Finite propagation speed for Leibenson's equation on Riemannian manifolds

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Abstract

We consider on arbitrary Riemannian manifolds the Leibenson equation

$$\partial_t u = \Delta_p u^q.$$

This equation is also known as doubly nonlinear evolution equation. It comes from hydrodynamics where it describes filtration of a turbulent compressible liquid in porous medium. We prove that that, under optimal restrictions on p and q, weak subsolutions to this equation have finite propagation speed.

Contents

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1 Introduction

We are concerned here with a non-linear evolution equation

$$\partial_t u = \Delta_p u^q \tag{1.1}$$

where $p > 1$, $q > 0$, $u = u(x,t)$ is an unknown non-negative function and Δ_p is the p-Laplacian

$$\Delta_p v = \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right).$$

Equation (1.1) was introduced by L. S. Leibenson [31, 32] in order to describe filtration of turbulent compressible fluid through a porous medium. The physical meaning of u is the *volumetric moisture content,* i.e. the (infinitesimal) fraction of volume of the medium taken by the liquid. Parameter p characterizes the turbulence of a flow while $q-1$ is the index of *polytropy* of the liquid, which determines the relation $PV^{q-1} = const$ between volume V and pressure P. The equation (1.1) is frequently referred to as a *doubly non-linear parabolic equation.*

The physically interesting values of the parameters p and q are as follows: $\frac{3}{2} \le p \le 2$ and $q \ge 1$. The case $p = 2$ corresponds to laminar flow (=absence of turbulence). In this case (1.1) becomes a *porous medium equation* $\partial_t u = \Delta u^q$, if $q > 1$, and the classical heat equation $\partial_t u = \Delta u$ if $q = 1$.

However, from the mathematical point of view, the entire range $p > 1, q > 0$ is interesting. For this range, G. I. Barenblatt [6] constructed spherically symmetric self-similar solutions of (1.1) in $\mathbb{R}^n$, that are nowadays called *Barenblatt solutions.*

Assume first that $q(p-1) > 1$. Then the Barenblatt solution is given by

$$u(x,t) = \frac{1}{t^{n/\beta}}\left(C - \varkappa\left(\frac{|x|}{t^{1/\beta}}\right)\right)_+^{\gamma}, \tag{1.2}$$

where $C > 0$ is any constant, and

$$\beta = p + n[q(p-1)-1], \quad \gamma = \frac{p-1}{q(p-1)-1}, \quad \varkappa = \frac{q(p-1)-1}{pq}\beta^{-\frac{1}{p-1}}. \tag{1.3}$$

The parameter β determines the space/time scaling and is analogous to the notion of a *walk dimension*, known for diffusions on fractals.

Clearly, for the Barenblatt solution (1.2), we have

$$u(x,t) = 0 \ \text{ whenever } \ |x| > ct^{1/\beta},$$

where c is a large enough constant; thus, $u(\cdot,t)$ has a bounded support for any $t > 0$. One says in this case that u has a *finite propagation speed.*

Assume now that $q(p-1) < 1$. In this case $\gamma, \varkappa < 0$, and the Barenblatt solution is given by a similar formula

$$u(x,t) = \frac{1}{t^{n/\beta}}\left(C + |\varkappa|\left(\frac{|x|}{t^{1/\beta}}\right)^{\frac{p}{p-1}}\right)^{\gamma}.$$

In the borderline case $q(p-1)=1$, the Barenblatt solution is given by

$$u(x,t)=\frac{1}{t^{n/p}}\exp\left(-\zeta\left(\frac{|x|}{t^{1/p}}\right)^{\frac{p}{p-1}}\right),$$

where $\zeta=(p-1)^2p^{-\frac{p}{p-1}}$. Hence, if $q(p-1)\leq 1$, then $u(x,t)>0$ for all $x\in\mathbb{R}^n$ and $t>0$, that is, u has an *infinite propagation speed.*

In the present paper, we prove the finite propagation speed for solutions of the Leibenson equation (1.1) on arbitrary Riemannian manifolds, under the optimal assumption

$$q(p-1)>1. \tag{1.4}$$

We understand solutions in a certain weak sense (see Section 2 for the definition). It is worth mentioning that existence results for weak solutions of (1.1) were obtained in various settings in the euclidean case in [4, 5, 8, 9, 30, 34, 37, 41] and on *Cartan-Hadamard manifolds* for the porous medium equation ($p=2$) in [23].

The main result of the present paper (cf. Theorem 5.1) is as follows.

Theorem 1.1. *Let M be a geodesically complete Riemannian manifold. Assume that (1.4) is satisfied and let u be a bounded non-negative solution to (1.1) in $M\times\mathbb{R}_+$ with an initial function $u_0=u(\cdot,0)$. If u_0 vanishes in a geodesic ball B_0 of radius R then*

$$u=0\quad\text{in }\frac{1}{2}B_0\times[0,t_0],$$

where

$$t_0=\eta R^p||u_0||_{L^\infty(M)}^{-[q(p-1)-1]},$$

and $\eta>0$ depends on the intrinsic geometry of B_0.

Hence, the solution u has a finite propagation speed inside B_0, and the speed of propagation is determined by the geometry of B_0 via the constant η. As a consequence, we obtain the following result (cf. Corollary 5.2).

Corollary 1.2. *Assume that $K=\operatorname{supp}u_0$ is compact. Then there exists an increasing continuous function $r:(0,T)\to\mathbb{R}_+$ for some $T\in(0,\infty]$ such that*

$$\operatorname{supp}u(\cdot,t)\subset K_{r(t)}\quad\textit{for all } t\in(0,T), \tag{1.5}$$

where $K_r=\{x\in M:d(x,K)\leq r\}$ denotes the closed r-neighborhood of K.

The function $r(t)$ is called the *propagation rate* of u. Hence, u has a finite propagation speed up to a certain time T.

Let us emphasize that these results are valid for an arbitrary geodesically complete Riemannian manifold, and the property of finite propagation speed depends on the *local* structure of the manifold. In particular, this is reflected in the fact that the value of T in (1.5) may be finite. It is an open question whether one can take $T=\infty$ on any geodesically complete manifolds.

In order to obtain a more detailed quantitative information about the propagation rate $r(t)$, one has to impose some restrictions on the global geometry of M, which may also help to ensure that $T=\infty$. For example, we prove the following result (cf. Corollary 5.3).

Corollary 1.3. *Let M be geodesically complete and non-compact. Assume that, for some $x_0 \in K$ and all large enough r,*

$$Ricci_{B(x_0,r)} \geq -\frac{c}{r^2},$$

where $c > 0$. Let u be a bounded non-negative solution in $M \times \mathbb{R}_+$ with the initial condition $u(\cdot, 0) = u_0$; set $K = \operatorname{supp} u_0$. Then, for all $t > 0$,

$$\operatorname{supp} u(\cdot, t) \subset K_{Ct^{1/p}},$$

where the constant C depends on $\|u_0\|_{L^\infty}, p, q, n, c$.

Let us emphasize that in this case the solution has a finite propagation speed for all $t > 0$, that is, $T = \infty$.

Let us recall some previous results about finite propagation speed of solutions of (1.1). Consider first the special case $q = 1$ when (1.1) becomes the parabolic p-Laplace equation

$$\partial_t u = \Delta_p u. \tag{1.6}$$

In this case the condition (1.4) amounts to $p > 2$. The aforementioned results of Theorem 5.1 and Corollaries 5.2, 5.3 were proved for the equation (1.6) by S. Dekkers [14]. In fact, the finite propagation speed was deduced in [14] from a certain non-linear version of the mean value inequality for solutions. We have borrowed this approach from [14], although the proof of the crucial mean value inequality in our case is carried out in an entirely different way.

Related results from the theory of the p-Laplace equation can be found, for instance, in [15, 17, 27, 28].

Consider now another special case $p = 2$ when (1.1) becomes the porous medium equation

$$\partial_t u = \Delta u^q. \tag{1.7}$$

The condition (1.4) amounts in this case to $q > 1$. A finite propagation speed for solutions of (1.7) in hyperbolic spaces was proved by Vazquez [43], in Cartan-Hadamard manifolds by Grillo and Muratori [22] and in manifolds with Ricci curvature bounded from below by De Ponti, Muratori and Orrieri [13].

Some related qualitative properties of solutions of (1.7) were proved in [11] in the setting of compact Riemannian manifolds, in [3, 7, 11] for solutions in $\mathbb{R}^n$, and in [19, 42] for solutions in bounded domains in $\mathbb{R}^n$ with Dirichlet boundary condition.

In the general case, when $p > 1$ and $q > 0$ satisfy (1.4), a finite propagation speed for solutions of (1.1) was proved by Andreucci and Tedeev [2], under the hypothesis that the underlying manifold M satisfies a certain isoperimetric inequality; for example, the latter is the case when M is a Cartan-Hadamard manifold. However, the hypothesis about isoperimetric inequality fails on general manifolds of non-negative Ricci curvature that are covered by our Corollary 5.3.

See also [35, 38, 40] for other results about the asymptotic behaviour of solutions of (1.1).

The structure of the paper is as follows. In Section 2, we define the notion of a weak solution of the Leibenson equation (1.1) and introduce the time mollification, which is then used to prove a *Caccioppoli type inequality* for weak subsolutions (Lemma 2.6). This inequality is one of the ingredients of the proof of the central technical result of this paper $-$ the *mean value inequality* for subsolution that is proved in Section 4 (Lemma 4.3). Another ingredient for the proof of the mean value inequality is introduced in Section 3 (Lemma 3.1)

Using Lemma 4.3, we prove in Section 5 our aforementioned results about finite propagation speed.

Let us make some comments on the mean value inequality of the key Lemma 4.3. It says the following. Let $q(p-1) \geq 1$ and let u be a non-negative bounded subsolution of (1.1) in a cylinder

$$Q = B \times [0,t]$$

where B is a precompact geodesic ball in M. Assume that $u(\cdot,0) = 0$ in B. Then, for the cylinder

$$Q' = \frac{1}{2}B \times [0,t]$$

and for any large enough constant $\sigma > 0$, we have

$$\|u\|_{L^\infty(Q')} \leq \left(\frac{CS_B}{R^{p(1+\nu)}}\right)^{\frac{1}{\sigma\nu}} \|u\|_{L^\infty(Q)}^{\frac{q(p-1)-1}{\sigma}} \|u\|_{L^\sigma(Q)},$$

where $C = C(p,q,\nu,\sigma)$. Here S_B and ν are positive constants that depend on the intrinsic geometry of the ball B, namely, on the Sobolev inequality in B (see Section 3).

Although the proof of Lemma 4.3 follows the classical Moser iteration argument [36], it has certain peculiarities due to the non-linearity of the equation, which is worth mentioning here. We consider a shrinking sequence of cylinders $\{Q_k\}_{k=0}^\infty$ interpolating between $Q_0 = Q$ and $Q_\infty = Q'$, and first prove that

$$\int_{Q_{k+1}} u^{\sigma(1+\nu)} \leq C(\cdots) \left(\int_{Q_k} u^\sigma\right)^{1+\nu}, \tag{1.8}$$

for some $\sigma > 1$ and $\nu > 0$, where ν come from the Sobolev inequality in B and "$\cdots$" stands for some terms that are unimportant for the present discussion (see Corollary 4.2 for details).

In the classical Moser argument, one proves (1.8) first for $\sigma = 2$ and then applies this inequality also to $u^{\sigma/2}$ with any $\sigma > 2$ because $u^{\sigma/2}$ is also a subsolution. This allows to set in (1.8) $\sigma = 2(1+\nu)^k$, reiterate (1.8) and to reach in the limit $\|u\|_{L^\infty(Q')}$ as $k \to \infty$. However, in our case this trick does not work as the powers of a subsolution are *not* necessarily subsolutions. Hence, we need to prove (1.8) directly for *any* σ and to compute carefully the constant $C = C(\sigma)$ in (1.8). It turns out that $C \simeq \sigma^{(2-p)\nu}$ and, surprisingly enough, this power growth of C with σ still allows to complete the iteration argument and to obtain (1.8).

Note also that similar mean value inequalities for subsolutions of the p-Laplacian (that is, in the case $q = 1$) were proved in [16, 18] in $\mathbb{R}^n$ and in [14] on manifolds. However, those proofs were carried out in an entirely different way by using instead of the powers of u the functions $(u-a)_+$ that are subsolutions of the p-Laplacian for any $a > 0$. However, that approach does not work for the general equation (1.1) because $(u-a)_+$ is not a subsolution in this case.

For mean value inequalities in various settings see also [1, 21, 24].

2 Weak subsolutions

2.1 Definition and basic properties

We consider in what follows the following evolution equation on a Riemannian manifold M:

$$\partial_t u = \Delta_p u^q. \tag{2.1}$$

By a *subsolution* of (2.1) we mean a non-negative function u satisfying

$$\partial_t u \leq \Delta_p u^q \tag{2.2}$$

in a certain weak sense as explained below.

We assume throughout that

$$p > 1 \quad \text{and} \quad q > 0.$$

Set

$$\delta = (p-1)q - 1.$$

Later we will assume that $\delta > 0$.

Let μ denote the Riemannian measure on M. For simplicity of notation, we frequently omit in integrations the notation of measure. All integration in M is done with respect to $d\mu$, and in $M \times \mathbb{R}$ – with respect to $d\mu dt$, unless otherwise specified.

Definition 2.1. Let Ω be an open subset of M and $0 < T \leq \infty$ and set $\Omega_T = \Omega \times [0, T)$. Then we call a non-negative function $u = u(x,t)$ a *weak subsolution* of (2.1) in Ω_T, if

$$u \in \mathcal{S}_{p,q}(\Omega_T) = C\left([0,T); L^2(\Omega)\right) \cap \left\{u^q \in L^p_{loc}\left([0,T); W^{1,p}(\Omega)\right)\right\} \tag{2.3}$$

and (2.2) holds weakly in Ω_T, which means that for all $0 \leq t_1 < t_2 < T$, and all non-negative functions

$$\psi \in \mathcal{T}_{p,q}(\Omega_T) = W^{1,2}_{loc}\left([0,T); L^2(\Omega)\right) \cap L^p_{loc}\left([0,T); W^{1,p}_0(\Omega)\right), \tag{2.4}$$

we have

$$\left[\int_\Omega u\psi\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega -u\partial_t\psi + |\nabla u^q|^{p-2}\langle \nabla u^q, \nabla\psi\rangle \leq 0. \tag{2.5}$$

Weak supersolutions and *weak solutions* of (2.1) are defined analogously. Note that the notion of weak solutions is standard (see [17, 26]).

If $u \in \mathcal{S}_{p,q}(\Omega_T)$, we define

$$\nabla u := \begin{cases} q^{-1}u^{1-q}\nabla(u^q), & u > 0, \\ 0, & u = 0. \end{cases}$$

Remark 2.2. It follows from (2.3) and (2.4) that the integrals in (2.5) are finite. Indeed, we have by Hölder's inequality

$$\begin{aligned} \int_{t_1}^{t_2}\int_\Omega |\nabla u^q|^{p-2}\,|\langle \nabla u^q, \nabla\psi\rangle| &\leq \int_{t_1}^{t_2}\int_\Omega |\nabla u^q|^{p-1}|\nabla\psi| \\ &\leq \left(\int_{t_1}^{t_2}\int_\Omega (|\nabla u^q|)^p\right)^{\frac{p-1}{p}} \left(\int_{t_1}^{t_2}\int_\Omega |\nabla\psi|^p\right)^{\frac{1}{p}}. \end{aligned}$$

Definition 2.3. Let $u = u(x,t)$ be a measurable function in Ω_T and $u(\cdot, 0) = u_0$. Then we define, for $h \in (0, T)$,

$$u^h(\cdot, t) = \frac{1}{h}\int_0^t e^{(s-t)/h}u(\cdot, s)ds$$

and

$$u_h(\cdot, t) = e^{-t/h}u_0 + \frac{1}{h}\int_0^t e^{(s-t)/h}u(\cdot, s)ds.$$

The properties of u^h and u_h in the following Lemma are proved in Lemma 2.2 in [29] and in Lemma B.1 and Lemma B.2 in [10].

Lemma 2.4. *Let $p \geq 1$ and suppose that $u \in L^p(\Omega_T)$. Then*

$$||u^h||_{L^p(\Omega_T)} \leq ||u||_{L^p(\Omega_T)}$$

and

$$||u_h||_{L^p(\Omega_T)} \leq ||u||_{L^p(\Omega_T)} + h^{1/p}||u_0||_{L^p(\Omega)},$$

Moreover, $u^h \to u$ and $u_h \to u$ in $L^p(\Omega_T)$ as $h \to 0$ and

$$\partial_t u_h = \frac{1}{h}(u - u_h) \in L^p(\Omega_T). \tag{2.6}$$

Lemma 2.5. *Let Ω be a precompact open subset of M and $u = u(x,t)$ be a bounded weak subsolution of (2.1) in Ω_T. Then*

$$\int_0^\tau \int_\Omega (\partial_t u_h)\psi + \langle [|\nabla u^q|^{p-2}\nabla u^q]^h, \nabla\psi\rangle \leq 0, \tag{2.7}$$

for all $\tau \in (0,T)$ and $\psi \in L^p\Big([0,\tau]; W_0^{1,p}(\Omega)\Big) \cap L^2(\Omega_\tau)$.

Proof. Let us first proof (2.7) in the case when ψ is a non-negative smooth function vanishing on the boundary $\partial\Omega \times [0,\tau]$. Fix some $s \in (0,\tau)$. By (2.5) with $t_1 = 0$, $t_2 = \tau - s$ and $\psi = \psi(x, t+s)$, we have

$$\left[\int_\Omega u(x,t)\psi(x,t+s)d\mu\right]_0^{\tau-s} + \int_0^{\tau-s}\int_\Omega -u\psi_t + |\nabla u^q|^{p-2}\langle\nabla u^q, \nabla\psi\rangle d\mu dt \leq 0.$$

Multiplying both sides by $h^{-1}e^{-s/h}$ and integrating over $[0,\tau]$ with respect to s, we get

$$\begin{aligned}&\frac{1}{h}\int_0^\tau\int_\Omega e^{-s/h}u(x,\tau-s)\psi(x,\tau)d\mu ds - \frac{1}{h}\int_0^\tau\int_\Omega e^{-s/h}u_0(x)\psi(x,s)d\mu ds\\ &+\frac{1}{h}\int_0^\tau\int_s^\tau\int_\Omega e^{-s/h}(-u(x,t-s)\psi_t + |\nabla u(x,t-s)^q|^{p-2}\langle\nabla u(x,t-s)^q, \nabla\psi\rangle)d\mu dt ds \leq 0.\end{aligned}$$

Noticing that

$$\frac{1}{h}\int_0^\tau e^{-s/h}u(\cdot,\tau-s)ds = u^h(\cdot,\tau)$$

and

$$\frac{1}{h}\int_0^\tau\int_s^\tau e^{-s/h}u(\cdot,t-s)dtds = \int_0^\tau u^h(\cdot,t)dt,$$

we deduce

$$\begin{aligned}&\int_\Omega u_h(x,\tau)\psi(x,\tau)d\mu - \int_\Omega e^{-\tau/h}u_0(x)\psi(x,\tau)d\mu - \int_\Omega u_0(x)\left(\frac{1}{h}\int_0^\tau e^{-s/h}\psi(x,s)ds\right)d\mu\\ &+\int_0^\tau\int_\Omega e^{-t/h}u_0\partial_t\psi d\mu dt - \int_0^\tau\int_\Omega u_h\partial_t\psi d\mu dt + \int_0^\tau\int_\Omega\langle[|\nabla u^q|^{p-2}\nabla u^q]^h, \nabla\psi\rangle d\mu dt \leq 0.\end{aligned}$$

By partial integration and using $u_h(\cdot,0) = u_0$, we have

$$\int_\Omega u_h(x,\tau)\psi(x,\tau)d\mu - \int_0^\tau\int_\Omega u_h\partial_t\psi d\mu dt = \int_\Omega u_0(x)\psi(x,0)d\mu + \int_0^\tau\int_\Omega(\partial_t u_h)\psi d\mu dt$$

and

$$\begin{aligned}\int_0^\tau \int_\Omega e^{-t/h} u_0 \partial_t \psi d\mu dt &= \left[\int_\Omega e^{-t/h} u_0(x)\psi(x,t) d\mu \right]_0^\tau + \frac{1}{h} \int_0^\tau \int_\Omega e^{-t/h} u_0(x)\psi(x,t) d\mu dt \\ &= \int_\Omega e^{-\tau/h} u_0(x)\psi(x,\tau) d\mu - \int_\Omega u_0(x)\psi(x,0) d\mu + \int_\Omega u_0(x) \left(\frac{1}{h} \int_0^\tau e^{-t/h} \psi(x,t) dt \right) d\mu,\end{aligned}$$

which implies (2.7).

Let us now prove (2.7) when ψ is in the class as in the statement. By Lemma 4.3 in [33], there exists a sequence $\{\psi_j\}_{j=1}^\infty$ of smooth functions such that $\psi_j \to \psi$ in $L^p\left([0,\tau]; W_0^{1,p}(\Omega)\right)$ as $j \to \infty$. This implies that, by Lemma 2.4 and Hölder's inequality,

$$\int_0^\tau \int_\Omega \langle [|\nabla u^q|^{p-2} \nabla u^q]^h, \nabla \psi_j \rangle \to \int_0^\tau \int_\Omega \langle [|\nabla u^q|^{p-2} \nabla u^q]^h, \nabla \psi \rangle \quad \text{as } j \to \infty.$$

Therefore, it remains to show that

$$\int_0^\tau \int_\Omega (\partial_t u_h) \psi_j \to \int_0^\tau \int_\Omega (\partial_t u_h) \psi \quad \text{as } j \to \infty. \tag{2.8}$$

If $p > 2$, we have $\psi_j \to \psi$ in $L^{\frac{p}{p-1}}(\Omega_\tau)$ since Ω is precompact and $\partial_t u_h \in L^p(\Omega_\tau)$ by (2.6), which implies (2.8) in this case. On the other hand, when $1 < p \le 2$, we have by the same argument $\partial_t u_h \in L^{\frac{p}{p-1}}(\Omega_\tau)$ and thus, (2.8) follows. This completes the proof of (2.7). ■

2.2 Caccioppoli type inequality

Let Ω be a precompact open subset of M and $0 < T \le \infty$.

Lemma 2.6. *Let $v = v(x,t)$ be a bounded non-negative subsolution to (2.1) in a cylinder Ω_T. Let $\eta(x,t)$ be a locally Lipschitz non-negative bounded function in Ω_T such that $\eta(\cdot,t)$ has compact support in Ω for all $t \in [0,T)$. Fix some real λ such that*

$$\lambda \ge \max(2, 1+q) \tag{2.9}$$

and set

$$\sigma = \lambda + \delta \quad \text{and} \quad \alpha = \frac{\sigma}{p}. \tag{2.10}$$

Choose $0 \le t_1 < t_2 < T$ and set $Q = \Omega \times [t_1, t_2]$. Then

$$\left[\int_\Omega v^\lambda \eta^p \right]_{t_1}^{t_2} + c_1 \int_Q |\nabla (v^\alpha \eta)|^p \le \int_Q \left[p v^\lambda \eta^{p-1} \partial_t \eta + c_2 v^\sigma |\nabla \eta|^p \right], \tag{2.11}$$

where c_1, c_2 are positive constants depending on p, q, λ.

In particular, if η does not depend on t, then

$$\left[\int_\Omega v^\lambda \eta^p \right]_{t_1}^{t_2} + c_1 \int_Q |\nabla (v^\alpha \eta)|^p \le c_2 \int_Q v^\sigma |\nabla \eta|^p. \tag{2.12}$$

Proof. Consider the function $\Phi_\alpha(u) = u^{\frac{\alpha}{q}}$. It follows from $\lambda \ge 1+q$, that $\frac{\alpha}{q} \ge 1$, whence Φ_α is a Lipschitz function on $[0, \sup v^q]$ and we obtain that $v^\alpha(\cdot,t) = \Phi_\alpha(v^q)(\cdot,t) \in W^{1,p}(\Omega)$

for all $t \in [0,T)$. Also, note that $\sigma \geq 1+q+(p-1)q-1 = pq$, so that all integrals in (2.11) are well-defined. Since v is a weak subsolution of (2.1), we obtain by (2.7),

$$\int_0^\tau \int_\Omega (\partial_t v_h)\psi + \langle [|\nabla v^q|^{p-2}\nabla v^q]^h, \nabla \psi\rangle \leq 0, \tag{2.13}$$

for all $h \in (0,T)$, $\tau \in (0,T)$ and $\psi \in L^p\left([0,\tau); W_0^{1,p}(\Omega)\right) \cap L^2(\Omega_\tau)$.

Claim:

$$\left[\int_\Omega v^\lambda \eta^p\right]_{t_1}^{t_2} \leq \int_Q -\lambda\langle |\nabla v^q|^{p-2}\nabla v^q, \nabla(v^{\lambda-1}\eta^p)\rangle + pv^\lambda \eta^{p-1}\partial_t \eta. \tag{2.14}$$

Let us consider, for $\nu < \frac{1}{4}(t_2-t_1)$, the function

$$\theta_\nu(t) = \begin{cases} 0, & t < t_1, \\ \frac{1}{\nu}(t-t_1), & t_1 \leq t < t_1+\nu, \\ 1, & t_1+\nu \leq t < t_2-\nu, \\ \frac{1}{\nu}(t_2-t), & t_2-\nu \leq t < t_2, \\ 0, & t \geq t_2 \end{cases}$$

(cf. [33]). We want to show that, for all $t \in [0,\tau]$,

$$v^{\lambda-1}(\cdot,t)\eta^p(\cdot,t)\theta_\nu(t) \in W_0^{1,p}(\Omega), \tag{2.15}$$

which will make this function admissible as a test function in (2.13). Using the function $\Phi_{\lambda-1}(u) = u^{\frac{\lambda-1}{q}}$, $\lambda \geq 1+q$ and the same argumentation as above, we obtain that $v^{\lambda-1} \in W^{1,p}(\Omega)$ and

$$\nabla(v^{\lambda-1}) = \Phi'_{\lambda-1}(v^q)\nabla(v^q) = (\lambda-1)q^{-1}v^{\lambda-(q+1)}\nabla(v^q) = (\lambda-1)v^{\lambda-2}\nabla v.$$

Hence, using this test function in (2.13),

$$\int_Q \partial_t v_h v^{\lambda-1}\eta^p\theta_\nu + \langle [|\nabla v^q|^{p-2}\nabla v^q]^h, \nabla(v^{\lambda-1}\eta^p)\rangle\theta_\nu \leq 0.$$

Let us write

$$\int_Q \partial_t v_h v^{\lambda-1}\eta^p\theta_\nu = \int_Q \partial_t v_h v_h^{\lambda-1}\eta^p\theta_\nu + \int_Q \partial_t v_h(v^{\lambda-1} - v_h^{\lambda-1})\eta^p\theta_\nu.$$

By (2.6), we see that

$$\int_Q \partial_t v_h(v^{\lambda-1} - v_h^{\lambda-1})\eta^p\theta_\nu = \frac{1}{h}\int_Q (v-v_h)(v^{\lambda-1} - v_h^{\lambda-1})\eta^p\theta_\nu \geq 0,$$

whence we obtain

$$\int_Q \partial_t v_h v_h^{\lambda-1}\eta^p\theta_\nu + \langle [|\nabla v^q|^{p-2}\nabla v^q]^h, \nabla(v^{\lambda-1}\eta^p)\rangle\theta_\nu \leq 0. \tag{2.16}$$

By using

$$\lambda\int_Q \partial_t v_h v_h^{\lambda-1}\eta^p\theta_\nu = \int_Q \partial_t v_h^\lambda \eta^p\theta_\nu = \left[\int_\Omega v_h^\lambda \eta^p\theta_\nu\right]_{t_1}^{t_2} - p\int_Q v_h^\lambda \eta^{p-1}\partial_t\eta\theta_\nu - \int_Q v_h^\lambda\eta^p\partial_t\theta_\nu,$$

we get, since $\theta_\nu(t_1) = \theta_\nu(t_2) = 0$,

$$-\int_Q v_h^\lambda \eta^p \partial_t \theta_\nu \leq \int_Q -\lambda \langle [|\nabla v^q|^{p-2}\nabla v^q]^h, \nabla(v^{\lambda-1}\eta^p)\rangle \theta_\nu + p v_h^\lambda \eta^{p-1}\partial_t \eta \theta_\nu. \tag{2.17}$$

We now want to let $h \to 0$ in (2.17) and apply Lemma 2.4 and then let $\nu \to 0$ to obtain (2.14). Note that $|\nabla v^q|^{p-1} \in L^{\frac{p}{p-1}}(Q)$, so that by Lemma 2.4, for $h \to 0$,

$$[|\nabla v^q|^{p-2}\nabla v^q]^h \to |\nabla v^q|^{p-2}\nabla v^q \quad \textit{in } L^{\frac{p}{p-1}}(Q).$$

Together with $|\nabla(v^{\lambda-1}\eta^p)|\theta_\nu \in L^p(Q)$, we obtain

$$\lim_{h\to 0}\int_Q -\lambda\langle [|\nabla v^q|^{p-2}\nabla v^q]^h, \nabla(v^{\lambda-1}\eta^p)\rangle\theta_\nu = \int_Q -\lambda\langle |\nabla v^q|^{p-2}\nabla v^q, \nabla(v^{\lambda-1}\eta^p)\rangle\theta_\nu.$$

For the convergence of the remaining terms in (2.17), we will use the boundedness of v. Note that by assumption $v \in L^2(Q)$ whence Lemma 2.4 implies that $v_h \to v$ in $L^2(Q)$. Since the function $u \mapsto u^\lambda$ is Lipschitz on any bounded subset of $[0,\infty)$, we get $v_h^\lambda \to v^\lambda$ in $L^2(Q)$ and thus,

$$\lim_{h\to 0}\int_Q p v_h^\lambda \eta^{p-1}\partial_t\eta\theta_\nu = \int_Q p v^\lambda \eta^{p-1}\partial_t \eta \theta_\nu.$$

The convergence

$$\lim_{h\to 0}\int_Q v_h^\lambda \eta^p \partial_t \theta_\nu = \int_Q v^\lambda \eta^p \partial_t\theta_\nu$$

follows by the same arguments. Hence,

$$-\int_Q v^\lambda \eta^p \partial_t\theta_\nu \leq \int_Q -\lambda\langle [|\nabla v^q|^{p-2}\nabla v^q], \nabla(v^{\lambda-1}\eta^p)\rangle\theta_\nu + p v^\lambda \eta^{p-1}\partial_t\eta\theta_\nu.$$

Sending now $\nu \to 0$, we deduce (2.14).

We have

$$\nabla(v^{\lambda-1}\eta^p) = (\lambda-1)\eta^p v^{\lambda-2}\nabla v + p\eta^{p-1}v^{\lambda-1}\nabla\eta. \tag{2.18}$$

Therefore, by (2.14) and (2.18), we obtain

$$\begin{aligned}\left[\int_\Omega v^\lambda \eta^p\right]_{t_1}^{t_2} &\leq \int_Q -\lambda(\lambda-1)v^{\lambda-2+(q-1)(p-1)}\eta^p|\nabla v|^p + \lambda p v^{\lambda-1+(q-1)(p-1)}|\nabla v|^{p-1}|\nabla\eta|\eta^{p-1} \\ &\quad + \int_Q p v^\lambda \eta^{p-1}\partial_t\eta \\ &= \int_Q -\lambda(\lambda-1)v^{p(\alpha-1)}\eta^p|\nabla v|^p + \lambda p v^{p(\alpha-1)+1}|\nabla v|^{p-1}|\nabla\eta|\eta^{p-1} + p v^\lambda\eta^{p-1}\partial_t\eta.\end{aligned} \tag{2.19}$$

Then by Young's inequality we have, for all $\varepsilon > 0$,

$$\begin{aligned} v^{p(\alpha-1)+1}|\nabla v|^{p-1}|\nabla\eta|\eta^{p-1} &= \left(v^{p(\alpha-1)\frac{p-1}{p}}|\nabla v|^{p-1}\eta^{p-1}\right)(v^\alpha|\nabla\eta|) \\ &\leq \varepsilon^{p'}v^{p(\alpha-1)}|\nabla v|^p\eta^p + \frac{1}{\varepsilon^p}v^{\alpha p}|\nabla\eta|^p,\end{aligned} \tag{2.20}$$

where $p' = \frac{p}{p-1}$. Combining this with (2.19), we deduce

$$\left[\int_\Omega v^\lambda\eta^p\right]_{t_1}^{t_2} \leq \int_Q -\lambda(\lambda-1-p\varepsilon^{p'})v^{p(\alpha-1)}|\nabla v|^p\eta^p + \frac{\lambda p}{\varepsilon^p}v^{\alpha p}|\nabla\eta|^p + p v^\lambda\eta^{p-1}\partial_t\eta.$$

Also,

$$|\nabla(v^\alpha\eta)|^p = \left|\alpha v^{\alpha-1}\eta\nabla v + v^\alpha\nabla\eta\right|^p \le 2^{p-1}\alpha^p|\nabla v|^p v^{p(\alpha-1)}\eta^p + 2^{p-1}v^{\alpha p}\,|\nabla\eta|^p\,,$$

which implies that

$$|\nabla v|^p v^{p(\alpha-1)}\eta^p \ge 2^{1-p}\alpha^{-p}\,|\nabla(v^\alpha\eta)|^p - \alpha^{-p}v^{\alpha p}\,|\nabla\eta|^p\,.$$

Therefore,

$$\begin{aligned}\left[\int_\Omega v^\lambda\eta^p\right]_{t_1}^{t_2} &\le \int_Q -\lambda(\lambda-1-p\varepsilon^{p'})2^{1-p}\alpha^{-p}\,|\nabla(v^\alpha\eta)|^p \\ &\quad + \int_Q \lambda\left(\left(\lambda-1-p\varepsilon^{p'}\right)\alpha^{-p}+\frac{p}{\varepsilon^p}\right)v^{\alpha p}|\nabla\eta|^p + pv^\lambda\eta^{p-1}\partial_t\eta \\ &= -c_1\int_Q|\nabla(v^\alpha\eta)|^p + c_2\int_Q v^{\alpha p}|\nabla\eta|^p + \int_Q pv^\lambda\eta^{p-1}\partial_t\eta,\end{aligned}$$

where

$$c_1 = \lambda\left(\lambda-1-p\varepsilon^{p'}\right)2^{1-p}\alpha^{-p}$$

and

$$c_2 = \lambda\left(\left(\lambda-1-p\varepsilon^{p'}\right)\alpha^{-p}+\frac{p}{\varepsilon^p}\right).$$

Hence, choosing ε small enough so that $c_1>0$, that is

$$p\varepsilon^{p'} < \lambda-1,$$

we obtain (2.11). Finally, let us specify c_1 and c_2. Let us choose ε so that

$$p\varepsilon^{p'} = \frac{1}{2}(\lambda-1)\,,$$

that is

$$c_1 = \lambda(\lambda-1)\,2^{-p}\alpha^{-p}. \tag{2.21}$$

It follows that

$$\begin{aligned} c_2 &= \frac{1}{2}\lambda(\lambda-1)\,\alpha^{-p} + \lambda\frac{p}{\varepsilon^p} \\ &= \frac{1}{2}\lambda(\lambda-1)\,\alpha^{-p} + \lambda\frac{p}{\left(\frac{1}{2}(\lambda-1)/p\right)^{p/p'}} \\ &= \frac{1}{2}\lambda(\lambda-1)\,\alpha^{-p} + \lambda\frac{2^{p/p'}p^{1+p/p'}}{(\lambda-1)^{p/p'}}.\end{aligned}$$

Since

$$\frac{p}{p'}+1 = \frac{p}{p/(p-1)}+1 = p$$

we have

$$c_2 = \frac{1}{2}\lambda(\lambda-1)\,\alpha^{-p} + \frac{\lambda 2^{p-1}p^p}{(\lambda-1)^{p-1}}. \tag{2.22}$$

which finishes the proof. ■

Remark 2.7. For the future we need the ratio $\frac{c_2}{c_1}$. It follows from (2.21) and (2.22) that

$$\begin{aligned}\frac{c_2}{c_1} &= 2^{p-1} + \lambda \frac{2^{p-1} p^p}{(\lambda-1)^{p-1} \lambda (\lambda-1) 2^{-p} \alpha^{-p}} \\ &= 2^{p-1} + \frac{2^{2p-1} \sigma^p}{(\lambda-1)^p},\end{aligned}$$

where we have used that $\alpha p = \sigma$. Since $\sigma = \lambda + \delta$, we obtain

$$\frac{c_2}{c_1} = 2^{p-1} + \frac{2^{2p-1} (\lambda+\delta)^p}{(\lambda-1)^p}.$$

It follows that, for all $\lambda \geq 2$,

$$\frac{c_2}{c_1} \leq C_{p,\delta},$$

where $C_{p,\delta}$ depend only on p and δ and does not depend on λ.

Remark 2.8. Let us obtain an upper bound of c_2. Using

$$\alpha = \frac{\sigma}{p} = \frac{\lambda+\delta}{p}$$

we obtain

$$c_2 = \frac{1}{2} \frac{\lambda(\lambda-1)}{(\lambda+\delta)^p} p^p + \frac{\lambda 2^{p-1} p^p}{(\lambda-1)^{p-1}}.$$

As $\lambda \geq 2$ and $\lambda + \delta \geq p > 1$, it follows that

$$c_2 \leq C_{p,\delta} \lambda^{2-p}. \tag{2.23}$$

Of course, if $p \geq 2$ then c_2 is uniformly bounded by a constant $C_{p,\delta}$ independently of λ, but if $p < 2$ then c_2 may grow with λ as in (2.23).

Lemma 2.9. *Let $v = v(x,t)$ be a bounded non-negative subsolution to (2.1) in M_T, and assume that M is geodesically complete. Then, for any $\lambda \geq \max(2, 1+q)$, including $\lambda = \infty$, the function*

$$t \mapsto \|v(\cdot,t)\|_{L^\lambda(M)}$$

is monotone decreasing.

Proof. Let $\eta(x,t) = \eta(x)$ be a bump function of some open geodesic ball B' (see Section 3) so that η has compact support in a larger ball B. Observe that the balls are precompact by the completeness of M. By Lemma 2.6 we obtain from (2.12), for any $0 \leq t_1 < t_2 < T$,

$$\left[\int_B v^\lambda \eta^p\right]_{t_1}^{t_2} \leq c_2 \int_{B\times[t_1,t_2]} v^\sigma |\nabla\eta|^p,$$

for some positive constant c_2. Therefore, sending $B \to M$, we conclude as then $\eta \to 1$ and $|\nabla\eta| \to 0$,

$$\left[\int_M v^\lambda\right]_{t_1}^{t_2} \leq 0$$

which proves the claim for finite λ. The case $\lambda = \infty$ then follows by sending $\lambda \to \infty$. ■

3 Sobolev and Moser inequalities

Let M be a connected Riemannian manifold of dimension n. Let d be the geodesic distance on M. For any $x \in M$ and $r > 0$, denote by $B(x,r)$ the geodesic ball of radius r centered at x, that is,

$$B(x,r) = \{y \in M : d(x,y) < r\}.$$

Let B be a precompact ball in M. The Sobolev inequality in B of order $p \geq 1$ says the following|: for any non-negative function $w \in W_0^{1,p}(B)$,

$$\left(\int_B w^{p\kappa}\right)^{1/\kappa} \leq S_B \int_B |\nabla w|^p, \tag{3.1}$$

where $\kappa > 1$ is some constant and S_B is called the *Sobolev constant* in B. The value of κ is independent of B and can be chosen as follows:

$$\kappa = \begin{cases} \dfrac{n}{n-p}, & \text{if } n > p, \\ \text{any number} > 1, & \text{if } n \leq p. \end{cases} \tag{3.2}$$

We always assume that S_B is chosen to be minimal possible. In this case the function $B \mapsto S_B$ is clearly monotone increasing with respect to inclusion of balls.

Dividing (3.1) by $\mu(B)^{1/\kappa}$, we obtain

$$\left(⨍_B w^{p\kappa}\right)^{1/\kappa} \leq \mu(B)^{1/\kappa'} S_B ⨍_B |\nabla w|^p, \tag{3.3}$$

where $\kappa' = \frac{\kappa}{\kappa-1}$ is the Hölder conjugate of κ and $⨍$ denotes the normalized integral. It follows from (3.2) that

$$\kappa' = \begin{cases} \dfrac{n}{p}, & \text{if } n > p, \\ \text{any number} > 1, & \text{if } n \leq p. \end{cases} \tag{3.4}$$

Denoting by $r(B)$ the radius of B, let us define a new quantity

$$\iota(B) := \frac{1}{\mu(B)} \left(\frac{r(B)^p}{S_B}\right)^{\kappa'} \tag{3.5}$$

so that

$$S_B = \frac{r(B)^p}{(\iota(B)\mu(B))^{1/\kappa'}} \tag{3.6}$$

and

$$\left(\mu(B)^{1/\kappa'} S_B\right)^{1/p} = \frac{r(B)}{\iota(B)^{\frac{1}{p\kappa'}}}.$$

Hence, (3.3) can be rewritten in the form

$$\left(⨍_B |\nabla w|^p\right)^{1/p} \geq \frac{\iota(B)^{\frac{1}{p\kappa'}}}{r(B)} \left(⨍_B w^{p\kappa}\right)^{1/p\kappa}. \tag{3.7}$$

It is clear from (3.7) that the value of κ can be always reduced (by modifying the value of $\iota(B)$). It is only important that $\kappa > 1$. In fact, the exact value of κ does not affect the results, although various constants do depend on κ.

The constant $\iota(B)$ is called the *normalized Sobolev* constant in B. It is known that if M is complete and $Ricci_B \geq -(n-1)k$ for some $k \geq 0$ then

$$\iota(B) \geq c e^{-C_n \sqrt{k} r(B)}, \tag{3.8}$$

for positive constants c, C_n (see [12], [20], [39]).

Let B be a precompact ball in M and $Q = B \times [0,T]$. Assume that the Sobolev inequality (3.7) holds in B with exponent $\kappa > 1$, and let κ' be its Hölder conjugate. Set

$$\nu = \frac{1}{\kappa'} = \frac{\kappa - 1}{\kappa}.$$

Lemma 3.1. *Let $w \in L^p\left([0,T]; W_0^{1,p}(B)\right)$ be a non-negative function. Then,*

$$\int_Q w^{p(1+\nu)} \leq S_B \left(\int_Q |\nabla w|^p\right) \sup_t \left(\int_B w^p\right)^\nu. \tag{3.9}$$

Proof. By the Hölder inequality, we have, for any $t \in [0,T]$

$$\begin{aligned} \int_B w^{p(1+\nu)} &= \int_B w^p w^{p\nu} \leq \left(\int_B w^{p\kappa}\right)^{1/\kappa} \left(\int_B w^{p\nu\kappa'}\right)^{1/\kappa'} \\ &= \left(\int_B w^{p\kappa}\right)^{1/\kappa} \left(\int_B w^p\right)^\nu \\ &\leq \left(\int_B w^{p\kappa}\right)^{1/\kappa} \sup_{t\in[0,T]} \left(\int_B w^p\right)^\nu, \end{aligned}$$

where we have used that $\nu\kappa' = 1$.

By the Sobolev inequality (3.1) we have

$$\left(\int_B w^{p\kappa}\right)^{1/\kappa} \leq S_B \int_B |\nabla w|^p.$$

It follows that

$$\int_B w^{p(1+\nu)} \leq S_B \left(\int_B |\nabla w|^p\right) \sup_t \left(\int_B w^p\right)^\nu.$$

Integrating this inequality in $t \in [0,T]$ gives (3.9). ∎

4 Estimates of subsolutions

4.1 Comparison in two cylinders

Here we assume that

$$p > 1 \quad \text{and} \quad \delta := q(p-1) - 1 \geq 0.$$

Lemma 4.1. *Consider two balls $B = B(x,r)$ and $B' = B(x,r')$ with $0 < r' < r$, and two cylinders*

$$Q = B \times [0,T], \quad Q' = B' \times [0,T].$$

Assume that B is precompact. Let λ be any real such that

$$\lambda \geq \max(2, 1+q). \tag{4.1}$$

Set

$$\sigma = \lambda + \delta.$$

Let v be a non-negative bounded subsolution of (2.1) in $B \times [0, T')$ for some $T' > T$, such that

$$v(\cdot, 0) = 0.$$

Then

$$\int_{Q'} v^{\sigma(1+\nu)} \leq \frac{C S_B \sigma^{(2-p)\nu}}{(r-r')^{p(1+\nu)}} \left(\int_Q v^{\sigma}\right) \left(\int_Q v^{\sigma+\delta}\right)^{\nu}, \tag{4.2}$$

where the constant C depends on p, δ and ν, but it is independent of σ.

Proof. As in Lemma 2.6, set $\alpha = \frac{\sigma}{p}$. Let η be a bump function of B' in B. Recalling the proof of Lemma 2.6, we see that $v^{\alpha}\eta \in L^p_{loc}\left([0, T'); W_0^{1,p}(B)\right)$. Applying (3.9) with

$$w = v^{\alpha}\eta$$

and using

$$w^p = v^{\sigma}\eta^p,$$

we obtain that, for any $t \in [0, T]$,

$$\int_Q v^{\sigma(1+\nu)}\eta^{p(1+\nu)} \leq S_B \left(\int_Q |\nabla(v^{\alpha}\eta)|^p\right) \sup_{t\in[0,T]} \left(\int_B v^{\sigma}\eta^p\right)^{\nu}.$$

By (2.12) we have

$$\int_Q |\nabla(v^{\alpha}\eta)|^p \leq \frac{c_2}{c_1}\int_Q v^{\sigma} |\nabla\eta|^p$$

and

$$\sup_{t\in[0,T]} \left(\int_B v^{\lambda}\eta^p\right) \leq c_2 \int_Q v^{\sigma} |\nabla\eta|^p.$$

Let us use the latter in the form

$$\sup_{t\in[0,T]} \left(\int_B v^{\lambda'}\eta^p\right) \leq c_2' \int_Q v^{\sigma'} |\nabla\eta|^p,$$

where

$$\lambda' = \sigma \quad \text{and} \quad \sigma' = \lambda' + \delta = \sigma + \delta.$$

Then we have

$$\sup_{t\in[0,T]} \left(\int_B v^{\sigma}\eta^p\right) \leq c_2' \int_Q v^{\sigma'} |\nabla\eta|^p.$$

It follows that

$$\int_Q v^{\sigma(1+\nu)}\eta^{p(1+\nu)} \leq S_B \frac{c_2}{c_1}\int_Q v^{\sigma} |\nabla\eta|^p \left(c_2' \int_Q v^{\sigma'} |\nabla\eta|^p\right)^{\nu}.$$

Using that $\eta = 1$ in B' and $|\nabla\eta| \leq \frac{1}{r-r'}$ we obtain

$$\int_{Q'} v^{\sigma(1+\nu)} \leq S_B \frac{c_2}{c_1}\frac{(c_2')^{\nu}}{(r-r')^{p(1+\nu)}} \left(\int_Q v^{\sigma}\right) \left(\int_Q v^{\sigma'}\right)^{\nu}.$$

By Remark 2.7 we have

$$\frac{c_2}{c_1} \le C_{p,\delta},$$

and, by the estimate (2.23) of Remark 2.8,

$$c_2' \le C_{p,\delta} \left(\lambda'\right)^{2-p} = C_{p,\delta}\sigma^{2-p}.$$

Hence, (4.2) follows. ∎

Corollary 4.2. *Under the hypotheses of* Lemma 4.1, *we have*

$$\int_{Q'} v^{\sigma(1+\nu)} \le \frac{CS_B\sigma^{(2-p)\nu} \left\|v\right\|_{L^\infty(Q)}^{\delta\nu}}{(r-r')^{p(1+\nu)}} \left(\int_Q v^\sigma\right)^{1+\nu}, \tag{4.3}$$

where $C = C(p,\delta,\nu)$.

4.2 Mean value inequality

We assume here that $p > 1$ and $\delta \ge 0$.

Lemma 4.3. *Let the ball* $B = B\left(x_0, R\right)$ *be precompact and* $T > 0$. *Let* u *be a non-negative bounded subsolution of* (2.1) *in* $B \times [0,T)$ *such that*

$$u\left(\cdot,0\right) = 0 \text{ in } B.$$

Choose $t \in (0,T)$ *and set*

$$Q = B \times [0,t] \quad \text{and} \quad Q' = \frac{1}{2}B \times [0,t]\,.$$

(see Fig. 1). *Then, for any large enough* $\sigma > 0$, *we have*

$$\left\|u\right\|_{L^\infty(Q')} \le \left(\frac{CS_B}{R^{p(1+\nu)}}\right)^{\frac{1}{\sigma\nu}} \left\|u\right\|_{L^\infty(Q)}^{\frac{\delta}{\sigma}} \left\|u\right\|_{L^\sigma(Q)}, \tag{4.4}$$

where $C = C\left(p,q,\nu,\sigma\right)$.

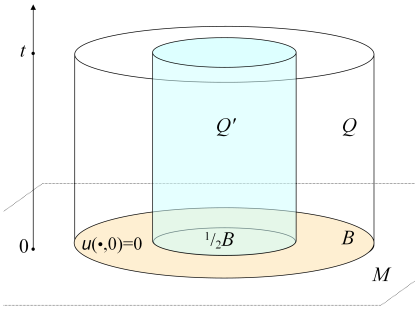

Figure 1: Cylinders Q and Q'

Proof. Consider a sequence of radii

$$r_k = \left(\frac{1}{2} + 2^{-k-1}\right)R$$

so that $r_0 = R$ and $r_k \searrow \frac{1}{2}R$ as $k \to \infty$. Set

$$B_k = B\left(x_0, r_k\right), \qquad Q_k = B_k \times [0, t]$$

so that

$$B_0 = B, \qquad Q_0 = Q \qquad \text{and} \qquad Q_\infty := \lim_{k\to\infty} Q_k = Q'$$

(see Fig. 2).

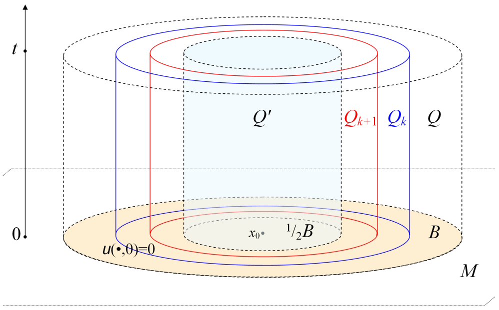

Figure 2: Cylinders Q_k

Set also

$$\sigma_k = \sigma\left(1+\nu\right)^k$$

and

$$J_k = \int_{Q_k} u^{\sigma_k}.$$

By (4.3) we have

$$\begin{aligned} J_{k+1} &\le \frac{C S_{B_k} \sigma_k^{(2-p)\nu} \left\|u\right\|_{L^\infty(Q_k)}^{\delta\nu}}{\left(r_k - r_{k+1}\right)^{p(1+\nu)}} J_k^{1+\nu} \\ &\le \frac{C 2^{kp(1+\nu)} \left(1+\nu\right)^{k(2-p)\nu} \sigma^{(2-p)\nu} S_B \left\|u\right\|_{L^\infty(Q)}^{\delta\nu}}{R^{p(1+\nu)}} J_k^{1+\nu} \\ &\le A^k \Theta^{-1} J_k^{1+\nu}, \end{aligned}$$

where

$$A = 2^{p(1+\nu)} \left(1+\nu\right)^{(2-p)_+\nu} \ge 1$$

and

$$\Theta^{-1} = \frac{C S_B \left\|u\right\|_{L^\infty(Q)}^{\delta\nu}}{R^{p(1+\nu)}},$$

where we have absorbed $\sigma^{(2-p)\nu}$ into C.

By Lemma 6.1 (see Appendix), we conclude that

$$\begin{aligned} J_k &\le \left(\left(A^{1/\nu}\Theta^{-1}\right)^{1/\nu} J_0\right)^{(1+\nu)^k} \left(A^{-1/\nu}\Theta\right)^{1/\nu} \\ &= A^{\frac{(1+\nu)^k - 1}{\nu^2}} \Theta^{-\frac{(1+\nu)^k - 1}{\nu}} J_0^{(1+\nu)^k}. \end{aligned}$$

It follows that

$$\left(\int_{Q_k} u^{\sigma_k}\right)^{1/\sigma_k} \leq A^{\frac{1-(1+\nu)^{-k}}{\sigma\nu^2}}\,\Theta^{-\frac{1-(1+\nu)^{-k}}{\sigma\nu}}\left(\int_{Q} u^{\sigma}\right)^{1/\sigma}.$$

As $k \to \infty$, we obtain

$$\begin{aligned}
\|u\|_{L^\infty(Q')} &\leq A^{\frac{1}{\sigma\nu^2}}\,\Theta^{-\frac{1}{\sigma\nu}}\,\|u\|_{L^\sigma(Q)} \\
&= A^{\frac{1}{\sigma\nu^2}}\left(\frac{CS_B\,\|u\|^{\delta\nu}_{L^\infty(Q)}}{R^{p(1+\nu)}}\right)^{\frac{1}{\sigma\nu}}\|u\|_{L^\sigma(Q)} \\
&= \left(\frac{CS_B}{R^{p(1+\nu)}}\right)^{\frac{1}{\sigma\nu}}\|u\|^{\frac{\delta}{\sigma}}_{L^\infty(Q)}\,\|u\|_{L^\sigma(Q)}\,,
\end{aligned}$$

where $A^{1/\nu}$ was absorbed into C. ■

Remark 4.4. Clearly, (4.4) implies

$$\|u\|_{L^\infty(Q')} \leq \left(\frac{CS_B}{R^{p(1+\nu)}}\right)^{\frac{1}{\sigma\nu}}(t\mu(B))^{\frac{1}{\sigma}}\,\|u\|^{1+\frac{\delta}{\sigma}}_{L^\infty(Q)}\,. \tag{4.5}$$

5 Finite propagation speed

In this section we assume that M is geodesically complete. In particular, all balls are precompact. We assume here that

$$p > 1 \quad \text{and} \quad \delta > 0.$$

5.1 Propagation speed inside a ball

The following theorem implies Theorem 1.1.

Theorem 5.1. *Let u be a bounded non-negative subsolution of (2.1) in M_T with the initial condition $u(\cdot,0) = u_0$. Let $B_0 = B(x_0,R)$ be a ball such that $u_0 = 0$ in B_0* (see Fig. 3). *Set*

$$t_0 = \eta\iota(B_0)R^p\,\|u_0\|^{-\delta}_{L^\infty(M)} \wedge T, \tag{5.1}$$

where η is a sufficiently small positive constant depending only on p, q, ν and $\iota(B_0)$ is the normalized Sobolev constant defined in (3.5). Then

$$u = 0 \quad in \quad \frac{1}{2}B_0 \times [0, t_0]\,.$$

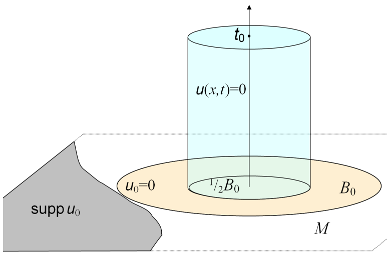

Figure 3: The support of u_0

Proof. Set $r = \frac{1}{2}R$ and fix for a while a point $x \in \frac{1}{2}B_0$ so that $B := B(x, r) \subset B_0$. Fix also some $t \in (0, T)$ and set

$$Q_k = 2^{-k}B \times [0, t] \quad \text{and} \quad J_k = \|u\|_{L^\infty(Q_k)}$$

(see Fig. 4).

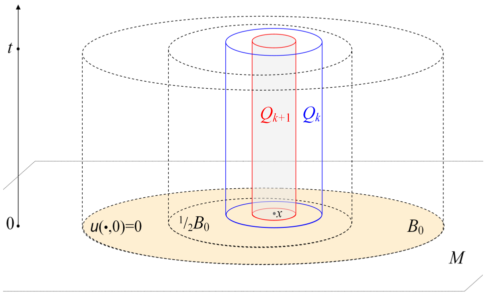

Figure 4: Cylinders Q_k

Choose and fix σ large enough as it is needed for Lemma 4.3. Then, by (4.5), we have

$$\begin{aligned} J_{k+1} &\le \left(\frac{CS_{2^{-k}B}}{(2^{-k}R)^{p(1+\nu)}}\right)^{\frac{1}{\sigma\nu}} \left(t\mu(2^{-k}B)\right)^{\frac{1}{\sigma}} J_k^{1+\frac{\delta}{\sigma}} \\ &\le 2^{k\frac{p(1+\nu)}{\sigma\nu}} \left(\frac{CS_B}{R^{p(1+\nu)}}\right)^{\frac{1}{\sigma\nu}} (t\mu(B))^{\frac{1}{\sigma}} J_k^{1+\frac{\delta}{\sigma}}. \end{aligned}$$

Observe that, by (3.6) and $\frac{1}{\nu} = \kappa'$,

$$\left(\frac{S_B}{R^{p(1+\nu)}}\right)^{\frac{1}{\nu}} \mu(B) = \frac{R^{p/\nu}}{R^{p\frac{(1+\nu)}{\nu}}\iota(B)\mu(B)}\mu(B) = \frac{1}{\iota(B)R^p},$$

so that

$$\begin{aligned} J_{k+1} &\le 2^{k\frac{p(1+\nu)}{\sigma\nu}} \left(\frac{Ct}{\iota(B)R^p}\right)^{\frac{1}{\sigma}} J_k^{1+\frac{\delta}{\sigma}} \\ &= A^k\Theta^{-1}J_k^{1+\omega}, \end{aligned}$$

where

$$\omega = \frac{\delta}{\sigma}, \qquad A = 2^{\frac{p(1+\nu)}{\sigma\nu}}$$

and

$$\Theta^{-1} = \left(\frac{Ct}{\iota(B)R^p}\right)^{\frac{1}{\sigma}}.$$

By Lemma 6.1, if

$$\Theta^{-1} \le A^{-1/\omega}J_0^{-\omega} \tag{5.2}$$

then, for all $k \ge 0$,

$$J_k \le A^{-k/\omega}J_0. \tag{5.3}$$

The condition (5.2) is equivalent to

$$\left(\frac{Ct}{\iota(B)R^p}\right)^{\frac{1}{\sigma}} \le A^{-1/\omega}J_0^{-\omega}$$

that is, to

$$t \leq C^{-1}\iota(B)R^p J_0^{-\delta}, \tag{5.4}$$

where A is absorbed to C. Since, by Lemma 2.9,

$$J_0 = \|u\|_{L^\infty(Q)} \leq \|u_0\|_{L^\infty(M)}$$

the condition (5.4) is satisfied for $t = t_0$, where t_0 is determined by (5.1) with $\eta = C^{-1}$. Hence, for $t = t_0$ we obtain from (5.3) that, for any k,

$$\|u\|_{L^\infty\left(2^{-k}B\times[0,t]\right)} \leq A^{-k/\omega}\,\|u_0\|_{L^\infty}\,.$$

For any k, we cover the ball $\frac{1}{2}B_0$ by a countable (or even finite) sequence of balls $B\left(x_i, 2^{-k}r\right)$ with $x_i \in \frac{1}{2}B_0$. Since for all i

$$\|u\|_{L^\infty\left(B(x_i,2^{-k}r)\times[0,t]\right)} \leq A^{-k/\omega}\,\|u_0\|_{L^\infty}\,,$$

we obtain that

$$\|u\|_{L^\infty\left(\frac{1}{2}B_0\times[0,t]\right)} \leq A^{-k/\omega}\,\|u_0\|_{L^\infty}\,.$$

Finally, letting $k \to \infty$, we obtain that $u = 0$ in $\frac{1}{2}B_0 \times [0,t]$, which was to be proved. ■

5.2 Propagation speed of support

As above, we assume here that

$$p > 1 \ \text{ and } \ \delta > 0.$$

For any set $K \subset M$ and any $r > 0$, denote by K_r a closed r-neighborhood of K.

Corollary 5.2. *Let $u\,(x,t)$ be a non-negative bounded subsolution of* (2.1) *in $M \times \mathbb{R}_+$ with the initial function $u_0 = u\,(\cdot,0)$. Assume that the support $K = \operatorname{supp} u_0$ is compact. Then there exists $T > 0$ and an increasing continuous function $\rho : (0,T) \to \mathbb{R}_+$ such that*

$$\operatorname{supp} u\,(\cdot,t) \subset K_{\rho(t)}$$

for all $t \in (0,T)$ (see Fig. 5).

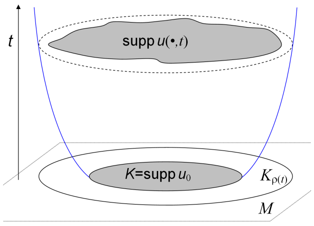

Figure 5: The support of $u(\cdot,t)$

Here T and $\rho\,(t)$ may depend on u. The function $\rho(t)$ is called the *propagation rate* of u.

Proof. Let us fix a reference point $x_0 \in K$ and define the following function for all $r > 0$:

$$\varphi\,(r) = \frac{\eta}{4^{p+p/\nu}}\iota(B(x_0,r))r^p\,\|u_0\|_{L^\infty(M)}^{-\delta}\,. \tag{5.5}$$

Denote $r_0 = \operatorname{diam} K$. Let us prove that, for any $r \geq r_0$,

$$t \leq \varphi\left(3r + r_0\right) \Rightarrow \operatorname{supp} u\left(\cdot, t\right) \subset K_r,$$

that is,

$$u(\cdot, t) = 0 \text{ in } M \setminus K_r.$$

Let us fix a point $x \in K_{2r} \setminus K_r$ (see Fig. 6). We have

$$d\left(x, K\right) \leq 2r \Rightarrow d\left(x, x_0\right) \leq 2r + r_0.$$

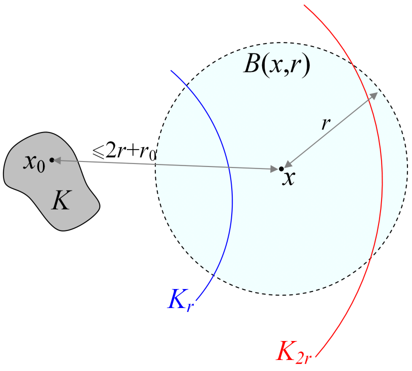

Figure 6: A point $x \in K_{2r} \setminus K_r$ and the ball $B(x, r)$

It follows that

$$B\left(x, r\right) \subset B\left(x_0, 3r + r_0\right) = B(x_0, R)$$

where

$$R := 3r + r_0.$$

The condition $r \geq r_0$ implies $R \leq 4r$. Since $B(x, r) \subset B(x_0, R)$, we have by the monotonicity of function (3.6) that

$$\frac{\iota(B(x,r))\mu(B(x,r))}{r^{p/\nu}} \geq \frac{\iota(B(x_0,R))\mu(B(x_0,R))}{R^{p/\nu}}.$$

It follows that

$$\begin{aligned} \iota(B(x,r))r^p &\geq \left(\frac{r}{R}\right)^{p+p/\nu} \iota(B(x_0,R))\frac{\mu(B(x_0,R))}{\mu(B(x,r))} R^p \\ &\geq \frac{1}{4^{p+p/\nu}} \iota(B(x_0,R))R^p. \end{aligned}$$

Therefore, the hypothesis $t \leq \varphi\left(R\right)$ implies that

$$t \leq \eta\iota(B(x,r)))r^p \left\|u_0\right\|_{L^\infty(M)}^{-\delta}.$$

Since $u(\cdot, 0) = 0$ in $B(x, r)$, we conclude by Theorem 5.1 that

$$u(\cdot, t) = 0 \text{ in } B(x, r/2).$$

Since this is true for any $x \in K_{2r} \setminus K_r$, we obtain that

$$u(\cdot, t) = 0 \text{ in } K_{2r} \setminus K_r. \tag{5.6}$$

Let us show that also

$$u(\cdot,t)=0 \ \text{ in } M\setminus K_r. \tag{5.7}$$

Fix some $s>>2r$ and let $\eta\left(x\right)$ be a bump function of $K_s\setminus K_{2r}$ in $K_{2s}\setminus K_r$; that is, η is the following function of $|x|:=d\left(x,K\right)$:

$$\eta\left(x\right)=\begin{cases}\left(\frac{|x|}{r}-1\right)_+, & |x|\leq 2r,\\ 1, \ \ |x|\in\left[2r,s\right], & \\ 2\left(1-\frac{|x|}{2s}\right)_+, & |x|\geq s\end{cases}$$

(see Fig. 7).

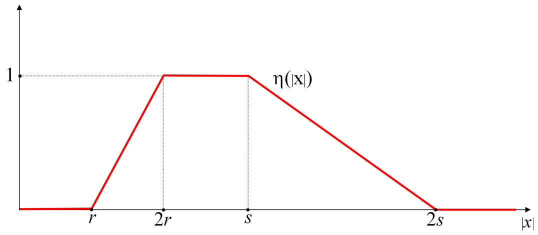

Figure 7: Function η

Applying the inequality (2.12) of Lemma 2.6 in some open neighborhood Ω_s of K_{2s} with some fixed λ , we obtain

$$\left[\int_{\Omega_s}u^{\lambda}\eta^p\right]_0^t\leq c_2\int_0^t\int_{\Omega_s}u^{\sigma}\left|\nabla\eta\right|^p. \tag{5.8}$$

Since $u\left(\cdot,0\right)=0$ on $\operatorname{supp}\eta$ and $\eta=1$ on $K_s\setminus K_{2r}$, the left hand side here is bounded below by

$$\int_{K_s\setminus K_{2r}}u^{\lambda}(\cdot,t).$$

Since $\eta=0$ in K_r, $u(\cdot,\tau)=0$ in $K_{2r}\setminus K_r$ for all $\tau\leq t$ (by (5.6)), and $\nabla\eta=0$ in $K_s\setminus K_{2r}$, the right hand side in (5.8) is equal to

$$c_2\int_0^t\int_{\Omega_s\setminus K_s}u^{\sigma}\left|\nabla\eta\right|^p.$$

Since

$$\left|\nabla\eta\right|\leq\frac{1}{s}\text{ in }\Omega_s\setminus K_s,$$

we obtain that

$$\int_{K_s\setminus K_{2r}}u^{\lambda}(\cdot,t)\leq c_2\int_0^t\int_{\Omega_s\setminus K_s}u^{\sigma}\left|\nabla\eta\right|^p\leq\frac{c_2}{s^p}\int_0^t\int_{\Omega_s\setminus K_s}u^{\sigma}.$$

The right hand side goes to 0 as $s\to\infty$, which implies that $u\left(\cdot,t\right)=0$ in $M\setminus K_{2r}$, thus proving (5.7).

Now let us define in $[r_0,\infty)$ a function

$$\psi\left(r\right)=\frac{1}{2}\sup_{s\in[r_0,r]}\varphi\left(3s+r_0\right)$$

so that $\psi(r)$ is monotone increasing. If $t \leq \psi(r)$ then $t \leq \varphi(3s + r_0)$ for some $s \in [r_0, r]$, which implies by the first part of the proof that

$$u(\cdot, t) = 0 \text{ in } M \setminus K_s$$

and, hence,

$$u(\cdot, t) = 0 \text{ in } M \setminus K_r.$$

It is unclear whether ψ is continuous or not. As a monotone function, ψ may have only jump discontinuities. By subtracting all these jumps, we obtain a continuous monotone function $\widetilde{\psi} \leq \psi$ with the same property:

$$t \leq \widetilde{\psi}(r) \Rightarrow u(\cdot, t) = 0 \text{ in } M \setminus K_r. \tag{5.9}$$

As a continuous monotone increasing function, $\widetilde{\psi}$ has an inverse $\rho = \widetilde{\psi}^{-1}$ on $[t_0, T)$ where

$$t_0 = \widetilde{\psi}(r_0) \text{ and } T = \sup \widetilde{\psi}.$$

Let us extend $\rho(t)$ to $t < t_0$ by setting $\rho(t) = \rho(t_0)$. Then $r = \rho(t)$ implies $t \leq \widetilde{\psi}(r)$, and by (5.9)

$$u(\cdot, t) = 0 \text{ in } M \setminus K_r,$$

which was to be proved. ■

5.3 Curvature and propagation rate

Corollary 5.3. *Let M be complete and non-compact. Let u be a bounded non-negative subsolution in $M \times \mathbb{R}_+$ with the initial condition $u(\cdot, 0) = u_0$. Set $K = \operatorname{supp} u_0$. Assume that for some $x_0 \in K$ and all large enough r, we have*

$$Ricci_{B(x_0, r)} \geq -\frac{c}{r^2}, \tag{5.10}$$

where $c > 0$. Then, for any $t > 0$,

$$\operatorname{supp} u(\cdot, t) \subset K_{Ct^{1/p}}$$

where C depends on $\|u_0\|_{L^\infty}, p, q, n, c$.

Proof. It follows from (3.8) and (5.10), that $\iota(B(x_0, r)) \geq \text{const} > 0$ for all $r > 0$. Hence, using the same notation as in Corollary 5.2, we obtain from (5.5),

$$\varphi(r) \geq c' r^p,$$

whence

$$\rho(t) \leq C t^{1/p},$$

which yields the claim. ■

Corollary 5.4. *Let M be a Cartan-Hadamard manifold. Let u be a bounded non-negative subsolution in $M \times \mathbb{R}_+$ with the initial condition $u(\cdot, 0) = u_0$. Set $K = \operatorname{supp} u_0$. Assume that for some $x_0 \in K$ and for all large enough r, we have*

$$\mu(B(x_0, r)) \leq c r^\alpha, \tag{5.11}$$

where $c > 0$ and $n \leq \alpha < n + p$. Then, for all large enough t,

$$\operatorname{supp} u(\cdot, t) \subset K_{Ct^{1/(n+p-\alpha)}}$$

where C depends on$\|u_0\|_{L^\infty}, p, q, n, \alpha, c$.

Note that the restriction $\alpha \geq n$ follows automatically from (5.11) because on Cartan-Hadamard manifolds always $\mu\left(B(x_0,r)\right) \geq \operatorname{const} r^n$.

Proof. Since M is a Cartan-Hadamard manifold, we have $S_B \leq \operatorname{const}$ for all geodesic balls $B \subset M$ (see [25]). It follows from (3.5) and (5.11) that, for large r,

$$\iota(B(x_0,r)) \geq \operatorname{const} r^{p\kappa'-\alpha}.$$

By (3.4) we have $k' \geq \frac{n}{p}$, whence

$$\iota(B(x_0,r)) \geq \operatorname{const} r^{n-\alpha}.$$

Using again the same notation as in Corollary 5.2, we obtain from (5.5) that

$$\varphi(r) \geq \operatorname{const} r^{n+p-\alpha},$$

which yields $\rho(t) \leq Ct^{1/(n+p-\alpha)}$. ■

Remark 5.5. The propagations rates of Corollaries 5.3 and 5.4 seem to be not sharp. Obtaining sharp estimates is a matter for future work.

6 Appendix: an auxiliary lemma

The following lemma was used in Sections 4 and 5.

Lemma 6.1. *Let a sequence* $\{J_k\}_{k=0}^{\infty}$ *of non-negative reals satisfy*

$$J_{k+1} \leq \frac{A^k}{\Theta} J_k^{1+\omega} \quad \text{for all } k \geq 0.$$

where $A, \Theta, \omega > 0$. *Then, for all* $k \geq 0$,

$$J_k \leq \left(\left(A^{1/\omega}\Theta^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^k} \left(A^{-k-1/\omega}\Theta\right)^{1/\omega}.$$

In particular, if $\Theta \geq A^{1/\omega} J_0^{\omega}$, *then* $J_k \leq A^{-k/\omega} J_0$ *for all* $k \geq 0$.

Proof. Consider the sequence

$$X_k = \left(\left(A^{1/\omega}\Theta^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^k} \left(A^{-k-1/\omega}\Theta\right)^{1/\omega}.$$

Then we have

$$X_0 = \left(A^{1/\omega}\Theta^{-1}\right)^{1/\omega} J_0 \left(A^{-1/\omega}\Theta\right)^{1/\omega} = J_0$$

and

$$\begin{aligned}
\frac{A^k}{\Theta} X_k^{1+\omega} &= \frac{A^k}{\Theta} \left(\left(A^{1/\omega}\Theta^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^{k+1}} \left(A^{-k-1/\omega}\Theta\right)^{\frac{1+\omega}{\omega}} \\
&= \left(\left(A^{1/\omega}\Theta^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^{k+1}} A^k\Theta^{-1} \left(A^{-k-1/\omega}\Theta\right) \left(A^{-k-1/\omega}\Theta\right)^{\frac{1}{\omega}} \\
&= \left(\left(A^{1/\omega}\Theta^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^{k+1}} A^{-1/\omega} \left(A^{-k-1/\omega}\Theta\right)^{1/\omega} \\
&= \left(\left(A^{1/\omega}\Theta^{-1}\right)^{1/\omega} J_0\right)^{(1+\omega)^{k+1}} \left(A^{-(k+1)-1/\omega}\Theta\right)^{1/\omega} = X_{k+1}.
\end{aligned}$$

Hence, by comparison we obtain $J_k \leq X_k$, which was to be proved. ■

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