Finite propagation speed for Leibenson's equation on Riemannian manifolds

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1 Introduction

We are concerned with an evolution equation

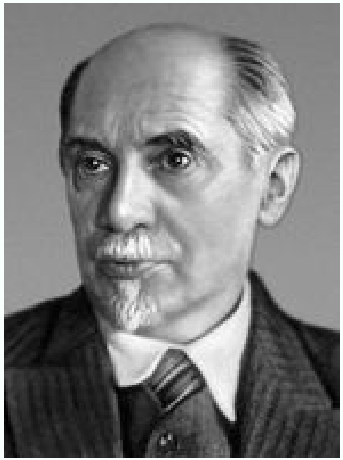
$$\partial_t u = \Delta_p u^q \tag{1}$$

where $p, q > 0$, $u(x,t)$ is an unknown non-negative function, and Δ_p is the p-Laplacian:

$$\Delta_p v = \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right).$$

Equation (1) was introduced by L. S. Leibenson in 1945 in order to describe filtration of turbulent compressible fluid through a porous medium. The physical meaning of u is the *volumetric moisture content*, i.e. the (infinitesimal) fraction of volume of the medium taken by the liquid.

Parameter p characterizes the turbulence of a flow while $q-1$ is the index of *polytropy* of the liquid, that determines relation $PV^{q-1} = \text{const}$ between volume V and pressure P.



Leonid Samuilovich Leibenson

The physically interesting values of p and q are as follows: $\frac{3}{2} \le p \le 2$ and $q \ge 1$.

The case $p = 2$ corresponds to laminar flow (=absence of turbulence). In this case (1) becomes a *porous medium* equation $\partial_t u = \Delta u^q$, if $q > 1$, and the classical heat equation $\partial_t u = \Delta u$ if $q = 1$.

From mathematical point of view, the entire range $p > 1$, $q > 0$ is interesting.

G.I.Barenblatt constructed in 1952 spherically symmetric self-similar solutions of (1) in $\mathbb{R}^n$ that are nowadays called *Barenblatt solutions.* Let us assume that

$$\boxed{q\,(p-1) > 1}.$$

Grigory Isaakovich Barenblatt

In this case the Barenblatt solution is as follows:

$$u\,(x,t) = \frac{1}{t^{n/\beta}} \left(C - \kappa \left(\frac{|x|}{t^{1/\beta}} \right)^{\frac{p}{p-1}} \right)_+^{\gamma},$$

where $C > 0$ is any constant, and

$$\beta = p + n\,[q\,(p-1) - 1]\ , \quad \gamma = \frac{p-1}{q(p-1)-1}, \quad \kappa = \frac{q(p-1)-1}{pq} \beta^{-\frac{1}{p-1}}. \tag{2}$$

Parameter β determines the space/time scaling and is analogous to the *walk dimension*.

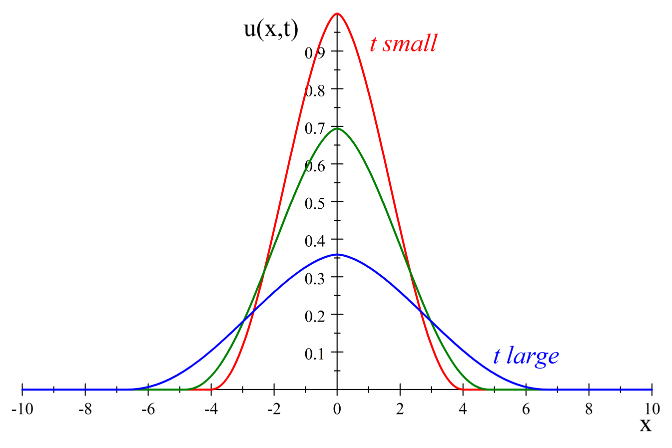
It is obvious that for the Barenblatt solution

$$u(x,t) = 0 \ \text{ for } \ |x| > ct^{1/\beta}$$

so that $u(\cdot, t)$ has a *compact support* for any t.

One says that u has a *finite propagation speed.*

Here are the graphs of function $x \mapsto u(x,t)$ for different values of t in the case $n = 1$.



In the case $q\,(p-1) < 1$, we have $\gamma, \kappa < 0$, and the Barenblatt solution

$$u\,(x,t) = \frac{1}{t^{n/\beta}} \left(C + |\kappa| \left(\frac{r}{t^{1/\beta}} \right)^{\frac{p}{p-1}} \right)^{\gamma}$$

is positive for all x, t. In the borderline case $q\,(p-1) = 1$, the Barenblatt solution is

$$u\,(x,t) = \frac{1}{t^{n/p}} \exp \left(-c \left(\frac{r}{t^{1/p}} \right)^{\frac{p}{p-1}} \right),$$

where $c = (p-1)^2\, p^{-\frac{p}{p-1}}$. Hence, if $q\,(p-1) \leq 1$ then u has infinite propagation speed.

2 p-Laplacian on Riemannian manifolds

Our goal is to investigate finite propagation speed for Leibenson's equation (1) on an arbitrary Riemannian manifold M. Solutions are understood in a certain weak sense.

Consider first the case $q = 1$, that is, the following evolution equation for p-Laplacian:

$$\partial_t u = \Delta_p u.$$

For this equation the following result was known.

Theorem 1 (S.Dekkers 2005) *Let $p > 2$ and let $u(x,t)$ be a bounded non-negative solution to $\partial_t u = \Delta_p u$ on $M \times \mathbb{R}_+$ with initial function $u_0 = u(\cdot, 0)$.*

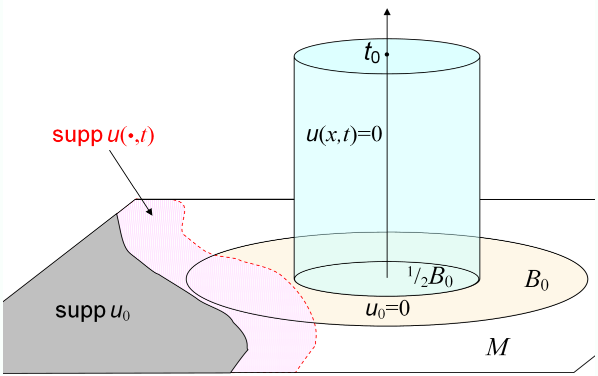
Let $B_0 = B(x_0, R)$ be a precompact ball in M such that $u_0 = 0$ in B_0. Then

$$u = 0 \text{ in } \tfrac{1}{2}B_0 \times [0, t_0],$$

where

$$t_0 = \eta R^p \|u_0\|_{L^\infty(M)}^{-(p-2)}$$

and $\eta > 0$ depends on the intrinsic geometry of B_0.



Hence, solution u has a finite propagation speed inside B_0, and the speed of propagation depends on the geometry of B_0 via the constant η.

For any set $K \subset M$, denote by K_r the open r-neighborhood of K.

Corollary 2 *Let M be complete and non-compact, $p > 2$, and $K = \operatorname{supp} u_0$ be compact.*

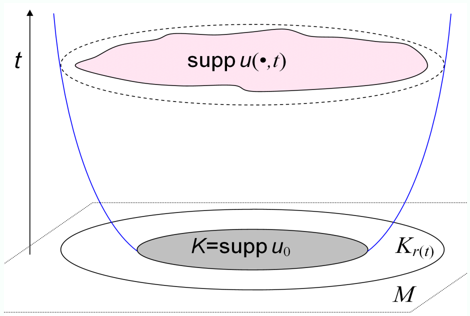
Then there exists an increasing function

$$r : (0, T) \longrightarrow \mathbb{R}_+$$

for some $T \in (0, \infty]$, such that

$$\operatorname{supp} u\,(\cdot, t) \subset K_{r(t)}$$

for all $t \in (0, T)$.



If $Ricci_M \geq 0$ then $r(t) = Ct^{\frac{1}{p}}$ and $T = \infty$.

The function $r(t)$ is called a *propagation rate* of u. Using the Barenblatt solution in $\mathbb{R}^n$, one obtains that a propagation rate in $\mathbb{R}^n$ for large t is

$$r(t) = Ct^{\frac{1}{p+n(p-2)}}$$

so that the result of Corollary 2 is not sharp in this case.

3 Main result

On an arbitrary manifold M of dimension n, consider Leibenson's equation

$$\partial_t u = \Delta_p u^q, \tag{3}$$

where we assume that $p > 2$ and $\frac{1}{p-1} < q \leq 1$. In particular, $q(p-1) - 1 > 0$.

Theorem 3 *Let u be a bounded non-negative subsolution of (3) in $M \times \mathbb{R}_+$, and assume that $u_0 := u(\cdot, 0) \in L^1(M)$. Let B_0 be a precompact ball of radius R s.t. $u_0 = 0$ in B_0. Then $u = 0$ in $\frac{1}{2}B_0 \times [0, t_0]$, where*

$$t_0 = \eta R^p \mu(B_0)^{\frac{q(p-1)-1}{\sigma}} \|u_0\|_{L^\sigma(M)}^{-[q(p-1)-1]}. \tag{4}$$

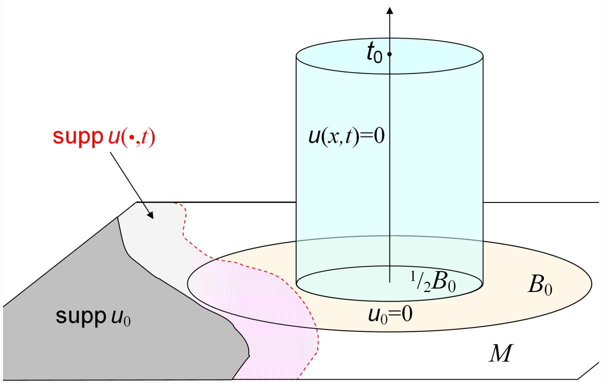
Here σ is any real such that

$$\boxed{\sigma \geq 1 \text{ and } \sigma > q(p-1) - 1}$$

and $\eta = \eta(B_0, p, q, n, \sigma) > 0$.

Besides, the value $\sigma = \infty$ is also included and in this case

$$t_0 = \eta R^p \|u_0\|_{L^\infty(M)}^{-[q(p-1)-1]}.$$



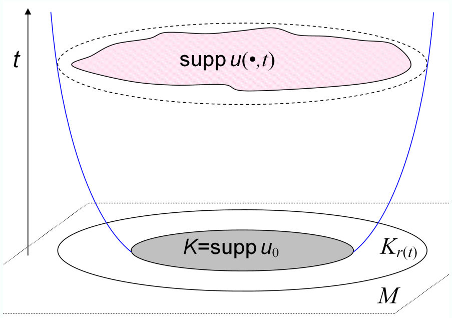
Theorem 1 is a particular case of Theorem 3 for $q = 1$ and $\sigma = \infty$.

In the next two statements M is complete and non-compact, and $K = \operatorname{supp} u_0$ is compact.

Corollary 4 *There exists $T \in (0, T]$ and an increasing function $r : (0, T) \to \mathbb{R}_+$ such that*

$$\operatorname{supp} u(\cdot, t) \subset K_{r(t)}$$

for all $t \in (0, T)$.



Let us refer to $r(t)$ as a *propagation rate* of solution u.

Corollary 5 *Assume that $Ricci_M \geq 0$. Fix $x_0 \in K$ and assume that*

$$\mu(B(x_0, r)) \geq cr^\alpha \quad \text{for all } r \geq r_0,$$

with some $c, \alpha > 0$. Then a propagation rate is $r(t) = Ct^{1/\beta}$ for $t \geq t_0$, where

$$\beta = p + \alpha \frac{q(p-1) - 1}{\sigma}. \tag{5}$$

Recall that in $\mathbb{R}^n$ a propagation rate is $r(t) = Ct^{1/\beta}$ where

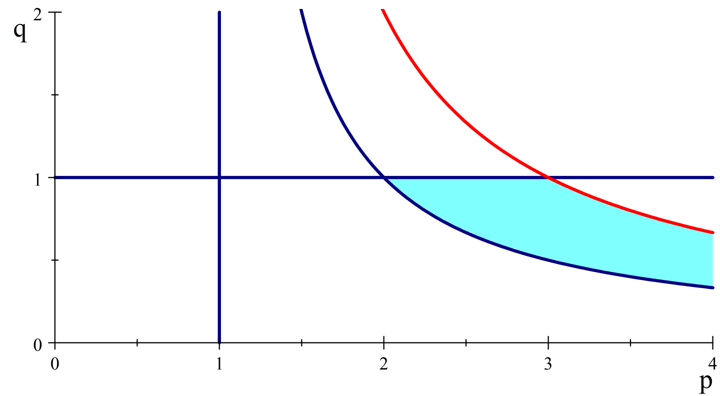
$$\beta = p + n\left[q(p-1) - 1\right].$$

Since in $\mathbb{R}^n$ $\alpha = n$, we see that the value of β in (5) is sharp if $\sigma = 1$.

We can take $\sigma = 1$ in Theorem 3 provided $q(p-1) - 1 < 1$, that is, when

$$1 < q(p-1) < 2.$$

This range of p, q is shown here:



For this range of p, q, we obtain a sharp propagation rate not only in $\mathbb{R}^n$ but also in a large class of models with *Ricci* ≥ 0, with any $\alpha \in (0, n]$.

Conjecture 6 *The statement of Theorem 3 holds for* $\sigma = 1$ *and for all*

$$p > 1 \quad and \quad q > \frac{1}{p-1}.$$

4 Mean value inequality

The main ingredient in the proof of Theorem is the following mean value inequality.

Theorem 7 *Let $B = B(x_0, r)$ be a precompact ball in M.*

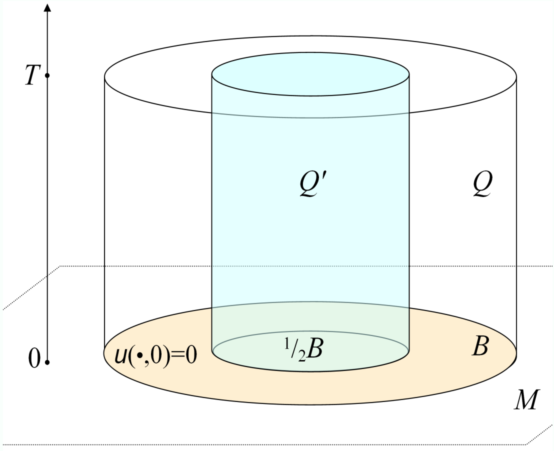
Let u be a non-negative bounded subsolution of (3) in the cylinder

$$Q = B \times [0, T],$$

and let $u(\cdot, 0) = 0$ in B. Then, for the cylinder

$$Q' = \tfrac{1}{2} B \times [0, T],$$

the following inequality holds:



$$\|u\|_{L^\infty(Q')} \leq \left(\frac{C}{\mu(B) r^p} \int_Q u^{\lambda + [q(p-1)-1]} \right)^{1/\lambda}, \tag{6}$$

where $\lambda > 0$ is any and C depends on p, q, λ and on the intrinsic geometry of B.

Proof of Theorem 7 starts with the following Lemma.

Lemma 8 *Let u be a non-negative subsolution of (3).*

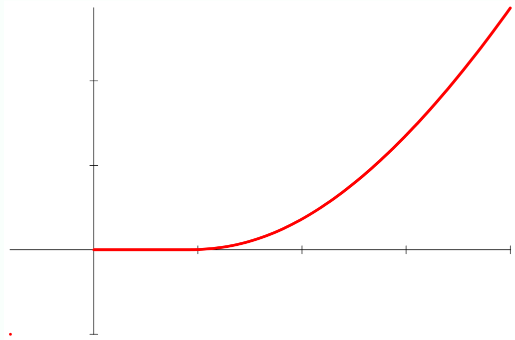
Set

$$a = \frac{q(p-1)-1}{p-2}.$$

If $0 < a \le 1$ then the function

$$v = (u^a - \theta)_+^{1/a}$$

is a subsolution for any $\theta > 0$.



Function $f_\theta(s) = (s^a - \theta)_+^{1/a}$

It satisfies $f_{\theta_1} \circ f_{\theta_2} = f_{\theta_1 + \theta_2}$

The condition $0 < a \le 1$ holds, in particular, in the case when

$$p > 2, \quad q(p-1) > 1 \text{ and } q \le 1.$$

For the p-Laplacian case, that is, when $q = 1$, we have $a = 1$. In this case it is well known that $v = (u - \theta)_+$ is a subsolution. If also $p = 2$ that is, if (3) is the heat equation then $v = f(u)$ is a subsolution for any convex f.

Sketch of proof of Theorem 7. Fix some $\theta > 0$ and define a sequence $\{u_k\}_{k=0}^{\infty}$ of functions:

$$u_0 = u, \quad u_k = \left(u_{k-1}^a - 2^{-k}\theta\right)_+^{1/a} \text{ for } k \geq 1$$

It is easy to see that $u_k = \left(u^a - \left(1 - 2^{-k}\right)\theta\right)_+^{1/a}$.

Consider a decreasing sequence of radii

$$r_k = \left(\tfrac{1}{2} + 2^{-k-1}\right) r$$

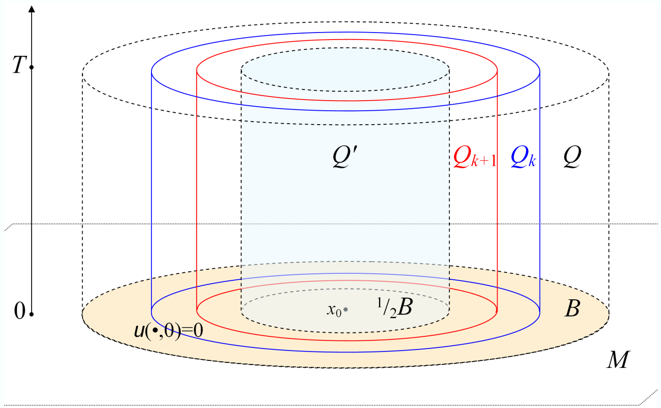
so that $r_0 = r \geq r_k \searrow \frac{1}{2}r$, and cylinders

$$Q_k = B\left(x_0, r_k\right) \times [0, t]$$

so that

$$Q_0 = Q \supset Q_k \searrow Q'$$

as $k \to \infty$.



Set

$$J_k = \int_{Q_k} u_k^{\lambda + [q(p-1)-1]}.$$

Clearly, $J_{k+1} \leq J_k$. Using a Caccioppoli type inequality for u_k and u_{k+1} as well as a certain Faber-Krahn type inequality for Δ_p in B (which reflects the intrinsic geometry of B), we prove that

$$J_{k+1} \leq \frac{CA^k}{\left(\mu(B)\theta^{\frac{\lambda}{a}} r^p\right)^{\nu}} J_k^{1+\nu},$$

where $\nu = p/n$ is the Faber-Krahn exponent for Δ_p and C, A are some constants.

Analyzing this recursive inequality we show that if

$$\theta \geq \left(\frac{CJ_0}{\mu(B)r^p}\right)^{\frac{a}{\lambda}}, \tag{7}$$

then $J_k \to 0$ as $k \to \infty$, which implies

$$\int_{Q'} \left[(u^a - \theta)_+^{1/a}\right]^{\lambda+q(p-1)-1} = 0,$$

that is, $u^a \leq \theta$ in Q'. Choosing the minimal value of θ in (7), we obtain

$$u \leq \left(\frac{CJ_0}{\mu(B)r^p}\right)^{\frac{1}{\lambda}} = \left(\frac{C}{\mu(B)r^p}\int_Q u^{\lambda+[q(p-1)-1]}\right)^{\frac{1}{\lambda}} \quad \text{in } Q'$$

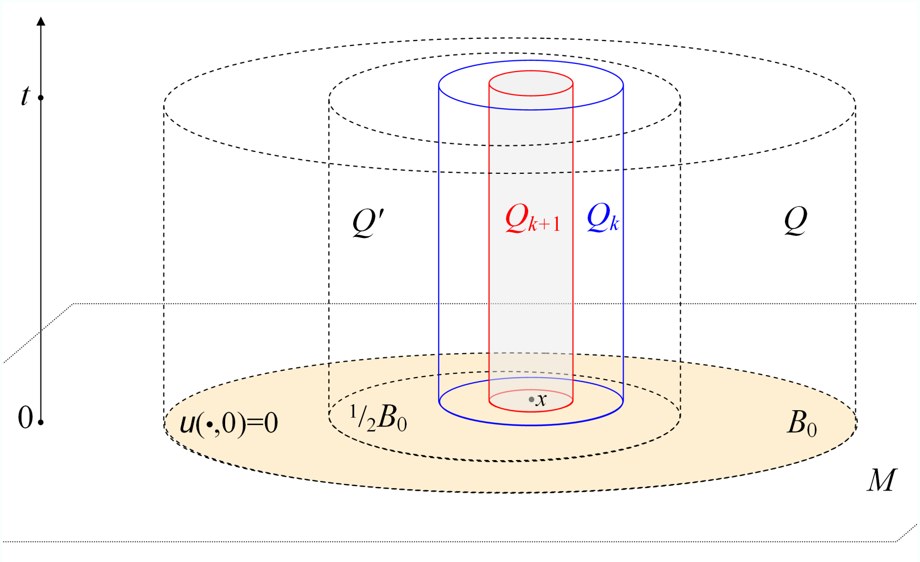
which proves (6).

This method works for $\lambda \geq 2$. The case $0 < \lambda < 2$ is obtained from $\lambda = 2$ using an additional iteration procedure. ■

5 From mean value to finite propagation speed

Sketch of proof of Theorem 3. Given a ball B_0 of radius R, set $r = \frac{1}{2}R$, fix some $t > 0$ and $x \in \frac{1}{2}B_0$, and set for any $k \in \mathbb{N}$

$$Q_k = B(x, 2^{-k}r) \times [0, t] \quad \text{and} \quad J_k = \int_{Q_k} u^\sigma.$$



Applying the mean value inequality (6) in Q_k and $Q'_k = Q_{k+1}$ with

$$\lambda = \sigma - [q(p-1) - 1] > 0$$

we obtain

$$\|u\|_{L^\infty(Q_{k+1})} \leq \left(\frac{C}{\mu\left(B(x, 2^{-k}r)\right)(2^{-k}r)^p} \int_{Q_k} u^\sigma \right)^{1/\lambda}.$$

Raising this to power σ and integrating over Q_{k+1}, we obtain

$$J_{k+1} \leq t\mu(B_0) \left(\frac{C^k}{\mu\left(B_0\right) R^p} J_k \right)^{\sigma/\lambda}.$$

Since $\sigma/\lambda > 1$, iteration of this inequality allows to prove that J_k decays *double* exponentially in k provided $t \leq t_0$ (where t_0 is determined by (4)):

$$J_k = \int_{B(x,2^{-k}r)\times[0,t]} u^\sigma \leq CA^{-(\sigma/\lambda)^k}, \tag{8}$$

and this is true for **all** $x \in \frac{1}{2}B_0$ and $k \geq 0$, with the same constants C and $A > 1$.

For any fixed k, let us cover $\frac{1}{2}B_0 = B(x_0, r)$ by a sequence of balls $B(x_i, 2^{-k}r)$ with some $x_i \in \frac{1}{2}B_0$. The minimal number of such balls is bounded by D^k for some constant D.

Hence, adding up (8) for all $x = x_i$, we obtain

$$\int_{Q'} u^\sigma \le CD^k A^{-(\sigma/\lambda)^k}.$$

This inequality holds for any k. Letting $k \to \infty$ and noticing that the right hand side $\to 0$ thanks to $\sigma/\lambda > 1$ and $A > 1$, we obtain that $u = 0$ in Q'. ■

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