_M u^\lambda \eta^p\right]_0^t \le c_2 \int_0^t \int_M u^\sigma |\nabla \eta|^p. \tag{12.4}$$

Since $u(\cdot, 0) = 0$ on $\operatorname{supp}\eta$ and $\eta = 1$ on $K_s \setminus K_{2r}$, the left hand side here is bounded below by

$$\int_{K_s \setminus K_{2r}} u^\lambda(\cdot, t).$$

Since $\eta = 0$ in K_r, $u(\cdot, \tau) = 0$ in $K_{2r} \setminus K_r$ for all $\tau \le t$ (by (12.2)), and $\nabla \eta = 0$ in $K_s \setminus K_{2r}$, the right hand side in (12.4) is equal to

$$c_2 \int_0^t \int_{M \setminus K_s} u^\sigma |\nabla \eta|^p.$$

Since

$$|\nabla \eta| \le \frac{1}{s} \text{ in } M \setminus K_s,$$

we obtain that

$$\int_{K_s \setminus K_{2r}} u^\lambda(\cdot, t) \le c_2 \int_0^t \int_{M \setminus K_s} u^\sigma |\nabla \eta|^p \le \frac{c_2}{s^p} \int_0^t \int_{M \setminus K_s} u^\sigma.$$

The right hand side goes to 0 as $s \to \infty$, which implies that $u(\cdot, t) = 0$ in $M \setminus K_{2r}$, thus proving (12.3).

Now let us define in $[r_0, \infty)$ a function

$$\psi(r) = \frac{1}{2} \sup_{r_0 \le s \le r} \varphi(3s + r_0)$$

so that $\psi(r)$ is monotone increasing. If $t \le \psi(r)$ then $t \le \varphi(3s + r_0)$ for some $s \in [r_0, r]$, which implies by the first part of the proof that

$$u(\cdot, t) = 0 \text{ in } M \setminus K_s$$

and, hence,

$$u(\cdot, t) = 0 \text{ in } M \setminus K_r.$$

It is unclear whether ψ is continuous or not. As a monotone function, ψ may have only jump discontinuities. By subtracting all these jumps, we obtain a continuous monotone function $\widetilde{\psi} \le \psi$ with the same property$|$:

$$t \le \widetilde{\psi}(r) \Rightarrow u(\cdot, t) = 0 \text{ in } M \setminus K_r. \tag{12.5}$$

As a continuous monotone increasing function, $\widetilde{\psi}$ has an inverse $\rho = \widetilde{\psi}^{-1}$ on $[t_0, T)$ where

$$t_0 = \widetilde{\psi}(r_0) \text{ and } T = \sup \widetilde{\psi}.$$

Let us extend $\rho(t)$ to $t < t_0$ by setting $\rho(t) = \rho(t_0)$. Then $r = \rho(t)$ implies $t \le \widetilde{\psi}(r)$, and by (12.5)

$$u(\cdot, t) = 0 \text{ in } M \setminus K_r,$$

which was to be proved. ∎

13   Curvature and propagation rate

In this section we assume again that M is geodesically complete and

$$\delta = q(p-1) - 1 > 0.$$

Theorem 11. *Let M be geodesically complete, non-compact, and let $Ricci_M \geq 0$. Let u be a bounded non-negative subsolution of* (4.1) *in $M \times \mathbb{R}_+$ with the initial condition $u(\cdot, 0) = u_0$. Set $K = \operatorname{supp} u_0$. Then, for any $t \geq 0$,*

$$\operatorname{supp} u(\cdot, t) \subset K_{Ct^{1/p}},$$

where C depends on $\|u_0\|_{L^\infty}$, p, q, n.

Proof. It is known that on such manifolds $\iota(B) \geq \text{const} > 0$ for all balls $B \subset M$.

Let $B = B(x, r)$ be any ball that is disjoint with K. It follows from Theorem 9, that is

$$t \leq cr^p \|u_0\|_{L^\infty(M)}^{-\delta},$$

where $c > 0$ is a small enough constant, then

$$u(\cdot, t) = 0 \;\; \text{in } \frac{1}{2}B.$$

Hence, if

$$r \geq Ct^{1/p}$$

where $C = c^{-1/p} \|u_0\|_{L^\infty(M)}^{\delta/p}$, then

$$\operatorname{supp} u(\cdot, t) \cap \frac{1}{2}B = \emptyset.$$

It follows that

$$\operatorname{supp} u(\cdot, t) \subset K_{\frac{1}{2}r},$$

whence the claim follows. ∎