Leibenson's equation on Riemannian manifolds

Alexander Grigor'yan
University of Bielefeld

Chemnitz, October 2023

Based on a joint work with Philipp Sürig

1 Introduction

We are concerned with an evolution equation

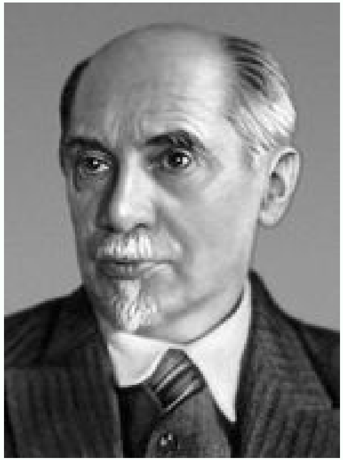
$$\partial_t u = \Delta_p u^q \tag{1}$$

where $p, q > 0$, $u(x,t)$ is an unknown non-negative function, and Δ_p is the p-Laplacian:

$$\Delta_p v = \operatorname{div}\left(|\nabla v|^{p-2}\, \nabla v\right).$$

Equation (1) was introduced by L. S. Leibenson in 1945 in order to describe filtration of turbulent compressible fluid through a porous medium. The physical meaning of u is the *volumetric moisture content*, i.e. the (infinitesimal) fraction of volume of the medium taken by the liquid.

Parameter p characterizes the turbulence of a flow while $q-1$ is the index of *polytropy* of the liquid, that determines relation $PV^{q-1} = \text{const}$ between volume V and pressure P.



Leonid Samuilovich Leibenson

The physically interesting values of p and q are as follows: $\frac{3}{2} \le p \le 2$ and $q \ge 1$.

The case $p = 2$ corresponds to laminar flow (=absence of turbulence). In this case (1) becomes a *porous medium* equation $\partial_t u = \Delta u^q$, if $q > 1$, and the classical heat equation $\partial_t u = \Delta u$ if $q = 1$.

From mathematical point of view, the entire range $p > 1$, $q > 0$ is interesting.

G.I.Barenblatt constructed in 1952 spherically symmetric self-similar solutions of (1) in $\mathbb{R}^n$ that are nowadays called *Barenblatt solutions.* Let us first assume that

$$\boxed{q\,(p-1) > 1}.$$

Grigory Isaakovich Barenblatt

In this case the Barenblatt solution is as follows:

$$u\,(x,t) = \frac{1}{t^{n/\beta}} \left(C - \kappa \left(\frac{|x|}{t^{1/\beta}} \right)^{\frac{p}{p-1}} \right)_+^{\gamma},$$

where $C > 0$ is any constant, and

$$\beta = p + n\,[q\,(p-1) - 1]\ , \quad \gamma = \tfrac{p-1}{q(p-1)-1}, \quad \kappa = \tfrac{q(p-1)-1}{pq} \beta^{-\frac{1}{p-1}}. \tag{2}$$

Parameter β determines the space/time scaling and is analogous to the *walk dimension.*

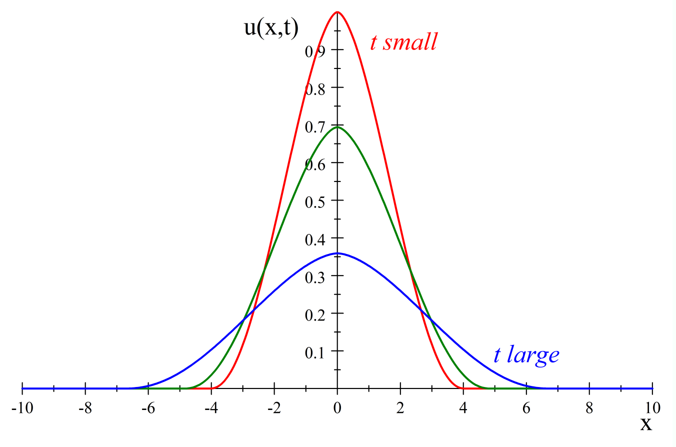
It is obvious that for the Barenblatt solution

$$u(x,t) = 0 \quad \text{for} \quad |x| > ct^{1/\beta}$$

so that $u(\cdot, t)$ has a *compact support* for any t.

One says that u has a *finite propagation speed.*

Here are the graphs of function $x \mapsto u(x,t)$ for different values of t in the case $n = 1$:



In the case $q(p-1) < 1$, we have $\gamma, \kappa < 0$, and the Barenblatt solution

$$u(x,t) = \frac{1}{t^{n/\beta}} \left(C + |\kappa| \left(\frac{r}{t^{1/\beta}} \right)^{\frac{p}{p-1}} \right)^{-|\gamma|}$$

is positive for all x, t. In the borderline case $q(p-1) = 1$, the Barenblatt solution is

$$u(x,t) = \frac{1}{t^{n/p}} \exp \left(-c \left(\frac{r}{t^{1/p}} \right)^{\frac{p}{p-1}} \right),$$

where $c = (p-1)^2 p^{-\frac{p}{p-1}}$. Hence, if $q(p-1) \leq 1$ then u has infinite propagation speed.

2 Propagation speed inside a ball

From now on let M be a geodesically complete Riemannian manifold of dimension n. Consider on M the Leibenson equation

$$\partial_t u = \Delta_p u^q, \tag{3}$$

where we assume that

$$\boxed{p > 1 \text{ and } q > \tfrac{1}{p-1}}, \tag{4}$$

that is, $\delta := q(p-1) - 1 > 0$. Solutions of (3) are understood in a certain weak sense.

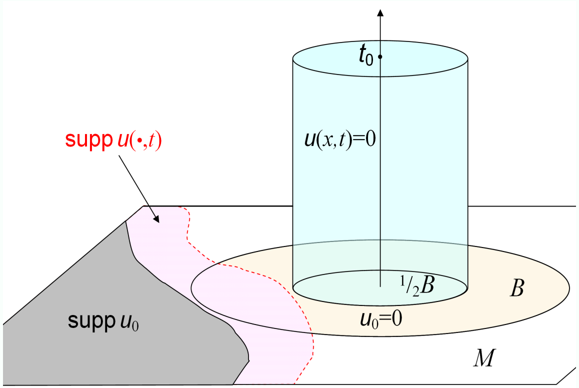
Theorem 1 *Let $u(x,t)$ be a bounded non-negative subsolution of (3) in $M \times \mathbb{R}_+$. Let B be a ball in M of radius R such that $u_0 := u(\cdot, 0) = 0$ in B. Then we have*

$$u(\cdot, t) = 0 \text{ in } \tfrac{1}{2}B \text{ for all } t \le t_0,$$

where

$$t_0 = \eta R^p \|u_0\|_{L^\infty(M)}^{-\delta}$$

and $\eta > 0$ depends on intrinsic geometry of B.



Note that the range (4) of parameters p, q is the same as that in the Barenblatt solutions with a finite propagation speed.

The only previously known case of Theorem 1 was when $p > 2$ and $q = 1$, that is, when (3) is the equation $\partial_t u = \Delta_p u$. In this case a finite propagation speed was proved by S. Dekkers in *Comm. Anal. Geom.* **14** (2005).

Another interesting case is when $p = 2$ and $q > 1$, that is, when (3) is a *porous medium* equation $\partial_t u = \Delta u^q$. Theorem 1 is new in this case.

Remark. The constant η depends on p, q, n as well as on the *normalized Sobolev constant* c_B in B: for any $u \in W_0^{1,p}(B)$

$$\left(\fint_B |\nabla u|^p \right)^{1/p} \geq \frac{c_B}{R} \left(\fint_B |u|^{p\kappa} \right)^{1/p\kappa} \tag{5}$$

where κ is the Sobolev exponent: $\kappa = \frac{n}{n-p}$ if $n > p$ and $\kappa > 1$ is any if $n \leq p$.

Remark. The Leibenson equation (3), that is, $\partial_t u = \Delta_p u^q$ can be equivalently rewritten in the form

$$\partial_t u = \operatorname{div} \left(u^{m-1} |\nabla u|^{p-2} \nabla u \right),$$

where $m = 1 + (q-1)(p-1) = \delta + 3 - p$. The condition $\delta > 0$ is, hence, equivalent to $m + p > 3$. Therefore, Theorem 1 holds for this equation when $p > 1$ and $m + p > 3$.

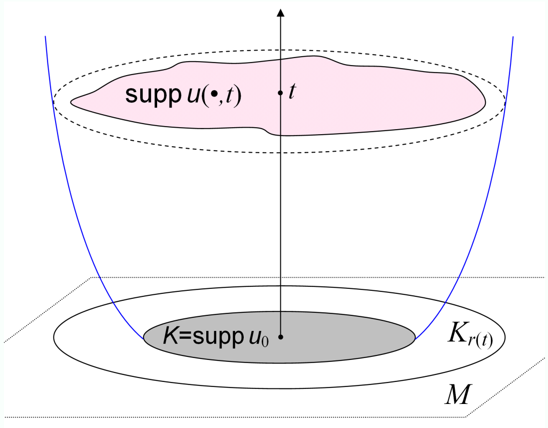
3 Finite propagation speed of support

Let u be a bounded non-negative subsolution of (3) with $u(\cdot, 0) = u_0$. For any set $K \subset M$ and any $r > 0$, denote by K_r the closed r-neighborhood of K.

Corollary 2 *Let* $K := \operatorname{supp} u_0$ *be a compact set. Then there an increasing positive function* $r : (0, T) \to \mathbb{R}_+$ *with some* $T \in (0, \infty]$, *such that*

$$\operatorname{supp} u(\cdot, t) \subset K_{r(t)}$$

for all $t \in (0, T)$.



Function $r(t)$ is referred to as a *propagation rate* of solution u.

Problem 3 Is it true that one can always have $T = \infty$? Either prove it or give a counterexample: a manifold and a solution u such that $\operatorname{supp} u_0$ is compact, while $\operatorname{supp} u(\cdot, t)$ is unbounded for large enough t.

Let M have non-negative Ricci curvature. Then the normalized Sobolev constant c_B in (5) can be taken the same for all balls. Hence, the constant η from Theorem 1 is also the same for all balls, which allows to obtain the following.

Corollary 4 *If $Ricci_M \geq 0$ then any bounded non-negative subsolution u with compactly supported u_0 has a propagation rate $r(t) = Ct^{1/p}$ for all $t > 0$.*

Recall that in $\mathbb{R}^n$ the propagation rate of the Barenblatt solution is

$$r(t) = Ct^{1/\beta},$$

where

$$\beta = p + n\left[q(p-1) - 1\right] = p + n\delta. \tag{6}$$

This implies that, for any bounded non-negative solution u in $\mathbb{R}^n$ with compactly supported u_0, the propagation rate is also $r(t) = Ct^{1/\beta}$ for large t.

Since $p < \beta$, we see that the propagation rate of Corollary 4 is not sharp in $\mathbb{R}^n$.

4 Sharp propagation rate

Instead of the previous conditions (4), we assume here more restricted hypotheses:

$$p > 2 \quad \text{and} \quad \frac{1}{p-1} < q \leq 1.$$

Theorem 5 *Let u be a bounded non-negative subsolution of (3) in $M \times \mathbb{R}_+$, with initial function $u_0 := u(\cdot, 0) \in L^1$. Let B be a ball in M of radius R such that $u_0 = 0$ in B. Then*

$$u(\cdot, t) = 0 \text{ in } \tfrac{1}{2}B \text{ for all } t \leq t_0$$

where

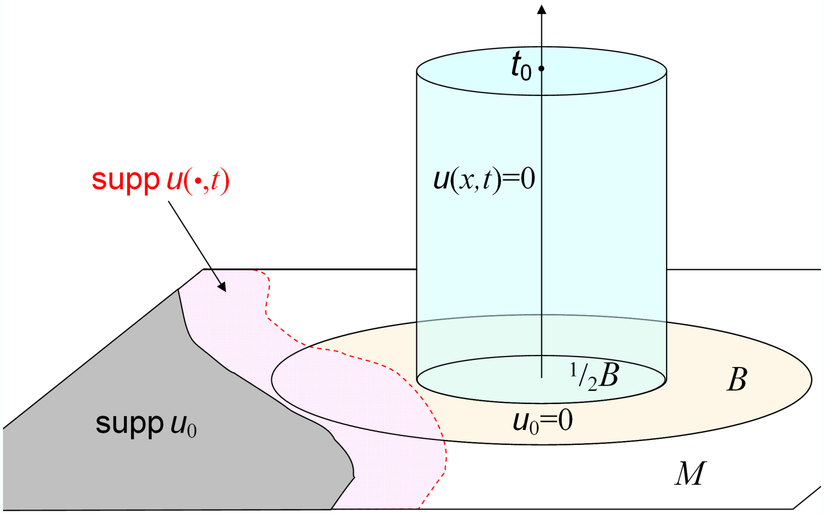
$$t_0 = \eta R^p \mu(B)^{\frac{\delta}{\sigma}} \|u_0\|_{L^\sigma(M)}^{-\delta}.$$

Here σ is any real number such that

$$\sigma \geq 1 \text{ and } \sigma > \delta, \qquad (*)$$

$\delta = q(p-1) - 1 > 0,$

$\eta = \eta(p, q, n, \sigma, c_B) > 0.$



Corollary 6 *Assume that* $Ricci_M \geq 0$. *Fix a point* $x_0 \in \operatorname{supp} u_0$ *and assume that*

$$\mu(B(x_0, r)) \geq cr^\alpha \quad \text{for all } r \geq r_0, \tag{7}$$

with some $\alpha, c > 0$. *Then* u *has a propagation rate* $r(t) = Ct^{1/\beta}$ *for large* t, *where*

$$\beta = p + \alpha \frac{\delta}{\sigma} \tag{8}$$

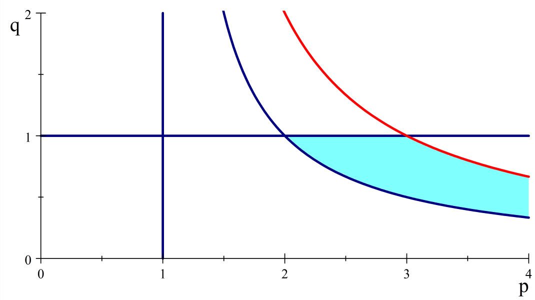
and σ *is as in* $(*)$.

In $\mathbb{R}^n$ we have $\alpha = n$. Setting $\sigma = 1$, we obtain $\beta = p + n\delta$ that matches propagation rate (6) in $\mathbb{R}^n$. However, we can take $\sigma = 1$ in $(*)$ only if $\delta < 1$, that is, if $q(p-1) < 2$.

The next diagram shows the following range of p, q:

$$p > 2 \quad \text{and} \quad 1 < q(p-1) < 2.$$

For these p, q, we obtain a sharp propagation rate not only in $\mathbb{R}^n$, but also in a large class of model manifolds satisfying $Ricci_M \geq 0$ as well as (7) with any $\alpha \in (0, n]$.



Conjecture 7 *The result of Theorem 5 holds for all* $p > 1$, $q > \frac{1}{p-1}$ *and for* $\sigma = 1$.

5 Mean value inequality

The main ingredient of the proof of Theorem 1 is the following mean value inequality. We assume here that $p > 1$ and $\delta \geq 0$.

Lemma 8 *Let $B = B(x_0, R)$ be a ball in M. Let $u(x,t)$ be a non-negative bounded subsolution of (3) in the cylinder*

$$Q = B \times [0, T]$$

such that $u_0 \equiv u(\cdot, 0) = 0$ *in* B.

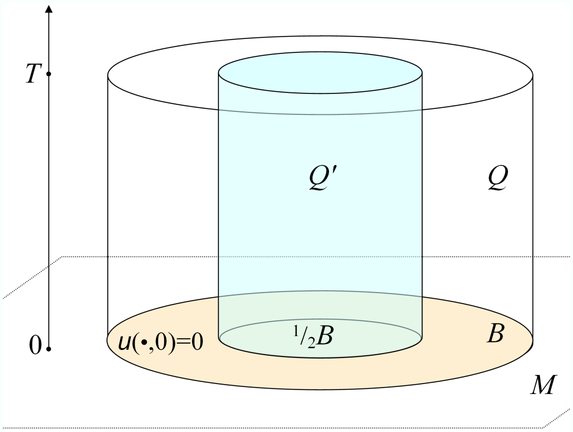
Then, for the cylinder

$$Q' = \tfrac{1}{2} B \times [0, T]$$

and for any

$$\lambda \geq \max(p, pq),$$

the following inequality holds:



$$\|u\|_{L^\infty(Q')} \leq C \left(\frac{T}{R^p}\right)^{1/\lambda} \|u\|_{L^\infty(Q)}^{\delta/\lambda} \left(⨍_Q u^\lambda \right)^{1/\lambda}, \tag{9}$$

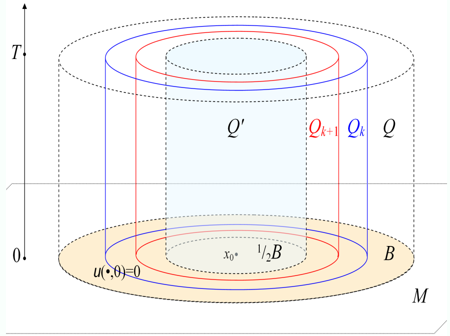
where $C = C(p, q, n, \lambda, c_B)$.

For the proof we use the Sobolev inequality inside B and Moser's iteration argument.

For that consider a shrinking sequence of cylinders $\{Q_k\}_{k=0}^{\infty}$ interpolating between $Q_0 = Q$ and $Q_\infty = Q'$, and prove that

$$\fint_{Q_{k+1}} u^{\sigma(1+\nu)} \leq C(\cdots) \left(\fint_{Q_k} u^\sigma\right)^{1+\nu} \qquad (*)$$

for $\sigma \gg 1$ and $\nu > 0$ that comes from the Sobolev inequality.



In the classical Moser argument, one proves $(*)$ first for $\sigma = 2$ and then applies this inequality also to $u^{\sigma/2}$ with any $\sigma > 2$ because $u^{\sigma/2}$ is also a subsolution. This allows to set in $(*)$ $\sigma = \lambda (1+\nu)^k$ and to reach $\|u\|_{L^\infty(Q')}$ by iterations as $k \to \infty$.

In our case this trick is not possible: *no power of subsolution is again a subsolution.* Hence, we need to prove $(*)$ directly for any σ and to compute carefully the constant $C = C(\sigma)$ in $(*)$. It turns out that $C \simeq \sigma^A$ for some A. Surprisingly, this moderate growth of C with σ still allows to complete the iteration argument and to obtain (9).

Using $\left(\fint_Q u^\lambda\right)^{1/\lambda} \leq \|u\|_{L^\infty(Q)}$, we obtain from (9)

$$\|u\|_{L^\infty(Q')} \leq C \left(\frac{T}{R^p}\right)^{1/\lambda} \|u\|_{L^\infty(Q)}^{1+\delta/\lambda}. \qquad (10)$$

6 From mean value to finite propagation speed

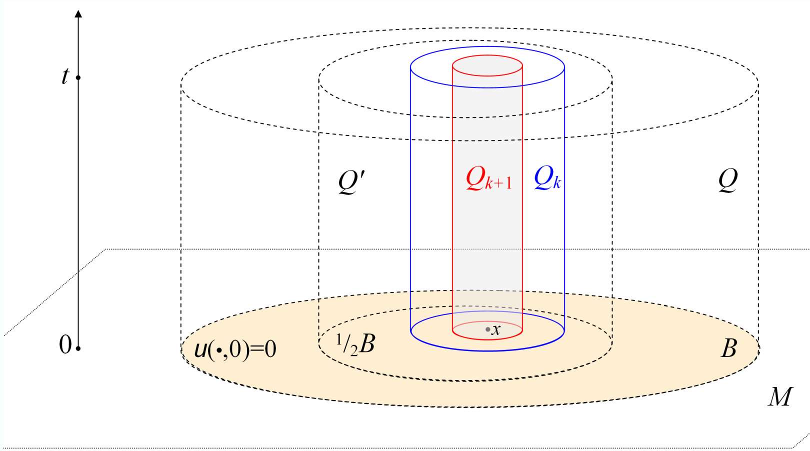
Sketch of proof of Theorem 1. Set $r = \frac{1}{2}R$ and fix for a while a point $x \in \frac{1}{2}B$.

Hence, we have $B(x, r) \subset B$.

Fix also some $t > 0$ and set

$$Q_k = B(x, 2^{-k}r) \times [0, t],$$

$$J_k = \|u\|_{L^\infty(Q_k)}.$$



Let λ be as it is needed for Lemma 8. Then by (10)

$$J_{k+1} \leq C \left(\frac{t}{(2^{-k}R)^p}\right)^{1/\lambda} J_k^{1+\frac{\delta}{\lambda}} = C2^{k/\lambda} \left(\frac{t}{R^p}\right)^{1/\lambda} J_k^{1+\frac{\delta}{\lambda}}.$$

Iterating this inequality, we obtain an upper bound of J_k via J_0 that implies the following: if

$$C \left(\frac{t}{R^p}\right)^{1/\lambda} \leq 2^{-1/\delta} J_0^{-\delta/\lambda} \tag{11}$$

then, for all k,

$$J_k \leq 2^{-k/\delta} J_0. \tag{12}$$

The condition (11) is equivalent to

$$t \leq \eta R^p J_0^{-\delta}. \tag{13}$$

Since $J_0 = \|u\|_{L^\infty(Q)} \leq \|u_0\|_{L^\infty(M)}$ and, hence,

$$t_0 = \eta R^p \|u_0\|_{L^\infty(M)}^{-\delta} \leq \eta R^p J_0^{-\delta}$$

we see that (13) is satisfied for $t \leq t_0$. For such t, we obtain from (12) that, for any k,

$$\|u\|_{L^\infty\left(B(x,2^{-k}r)\times[0,t]\right)} \leq 2^{-k/\delta} \|u_0\|_{L^\infty}.$$

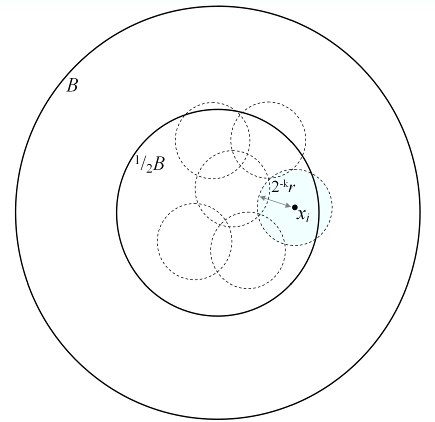
For any k, we cover the ball $\frac{1}{2}B$ by a finite sequence of balls $B(x_i, 2^{-k}r)$ with $x_i \in \frac{1}{2}B$. Since for all i

$$\|u\|_{L^\infty\left(B(x_i,2^{-k}r)\times[0,t]\right)} \leq 2^{-k/\delta} \|u_0\|_{L^\infty}$$

we obtain that

$$\|u\|_{L^\infty\left(\frac{1}{2}B\times[0,t]\right)} \leq 2^{-k/\delta} \|u_0\|_{L^\infty}.$$

As $k \to \infty$, we obtain that $u = 0$ in $\frac{1}{2}B \times [0,t]$, which was to be proved. ■



7 Mean value inequality 2

The main ingredient in the proof of Theorem 5 is the following version of the mean value inequality. We assume here that $p > 2$ and $\frac{1}{p-1} < q \leq 1$.

Lemma 9 *Let $B = B(x_0, R)$ be a precompact ball in M.*

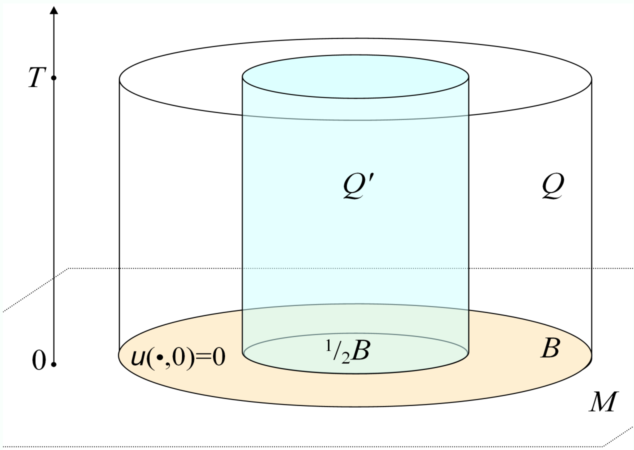
Let u be a non-negative bounded subsolution of (3) in the cylinder

$$Q = B \times [0, T],$$

and let $u(\cdot, 0) = 0$ in B. Then, for the cylinder

$$Q' = \tfrac{1}{2}B \times [0, T],$$

the following inequality holds:



$$\|u\|_{L^\infty(Q')} \leq C \left(\tfrac{T}{R^p}\right)^{1/\lambda} \left(⨍_Q u^{\lambda+\delta}\right)^{1/\lambda}, \tag{14}$$

where $\lambda > 0$ is any, $\delta = q(p-1) - 1$, and $C = C(p, q, n, \lambda, c_B)$.

In the proof of Lemma 9 we use the following lemma.

Lemma 10 *Let u be a non-negative subsolution of* (3)*.*

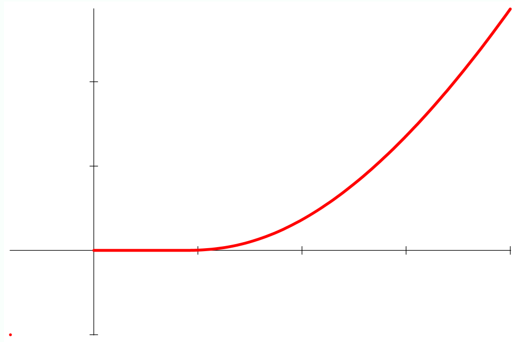
Set

$$a = \frac{q(p-1)-1}{p-2}.$$

If $0 < a \leq 1$ *then the function*

$$v = (u^a - \theta)_+^{1/a}$$

is a subsolution for any $\theta > 0$*.*



Function $f_\theta(s)=(s^a-\theta)_+^{1/a}$

It satisfies $f_{\theta_1} \circ f_{\theta_2} = f_{\theta_1+\theta_2}$

The condition $0 < a \leq 1$ holds, in particular, in the case when

$$p > 2 \quad \text{and} \quad \frac{1}{p-1} < q \leq 1$$

For the p-Laplacian case, that is, when $q = 1$, we have $a = 1$. In this case it is well known that $v = (u - \theta)_+$ is a subsolution. If also $p = 2$ that is, if (3) is the heat equation, then $v = f(u)$ is a subsolution for any convex f.

Sketch of proof of Lemma 9. Fix some $\theta > 0$ and define a sequence $\{u_k\}_{k=0}^{\infty}$ of functions:

$$u_0 = u, \quad u_k = \left(u_{k-1}^a - 2^{-k}\theta\right)_+^{1/a} \text{ for } k \geq 1$$

It is easy to see that $u_k = \left(u^a - \left(1 - 2^{-k}\right)\theta\right)_+^{1/a}$.

Consider a decreasing sequence of radii

$$r_k = \left(\tfrac{1}{2} + 2^{-k-1}\right) R$$

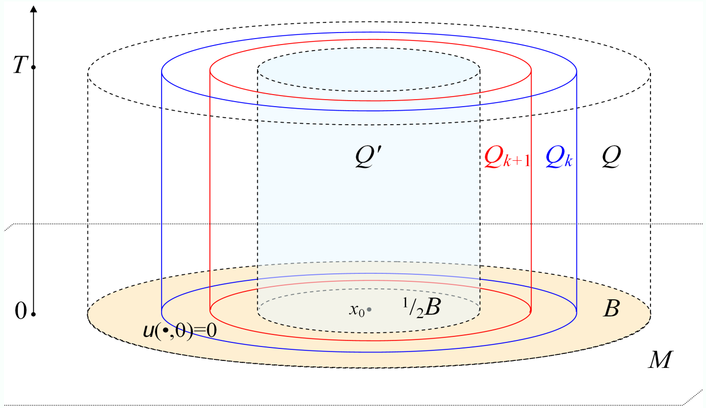
so that $r_0 = r \geq r_k \searrow \frac{1}{2}R$, and cylinders

$$Q_k = B\left(x_0, r_k\right) \times [0, t]$$

so that

$$Q_0 = Q \supset Q_k \searrow Q'$$

as $k \to \infty$.



Set

$$J_k = \int_{Q_k} u_k^{\lambda+\delta}.$$

Clearly, $J_{k+1} \leq J_k$. Using a Caccioppoli type inequality for u_k and u_{k+1} as well as a certain Faber-Krahn type inequality for Δ_p in B (which reflects the intrinsic geometry of B), we prove that

$$J_{k+1} \leq \frac{CA^k}{\left(\mu(B)\theta^{\frac{\lambda}{a}} r^p\right)^{\nu}} J_k^{1+\nu},$$

where $\nu > 0$ is the Faber-Krahn exponent for Δ_p, and C, A are some constants.

Analyzing this recursive inequality, we show that if

$$\theta \geq \left(\frac{CJ_0}{\mu(B)r^p}\right)^{\frac{a}{\lambda}}, \tag{15}$$

then $J_k \to 0$ as $k \to \infty$, which implies

$$\int_{Q'} \left[(u^a - \theta)_+^{1/a}\right]^{\lambda+\delta} = 0,$$

that is, $u^a \leq \theta$ in Q'. Choosing the minimal value of θ in (15), we obtain

$$u \leq \left(\frac{CJ_0}{\mu(B)r^p}\right)^{\frac{1}{\lambda}} = \left(\frac{C}{\mu(B)r^p}\int_Q u^{\lambda+\delta}\right)^{\frac{1}{\lambda}} \quad \text{in } Q'$$

which proves (14).

This method works for $\lambda \geq 2$. The case $0 < \lambda < 2$ is obtained from $\lambda = 2$ using an additional iteration procedure. ■