# PARABOLICITY AND STOCHASTIC COMPLETENESS OF MANIFOLDS IN TERMS OF THE GREEN FORMULA

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Contents



## 1. Introduction

Let $(M, g)$ be a connected smooth Riemannian manifold without boundary. A weighted manifold is a triple $(M, g, \mu)$ with underlying manifold $M$, the Riemannian metric $g$, and a measure

$$d\mu = \psi dv_g,$$

where $dv_g$ is the Riemannian volume and $\psi$ is a positive smooth function on $M$.

A weighted manifold[1] $M$ carries a natural second-order elliptic operator called the *(weighted) Laplace operator*

$$\Delta = \mathrm{div}_\mu \nabla,$$

where $\nabla$ is the gradient associated with $g$ and $\mathrm{div}_\mu$ is the weighted divergence, that is defined as the adjoint operator to $\nabla$ with respect to measure $\mu$ (see (9) below).

We say that the weighted manifold $M$ *is parabolic* if $\Delta$ does not admit a positive fundamental solution. We say that $M$ *is stochastically complete* if any bounded solution $u(t, x)$ in $[0, +\infty) \times M$ of the associated heat equation $\frac{\partial u}{\partial t} = \Delta u$ is uniquely determined by the initial value $u|_{t=0}$. Equivalently, this means that $e^{t\Delta_D} 1 \equiv 1$ where $\Delta_D$ is the Dirichlet Laplacian and $e^{\Delta_D t}$ is the associated heat semigroup (see Section 2 for details).

Any parabolic manifold is stochastically complete but the opposite implication is not true. For example, all spaces $\mathbb{R}^n$ (with Euclidean measure) are stochastically complete, whereas $\mathbb{R}^n$ is parabolic if and only if $n = 1, 2$.

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[1] We frequently write $M$ for both $(M, g)$ and $(M, g, \mu)$ when this does not cause a confusion.

Let $\{X_t\}$ be the minimal Brownian motion on $M$, that is, the diffusion process, generated by $\Delta_D$. Then it is well known, that the parabolicity of $M$ is equivalent to the recurrence of $X_t$, and the stochastic completeness of $M$ is equivalent to the non-explosion property of $X_t$, that is, to the fact that the lifetime of the process is $\infty$.

If $(M,g)$ is geodesically complete, then one can state sufficient conditions for the parabolicity and stochastic completeness in terms of the volume function

$$V(r)=\mu\left(B\left(x_0,r\right)\right),$$

where $B\left(x_0,r\right)$ is the geodesic ball of radius $r$ centered at a fixed point $x_0\in M$. Namely, the following implications are true:

$$\int^{\infty}\frac{rdr}{V(r)} = \infty \Rightarrow \text{ the parabolicity of } M \tag{1}$$

$$\int^{\infty}\frac{rdr}{\log V(r)} = \infty \Rightarrow \text{ the stochastic completeness of } M. \tag{2}$$

For example, (1) holds provided $V(r)\leq Cr^2$, and (2) holds if $V(r)\leq\exp\left(Cr^2\right)$. That the condition $V(r)\leq Cr^2$ implies the parabolicity was first proved by S.Y.Cheng and S.T.Yau [5]. The sharp sufficient condition (1) for parabolicity was proved in [16],[17],[28],[42]. The sufficient condition $V(r)\leq\exp\left(Cr^2\right)$ for the stochastic completeness was proved in [8],[25],[29],[41] (see also an earlier result [15]), and the sharp result (2) was established in [18].

For a model manifold with the pole at $x_0$, both the parabolicity and stochastic completeness can be characterized solely in terms of the function $V(r)$ and its derivative (see Proposition 4.1 and [22]).

Let $d$ be the Riemannian distance of $M$ and $(\overline{M},d)$ be the completion of the metric space $(M,d)$. The *Cauchy boundary* of $M$ is defined by

$$\partial_C M=\overline{M}\setminus M.$$

Note that $M$ is geodesically complete if and only if $\partial_C M=\phi$.

We will define the notion of capacity of $\partial_C M$ in Section 2 and say $\partial_C M$ is *polar* if it has capacity 0. The stochastic completeness and parabolicity can be violated for two reasons:
- a fast volume growth at $\infty$;
- the non-polarity of $\partial_C M$.

It is easy to see that if $\partial_C M$ is bounded and polar, then the volume tests (1) and (2) for the parabolicity and stochastic completeness, respectively, remain the same (see Remark 2.3).

There are several ways to characterize the parabolicity and the stochastic completeness in a uniform way; for instance, using the Liouville property for Schrödinger operators (see for e.g., [21]), curvature bounds [2],[9],[26],[27], [43], and the existence of cut-off functions satisfying certain properties [35].

The main purpose of the present paper is to present and prove a new characterization of these properties in terms of Green's formula with the boundary at infinity. In the statements below we understand the Laplace operator $\Delta$ in the distributional (weak) sense. We denote $L^p=L^p\left(M,\mu\right)$ and suppress the $M$ and $d\mu$ from the integrals when it does not create a confusion.

**Theorem 1.1.** *$M$ is parabolic if and only if*

$$\int\Delta u=0 \textit{ for all } u\in L^{\infty} \textit{ such that } \Delta u\in L^1. \tag{3}$$

Let $M$ be a bounded open subset of $\mathbb{R}^n$ with a smooth boundary $\partial M$. Then, for any function $u \in C^2\left(\overline{M}\right)$, we have by the classical Green formula

$$\int_M \Delta u \, d\mu = \int_{\partial M} \frac{\partial u}{\partial \nu} d\sigma$$

where $\nu$ is the outward normal vector field on $\partial M$ and $\sigma$ is the area on $\partial M$. We see that the condition (3) never holds, and the reason is the presence of the Dirichlet boundary $\partial M$.

In this example $M$ is not geodesically complete. However, even if $M$ is geodesically complete, still one can have a non-zero value for $\int \Delta u$ due to certain properties of $M$ at $\infty$. For example, in $\mathbb{R}^3$ it is easy to construct a bounded super-harmonic function $u(x)$ such that $u(x) = |x|^{-1}$ for large $|x|$. For this function we have $\int \Delta u < 0$ so that (3) fails.

Let $W^{1,2}$ be the space of $L^2$ functions $u$ whose gradient $\nabla u$ is also in $L^2$. The space $W_0^{1,2}$ is the closure of the set $C_0^\infty$ of smooth functions with compact support in $W^{1,2}$. The restriction of $\Delta$ to $\{u \in W_0^{1,2} : \Delta u \in L^2\}$ is referred to as the *Dirichlet Laplacian* $\Delta_D$.

**Theorem 1.2.** *$M$ is stochastically complete if and only if*

$$\int \Delta u = 0 \quad \textit{for all } u \in D(\Delta_D) \cap L^1 \textit{ such that } \Delta u \in L^1. \tag{4}$$

Examples to this theorem will be given below after Corollaries 1.3 and 1.6, and an extension will be given in Proposition 3.2. Propositions 4.2 and 4.3 show that the conditions $u \in L^1$ and $\nabla u \in L^2$ in (4) cannot be dropped.

Since on a geodesically complete manifold $D(\Delta_D) = \{u \in L^2 : \Delta u \in L^2\}$, we obtain from Theorem 1.2 the following consequence.

**Corollary 1.3.** *If $M$ is geodesically complete, then it is stochastically complete if and only if*

$$\int \Delta u = 0 \quad \textit{for all } u \in L^1 \cap L^2 \textit{ such that } \Delta u \in L^1 \cap L^2.$$

A similar example to Theorem 1.1 can be obtained on a model manifold (see Section 4). Indeed, the Green function $g(x, y)$ of a stochastic incomplete model manifold is integrable at $\infty$. This allows to construct a bounded super-harmonic function $u(x)$ such that $u(x) = g(x, y)$ for a fixed $y \in M$ and large enough $d(x, y)$. It follows that $u \in L^1 \cap L^\infty$ (this implies that $u \in L^2$ ) and $\Delta u$ has compact support in particular, $\Delta u \in L^1 \cap L^2$, while $\int \Delta u < 0$. (See Proposition 4.1 for more detail.) The assumption of geodesic completeness in Corollary 1.3 can not be replaced by the condition that $\partial_C M$ is compact and polar (see Proposition 4.3).

**Theorem 1.4.** *If $M$ is geodesically complete, then it is stochastically complete if and only if*

$$\int \Delta u = 0 \quad \textit{for all } u \in L^1 \cap L^2 \textit{ such that } \nabla u \in L^1 \cup L^2 \textit{ and } \Delta u \in L^1. \tag{5}$$

Theorem 1.4 remains true for a geodesically incomplete manifold if $\partial_C M$ is polar and if $u$ satisfies in addition to (5) that $u \in L^\infty(B)$ on a neighborhood $B$ of $\partial_C M$; however, the condition $u \in L^1$ from (5) can not be removed (see Proposition 4.2).

In the next theorem, we are concerned with conditions for the identity $W^{1,2} = W_0^{1,2}$. It is known that this is satisfied for geodesically complete manifolds [1]. The relation with parabolicity and stochastic completeness is given by

$$\text{parabolicity} \quad \Rightarrow \quad \text{stochastic completeness} \quad \Rightarrow W^{1,2} = W_0^{1,2},$$

where the last implication follows from Theorem 1.7 below. We set $A(k) = B(x_0, 2k) \setminus B(x_0, k)$ and $A = \cup_n A(k(n))$ with $k, n \geq 1$ and a sequence $\{k(n)\}_{n>0}$ which goes to $\infty$ as $n \to \infty$. Let $D(\Delta_p)$ with $1 \leq p \leq \infty$ denote the closure of the space $C_0^\infty$ of smooth functions with compact support with respect to $\Delta$-graph norm in $L^p$ and $D = \cup_{1 \leq p \leq \infty} D(\Delta_p)$. Other two spaces are $D\prime = \cap_{k \geq 1} D(\Delta_D^k)$ and $L = \cap_{1 \leq p \leq \infty} L^p$.

**Theorem 1.5.** *Suppose that $V(r) < \infty$ for every $r > 0$.*

(*a*) *If $W_0^{1,2} = W^{1,2}$, then*

$$\int \Delta u = 0 \text{ for all } u \in L^1_{\mathrm{loc}} \text{ such that } \nabla u \in L^2_{\mathrm{loc}} \cap L^2(B) \cap L^1(A) \text{ and } \Delta u \in L^1 \tag{6}$$

*for some open set $B \supset \partial_C M$ and a sequence $\{k(n)\}_{n>0}$.*

(*b*) *If $\partial_C M$ is finite and*

$$\int \Delta u = 0 \quad \text{for all } u \in D' \cap L \text{ such that } \nabla u \in L^1 \text{ and } \Delta u \in L, \tag{7}$$

*then $W_0^{1,2} = W^{1,2}$.*

If $M$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary, then $W^{1,2} \neq W_0^{1,2}$ and (6) and (7) never hold true. Other example $M$ is a complete manifold $N$ punctured a compact submanifold $\Sigma \subset N$ of co-dimension 1. Indeed, the Cauchy boundary of $M = N \setminus \Sigma$ is $\Sigma$, and the solution to the boundary value problem: $u|_\Sigma = 1$, $u|_B = 0$ for some $B \supset \Sigma$ and $\Delta u|_{B \setminus \Sigma} = 0$, allows to construct $u \in C^\infty(M)$ with support in $B$ and $\int \Delta u < 0$. Propositions 4.2 and 4.3 show that we can not remove the condition $\nabla u \in L^2 \cap L^1$ from (6).

If $M$ is geodesically complete, then the statement reduces to

**Corollary 1.6.** *If $M$ is geodesically complete, then*

$$\int \Delta u = 0 \quad \text{for all } u \in L^1_{\mathrm{loc}} \text{ such that } \nabla u \in L^1(A) \text{ and } \Delta u \in L^1 \tag{8}$$

*with some sequence $\{k(n)\}_{n>0}$.*

This result was proved in [14] in the case that $k(n) = n$. Proposition 4.2 shows that we can not remove the condition $\nabla u \in L^1$ from (8). A weaker statement of the opposite implication in the case that the Riemannian metric extends to $\partial_C M$ can be found in Proposition 2.4.

We denote by $\nabla_D$ and $\nabla_N$ the gradient operators with domains $W_0^{1,2}$ and $W^{1,2}$, respectively. The minimal Laplacian $\Delta_M$, the Dirichlet Laplacian $\Delta_D$, the Neumann Laplacian $\Delta$, and the Gaffney[2] Laplacian $\Delta_G$ are the restrictions of the distributional Laplacian $\Delta$ to the following domains:

$$\begin{aligned} D(\Delta_M) &= \text{the closure of } C_0^\infty \text{ with } \Delta\text{-graph norm} \\ D(\Delta_D) &= \{u \in W_0^{1,2} : \nabla u \in D(\nabla_D^*)\} \\ D(\Delta_N) &= \{u \in W^{1,2} : \nabla u \in D(\nabla_N^*)\} \\ D(\Delta_G) &= \{u \in W^{1,2} : \nabla u \in D(\nabla_D^*)\}, \end{aligned}$$

where $\nabla^*$ is the adjoint operator of $\nabla$. The following inclusions are obvious:

$$\Delta_M \subset \Delta_D \subset \Delta_G$$
$$\Delta_M \subset \Delta_N \subset \Delta_G.$$

[2]M.P. Gaffney studied the essential self-adjointness of the Hodge-Laplacian acting on the space of differentiable forms [12]. If we restrict that Laplacian to the space of functions, then its essential self-adjointness is equivalent to the self-adjointness of $\Delta_G$.

Note that $\Delta_D$ and $\Delta_N$ are self-adjoint on an arbitrary weighted manifold. If $M$ is geodesically complete, then all four Laplacians coincide. In general, $\Delta_M$ and $\Delta_G$ do not need to be self-adjoint. For instance, $\Delta_M$ on $\mathbb{S}^2 \setminus \{p\}$ has infinitely many self-adjoint extensions; $\Delta_G$ is not even symmetric on a manifold with boundary due to the presence of the boundary term in Green's formula.

A self-adjoint operator $A$ is called *Markovian* if the semigroup $T_t = e^{tA}$ is Markovian, i.e., $0 \le T_t u \le 1$ $\mu$-a.e., whenever $0 \le u \le 1$ $\mu$-a.e. Let $\mathcal{A}(\Delta_M)$ be the set of Markovian extensions of $\Delta_M$. Every $A \in \mathcal{A}(\Delta_M)$ generates a Brownian motion on $M$ according to the boundary condition; in particular, the Dirichlet and Neumann Laplacians are Markovian on arbitrary weighted manifolds (Proposition 3.4) and the associated Brownian motions satisfy the absorbing and reflecting boundary conditions, respectively. The set $\mathcal{A}(\Delta_M)$ is furnished with a natural semi-order (see Subsection 3.3), and we consider the minimum and maximum elements, that are used in the following statement.

**Theorem 1.7.** $(a)$ *The Dirichlet Laplacian $\Delta_D$ and the Neumann Laplacian $\Delta_N$ are the minimum and maximum Markovian operators in $\mathcal{A}(\Delta_M)$, respectively.*

$(b)$ *The following three conditions are equivalent.*

$(i)$ $W_0^{1,2} = W^{1,2}$ *(that is, $\Delta_D = \Delta_N$).*

$(ii)$ $\Delta_G$ *is self-adjoint.*

$(iii)$ $\Delta_M$ *has a unique Markov extension.*

$(c)$ *If $M$ is either stochastically complete, or geodesically complete, or $\partial_C M$ is polar, then each of the conditions $(i)$, $(ii)$, and $(iii)$ is satisfied.*

Note that neither the parabolicity, nor the stochastic completeness, nor, the polarity and compactness of $\partial_C M$ imply the self-adjointnes of $\Delta_M$. For instance, $M = \mathbb{S}^2 \setminus \{p\}$ is parabolic (in particular stochastically complete) and the Cauchy boundary $\{p\}$ is polar, but $\Delta_M$ is not self-adjoint as explained above. Therefore, among all those infinitely many self-adjoint extensions, $\Delta_D (= \Delta_N = \Delta_G)$ is the only Markovian extension.

In this paper we consider a manifold without boundary however all our results remain true for a manifold with boundary imposed Neumann boundary condition.

We arrange the article as follows. Section 2 is the preliminaries. In particular, we discuss the relationship between the polarity of $\partial_C M$ and the Sobolev spaces $W^{1,2}$ and $W_0^{1,2}$. We prove all theorems in Section 3. In Section 4, we present and discuss some examples. Some examples demonstrate that certain conditions in the main theorems can not be removed, and other examples are related to the condition of the Cauchy boundary to be polar and the manifold to be parabolic. They will show that the Minkowski codimension of $\partial_C M$ equals 2 does not imply the polarity, that if $\partial_C M$ has infinite capacity, then both $W^{1,2} = W_0^{1,2}$ and $W^{1,2} \neq W_0^{1,2}$ may occur, and that the $V(r) \sim r^2$ at infinity does not imply the parabolicity of a geodesically complete manifold.

## 2. Preliminaries

Let $W^{1,2}$ be the space of all functions $u \in L^2 = L^2(M, \mu)$, whose distributional gradient $\nabla u$ is also in $L^2$. Then $W^{1,2}$ is a Hilbert space with the inner product

$$(u, v)_{1,2} = \int_M uv \, d\mu + \int_M (\nabla u, \nabla v) \, d\mu.$$

The space of smooth functions:

$$C^\infty \cap W^{1,2}$$

is dense in $W^{1,2}$ [33] [4]. Let $W_0^{1,2}$ be the closure of the space $C_0^\infty$ of smooth functions with compact support in $W^{1,2}$. The weighted divergence $\mathrm{div}_\mu$ is the negative of the formal adjoint operator of $\nabla$ determined as

$$\int_M u\, \mathrm{div}_\mu X\, d\mu = -\int_M Xu\, d\mu, \tag{9}$$

for smooth function $u$ and vector field $X$ with compact support. The *weighted Laplacian* $\Delta(=\Delta_\mu)$ is

$$\Delta u(x) = \mathrm{div}_\mu \nabla u(x), \quad \text{for } u \in C^\infty \text{ and any } x \in M.$$

A local expression shows that $\Delta$ is a second-order elliptic differential operator. As in Introduction, we denote the Dirichlet Laplacian by $\Delta_D$ and the associated semigroup in $L^2$ by

$$T_t = e^{t\Delta_D}, \quad \text{for all } t > 0.$$

The semigroup $T_t$ can be uniquely extended to a bounded operator in all $L^p$ with any $1 \le p \le \infty$ and it has a smooth integral kernel $k$:

$$T_t u(x) = \int k(t,x,y)u(y)\,\mu(dy), \quad \text{for } u \in L^p \text{ with } 1 \le p \le \infty,\ t > 0, \text{ and } x \in M.$$

The function $k$ is the smallest positive fundamental solution to the heat equation on $M$. We say that the manifold $M$ is stochastically complete if and only if

$$T_t 1(x) \equiv 1, \quad \mu\text{-a.e. for every } t > 0.$$

The *Cauchy boundary* is

$$\partial_C M = \overline{M} \setminus M,$$

where $\overline{M}$ is the completion of $M$ with respect to the Riemannian distance. The associated 1-capacity is defined as follows. Let $\mathcal{O}$ denote the family of all open subsets of $\overline{M}$. We define for $\Omega \in \mathcal{O}$ that

$$\mathrm{Cap}(\Omega) := \inf_{u \in \mathcal{L}(\Omega)} \int_M u^2 + |\nabla u|^2\, d\mu, \quad \text{if } \mathcal{L}(\Omega) \neq \phi,$$

where $\mathcal{L}(\Omega)$ is a set of $u \in W^{1,2}$ such that $0 \le u \le 1$ and $u|_{\Omega \cap M} = 1$. We let $\mathrm{Cap}(\Omega) = \infty$ if $\mathcal{L}(\Omega) = \phi$, and $\mathrm{Cap}(\phi) = 0$. We define the capacity for an arbitrary set $\Sigma \subset \overline{M}$ as

$$\mathrm{Cap}(\Sigma) := \inf_{\Omega \in \mathcal{O},\ \Sigma \subset \Omega} \mathrm{Cap}(\Omega).$$

We say $\Sigma$ is *polar* if $\mathrm{Cap}(\Sigma) = 0$. If $\Sigma = \phi$, then $\mathrm{Cap}(\Sigma) = 0$. The following can be proven in the same way for the standard capacity (see for e.g., [11])

**Lemma 2.1.** *The capacity defined above is a* Choquet capacity[3]*; namely, it satisfies*

(*a*) $A \subset B \quad \Rightarrow \quad \mathrm{Cap}(A) \le \mathrm{Cap}(B)$.
(*b*) *If* $\Omega_n \subset \Omega_{n+1}$, *then* $\mathrm{Cap}(\cup \Omega_n) = \sup \mathrm{Cap}(\Omega_n)$.
(*c*) *If* $\Omega_{n+1} \subset \Omega_n$ *and* $\Omega_n$ *is compact, then* $\mathrm{Cap}(\cap \Omega_\mathrm{n}) = \inf \mathrm{Cap}(\Omega_\mathrm{n})$.

Let $\Omega$ be a pre-compact open set in $M$ and $K$, a compact subset in $\Omega$. We define the relative capacity $\mathrm{cap}(K,\Omega)$ for the pair $(K,\Omega)$ by

$$\mathrm{cap}(K,\Omega) := \inf_{u \in \mathcal{L}(K,\Omega)} \int_\Omega |\nabla u|^2\, d\mu,$$

where $\mathcal{L}(K,\Omega)$ is a set of $u \in W^{1,2}$ with support in $\overline{\Omega}$ such that $0 \le u \le 1$ and $u|_K = 1$. We let $\mathrm{cap}(K,\Omega) = \infty$ if $\mathcal{L}(K,\Omega) = \phi$.

For an open pre-compact set $K \subset \Omega$, we define its relative capacity by

$$\mathrm{cap}(K,\Omega) := \mathrm{cap}(\overline{K},\Omega).$$

[3] A Choquet capacity is usually defined for a subset of $M$.

The following shows the relationship between the polarity of $\partial_C M$ and the Sobolev spaces $W^{1,2}$ and $W_0^{1,2}$. Let $\{B_k\}$ be an exhaustion of $\overline{M}$.

**Lemma 2.2.** *$(a)$ If $\partial_C M$ is polar, then $W_0^{1,2} = W^{1,2}$.*

*$(b)$ If $\mathrm{Cap}(B_k \cap \partial_C M) < \infty$ for every $k \geq 1$ and $W_0^{1,2} = W^{1,2}$, then $\partial_C M$ is polar. In particular, if $W_0^{1,2} = W^{1,2}$, then either $\partial_C M$ is polar or there is a pre-compact open set $B$ of $\overline{M}$ such that $\mathrm{Cap}(B \cap \partial_C M) = \infty$.*

*$(c)$ If $V(r) < \infty$ with any $r > 0$ and $W_0^{1,2} = W^{1,2}$, then $\partial_C M$ is polar.*

*Proof.* $(a)$ For $u \in W^{1,2}$, we construct $u_n \in W_0^{1,2}$ converging to $u$ in $W^{1,2}$. Since $W^{1,2} \cap L^\infty$ is dense in $W^{1,2}$, we may assume $u$ to be bounded without loss of generality. Let $\Omega_k$ with $k = 1, 2, \cdots$ be an open set of $\overline{M}$ such that

$$\partial_C M \subset \Omega_{k+1} \subset \Omega_k \text{ for every } k > 1$$
$$\mathrm{Cap}(\Omega_k) \to 0 \text{ as } k \to \infty.$$

For each $k > 0$, let $\phi_n^{(k)} \in \mathcal{L}(\Omega_k)$ satisfy that $\|\phi_k^{(n)}\|_{W^{1,2}} \to \mathrm{Cap}(\Omega_k)$ as $n \to \infty$. Put $\phi_n = \phi_n^{(n)}$ and $u_n = (1 - \phi_n)u$. Then $u_n \in W_0^{1,2}$ and since $1 - \phi_n \uparrow 1$ $\mu$-a.e,

$$u_n \to u \text{ in } L^2 \text{ as } n \to \infty.$$

Since $\phi_n \downarrow 0$ $\mu$-a.e., $\nabla u \in L^2$, $\nabla \phi_n \to 0$ in $L^2$ and $u \in L^\infty$,

$$\nabla u_n = (1 - \phi_n)\nabla u - u\nabla\phi_n \to \nabla u \text{ in } L^2 \text{ as } n \to \infty.$$

This proves $(a)$.

$(b)$ If $\mathrm{Cap}(B \cap \partial_C M) < \infty$ for a pre-compact set $B$ in $\overline{M}$, then there exist an open set $O$ of $\overline{M}$ such that $B \cap \partial_C M \subset O$ and a function $u \in W^{1,2}$ such that $u|_{O \cap M} = 1$. Since $W_0^{1,2} = W^{1,2}$, then there exists a sequence $u_n \in C_0^\infty$ which converges to $u$ in $W^{1,2}$. Since $u_n$ has compact support in $M$, there is an open set $U_n$ in $\overline{M}$ such that $B \cap \partial_C M \subset U_n$ and $u_n(x) = 0$ if $x \in U_n \cap M$.

Set $v_n = u - u_n$. If $x \in O \cap U_n \cap M$, then $v_n(x) = u(x) - u_n(x) = 1$. Hence $v_n \in \mathcal{L}(O \cap U_n)$, and

$$\mathrm{Cap}(B \cap \partial_C M) \leq \|v_n\|_{W^{1,2}} \to 0 \text{ as } n \to \infty,$$

in particular, $\mathrm{Cap}(B_k \cap \partial_C M) = 0$ for every $k > 0$ due to the assumption. Because $\cup_{k \geq 1} (B_k \cap \partial_C M) = \partial_C M$ and by Lemma 2.1

$$0 = \sup_{k \geq 1} \mathrm{Cap}\,(B_k \cap \partial_C M) = \mathrm{Cap}(\partial_C M)$$

we conclude that $\partial_C M$ is polar.

$(c)$ Let $B$ be a pre-compact open set of $\overline{M}$ and $r > 0$ be such that $B \cap M \subset B(x_0, r)$. Consider the function $u = ((2 - d(x_0, \cdot)r^{-1}) \wedge 1)_+$. Since $V(2r) < \infty$, $u \in \mathcal{L}(B(x_0, r))$, and hence, $\mathrm{Cap}(B \cap \partial_C M) \leq \mathrm{Cap}(B(x_0, r)) < \infty$. The assertion follows from $(b)$. □

**Remark 2.3.** *(1) If $Cap(\partial_C M) = \infty$, then both cases $W^{1,2} = W_0^{1,2}$ and $W^{1,2} \neq W_0^{1,2}$ may occur. See Proposition 4.5.*

*(2) Suppose $\partial_C M$ is bounded and almost polar. If we also have (1), then split $M$ into $M_1$ and $M_2$ in a way such that they have compact intersection, $\partial_C M \subset \partial_C M_1$, and $\mu(M_1) < \infty$. Then $M_1$ and $M_2$ with the Neumann boundary condition are parabolic by* [32] *and (1), respectively, and so is $M$ by Proposition 14.1 (e)* [22]*. The same argument together with Proposition 6.1 and Theorem 6.2* [22] *shows that $M$ is stochastically complete under the volume test (2). The same results can be achieved by using Theorems 1.1 and 1.2.*

In the following we study other sufficient condition for the identity $W^{1,2} = W_0^{1,2}$. The results to the end of this section will not be used in the later sections. Let $\vec{W}^{1,1}$ be the set of all vector fields $X \in L^1$ whose distributional divergence $\mathrm{div}_\mu X$ is in $L^1$. $\vec{W}^{1,1}$ is a Banach space with the norm

$$\|X\|_{1,1} = \int_M |X| + |\mathrm{div}_\mu X| \, d\mu.$$

Let $\vec{W}_0^{1,1}$ be the closure of the space of smooth vector fields with compact support in $\vec{W}^{1,1}$.

It is easy to see that the condition $\vec{W}^{1,1} = \vec{W}_0^{1,1}$ implies that $W^{1,2} = W_0^{1,2}$. Indeed, let $u \in W^{1,2}$ and $X \in \vec{W}^{1,2}$. If $\vec{W}^{1,1} = \vec{W}_0^{1,1}$, then $(\nabla u, X) = (u, -\mathrm{div}_\mu X)$, which shows $W^{1,2} = W_0^{1,2}$. It is known that $\vec{W}^{1,1} = \vec{W}_0^{1,1}$ if $M$ is geodesically complete [14]. The opposite implication is also true if the Riemannian metric extends to the Cauchy boundary:

**Proposition 2.4.** *Let $\Sigma$ be a closed subset of a geodesically complete manifold $M$. If $\Sigma$ is not empty, then*

$$\vec{W}_0^{1,1}(M \setminus \Sigma) \subsetneq \vec{W}^{1,1}(M \setminus \Sigma).$$

*Proof.* Let $\Omega \subset M$ be a pre-compact open set with smooth boundary such that $B(\partial\Omega) \cap (M \setminus \Omega) \neq \phi$. Let $g$ be Green's function of $\Omega$ with Dirichlet boundary condition. Extend $g$ to $M$ by setting $g(x) = 0$ if $x \in M \setminus \Omega$.

Set $h = g(x_o, \cdot)$ with some $x_o \in \Sigma$. Let $\epsilon > 0$ and $\psi \in C^\infty(\mathbb{R})$ be a convex function such that

$$\psi(t) = \begin{cases} 0, & \text{if } t < \epsilon, \\ t - 2\epsilon, & \text{if } t > 3\epsilon. \end{cases}$$

Set $u = \psi(h)$ and a smooth vector field, $X = \nabla u$. Recall that $h$ has the same magnitude of the singularity of that of Green's function of the Euclidean space of the same dimension as $M$; namely, if $M$ has dimension $n$, then $h(x) = g_{\mathbf{R}^n}(0, x) + f(x)$, with Green's function $g_{\mathbf{R}^n}$ of $\mathbf{R}^n$ and a smooth function $f$ in $\Omega$. Therefore, $X \in \vec{W}^{1,1}$. It follows from the identity

$$\Delta u = \psi''(h)|\nabla h|^2 + \phi'(h)\Delta h,$$

that

$$\int_M \mathrm{div}_\mu(X) < 0.$$

By choosing $\epsilon > 0$ sufficiently small, we obtain also $\int_{M\setminus\Sigma} \mathrm{div}_\mu(X) < 0$.

Suppose $X \in \vec{W}_0^{1,1}(M \setminus \Sigma)$. Then there exists a sequence of smooth vector field $X_n$ with compact support in $M \setminus \Sigma$ such that $X_n \to X$ in $\vec{W}^{1,1}(M \setminus \Sigma)$, and

$$\int_{M\setminus\Sigma} \mathrm{div}_\mu(X) = \lim_{n\to\infty} \int_{M\setminus\Sigma} \mathrm{div}_\mu(X_n) = 0.$$

Therefore, $X \in \vec{W}^{1,1}(M \setminus \Sigma) \setminus \vec{W}_0^{1,1}(M \setminus \Sigma)$. $\square$

We summarize some facts regarding to the polarity of $\partial_C M$ and the "completeness" in terms of geodesics, Brownian motion, and some Sobolev spaces:

$$\begin{tikzcd}
 & 0<\mathrm{Cap}(\partial_C M)<\infty \arrow[r, Rightarrow] \arrow[d, Rightarrow, "(1)"'] & \Delta_G \neq \Delta_G^* \arrow[r, Leftrightarrow] \arrow[dl, Leftrightarrow, "(5)"] & \exists u \neq 0 \in W^{1,2},\ \Delta u = u \\
\vec{W}^{1,1} \neq \vec{W}_0^{1,1} \arrow[ddr, Rightarrow, "(3)"'] & W^{1,2} \neq W_0^{1,2} \arrow[l, Rightarrow, "(2)"] \arrow[r, Rightarrow, "(6)"'] \arrow[d, Rightarrow, "(4)"'] & T_t 1 < 1 \arrow[r, Leftrightarrow] & \exists u \neq 0 \in L^\infty,\ \Delta u = u \\
 & W^{2,2} \neq W_0^{2,2} \arrow[r, Leftrightarrow, "(8)"] & \Delta_M \neq \Delta_M^* \arrow[r, Leftrightarrow] \arrow[dl, Rightarrow, "(7)"] & \exists u \neq 0 \in L^2,\ \Delta u = u \\
 & \partial_C M \neq \phi. & &
\end{tikzcd}$$

- (1) is Lemma 2.2. The opposite implication does not hold by Proposition 4.5.
- (2) was explained above.
- (3) is included in the proof of Theorem 1.5. Indeed, for $X \in \vec{W}^{1,1}$ one may construct $X_n \in \vec{W}_0^{1,1}$ converging to $X$ from $\chi^l_{k(n)}X$. See also [13]. The opposite implication holds true if the Riemannian metric extends to $\partial_C M$ by Proposition 2.4.
- (4), (5), and (8) follow from the definitions.
- (6) is (c) of Theorem 1.7.
- (7) was first proved in [7]. See also [38] [24]. The opposite implication is not true. Indeed, if $N$ is geodesically complete and $\Sigma$ is a closed submanifold of $N$, then $W^{2,2}(N \setminus \Sigma) = W_0^{2,2}(N \setminus \Sigma)$ if and only if $\mathrm{codim}(\Sigma) \geq 4$ [31]. This fact together with Proposition 2.4 shows that $W^{2,2} = W_0^{2,2}$ does not imply $\vec{W}^{1,1} = \vec{W}_0^{1,1}$.
- There are no implications between the stochastic completeness and the self-adjointness of $\Delta_M$ as we explained in Introduction.
- The characterizations of the self-adjointness of $\Delta_G$ and $\Delta_M$ in terms of 1-harmonic functions follows by (5) and a standard argument. See [22] for the characterization of the stochastic completeness in terms of 1-harmoinc functions.

Other characterization is the uniqueness of the solution to the heat equation in a certain class.

**Remark 2.5.** *Consider the Cauchy problem to find a smooth function $u(t,x)$ on $\mathbb{R}_+ \times M$ such that*

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u, \\ u(0+, \cdot) \equiv 0. \end{cases} \tag{10}$$

*Then*

$$\text{(10) has unique } L^\infty\text{-solutions} \Rightarrow \Delta_G \text{ is self-adjoint.} \tag{11}$$

$$\text{(10) has unique } D(\Delta_M^*)\text{-solutions} \Leftrightarrow \Delta_M \text{ is self-adjoint.} \tag{12}$$

*The implication (11) follows from Proposition 3.7 together with the fact that the stochastic completeness of $M$ is equivalent to the uniqueness of $L^\infty$-solutions to (10) (see e.g. [22]). The equivalence (12) was proved in [3] (see also [7] and [37]).*

**Remark 2.6.** *If $M$ is stochastically complete, then there exists a sequence $\chi_n \in D(\Delta_D^k) \cap L^p \cap C^\infty$ for any $k > 0$ and $1 \leq p \leq \infty$ such that $0 \leq \chi_n \leq 1$, $\chi_n \uparrow 1$ and*

$$\Delta \chi_n \to 0, \ \text{as } n \to \infty.$$

*If $M$ is geodesically complete, there exists a sequence $\chi_n \in C_0^\infty$ such that $0 \le \chi_n \le 1$, $\chi_n \uparrow 1$ and*

$$\nabla \chi_n \to 0, \ \text{as } n \to \infty.$$

*Moreover, if $M$ is geodesically complete, then for any $u \in D(\Delta_D)$ there exists $u_n \in C_0^\infty(M)$ such that*

$$u_n \to u, \ \nabla u_n \to \nabla u, \ \text{and } \Delta u_n \to \Delta u \ \text{in } L^2 \ \text{as } n \to \infty.$$

*This is a consequence of the self-adjointness of $\Delta_M$. A direct proof for this fact seems to construct $\chi_n \in C_0^\infty$ such that $0 \le \chi_n \le 1$, $\chi_n \uparrow 1$ and*

$$\nabla \chi_n \to 0 \ \text{and } \Delta \chi_n \to 0, \ \text{as } n \to \infty. \tag{13}$$

*If $M$ is geodesically complete and bounded geometry then there exists $\chi_n \in C_0^\infty$ satisfying that (13) (for e.g. [37]).*

## 3. Proofs

In this section we prove the main theorems. The proof of each theorem is contained in individual subsection.

### 3.1. **Proof of Theorem 1.1: Parabolicity.** The proof is taken from [19].

*Proof of Theorem 1.1.* Let $u \in L^\infty$ be non-constant and $\Delta u \in L^1$. Let $B \subset B' \subset M$ be arbitrary pre-compact open sets with smooth boundaries such that $\overline{B} \subset B'$. Set $\Omega = B' \backslash \overline{B}$. We assume without loss of generality that

$$\sup_M u < 1 \ \text{and} \ \inf_\Omega u > 0.$$

Additionally, we first assume that $u \in C^2$. Let $\phi$ be the solution to the boundary value problem: $\phi|_{\partial B} = 1$, $\phi|_{\partial B'} = 0$, and $\Delta \phi = 0$ in $\Omega$. The function $\phi$ is the equilibrium potential of $\Omega$; namely,

$$\mathrm{cap}(B, B') = \int_\Omega |\nabla \phi|^2.$$

Set $w = \phi - u$. Since

$$\begin{aligned} w(x) &= 1 - u(x) > 0, \quad \text{if } x \in \partial B, \\ w(x) &= -u(x) < 0, \quad \text{if } x \in \partial B', \end{aligned}$$

there exists a regular value $\epsilon > 0$ for $w$ such that $\Gamma = \{w = \epsilon\} \subset \Omega$ and

$$\int_\Gamma \frac{\partial w}{\partial \nu}\, d\sigma \ge 0,$$

where $\nu$ is the normal vector to $\Gamma$ and $\sigma$ is the surface measure on $\Gamma$. Thus,

$$\int_\Gamma \frac{\partial \phi}{\partial \nu}\, d\sigma \ge \int_\Gamma \frac{\partial u}{\partial \nu}\, d\sigma.$$

By Green's formula and the fact that $\Delta \phi = 0$ in $\Omega$,

$$\begin{aligned} \mathrm{cap}(B, B') &= \int_\Omega |\nabla \phi|^2 \, d\mu \\ &= \int_{\partial B} \frac{\partial \phi}{\partial \nu}\, d\sigma = \int_\Gamma \frac{\partial \phi}{\partial \nu}\, d\sigma + \int_{\Omega \cap \{w > \epsilon\}} \Delta \phi \, d\mu \\ &= \int_\Gamma \frac{\partial \phi}{\partial \nu}\, d\sigma \\ &\ge \int_\Gamma \frac{\partial u}{\partial \nu}\, d\sigma. \end{aligned}$$

Let $\{B_i\}$ be an exhaustion of $M$ such that $B_i$ has smooth boundary and $\overline{B}_{i-1} \subset B_i$ for every $i > 1$. Since $M$ is parabolic,

$$\mathrm{cap}(B_k, B_m) \to 0 \text{ as } m \to \infty.$$

For each $k$, let $m(k)$ be such that $\mathrm{cap}(B_k, B_{m(k)}) \to 0$ as $k \to \infty$. Let $B = B_k$, $B' = B_{m(k)}$, and $\Omega_k \supset B_k$ be an open set such that $\partial\Omega_k = \Gamma$ as in the notations above. We deduce that

$$\int_M \Delta u = \lim_{k\to\infty} \int_{\Omega_k} \Delta u = \lim_{k\to\infty} \int_{\partial\Omega_k} \frac{\partial u}{\partial \nu}\, d\sigma \le \lim_{k\to\infty} \mathrm{cap}(B_k, B_{m(k)}) = 0.$$

If we apply this argumentation for $1 - u$, then we find

$$\int_M \Delta(-u) \le 0,$$

and hence,

$$\int_M \Delta u = 0.$$

We can remove the assumption of the smoothness of the function by applying the Friedrichs mollifier (see for e.g., [4]).

Assume that $M$ is not parabolic. Then $M$ admits a positive Green function $g$. Let $u \in C_0^\infty$ be $u \ge 0$ and not identically 0. Then $v = \int gu \in L^\infty$ and

$$\int \Delta v = \int u > 0.$$

□

3.2. **Proof of Theorem 1.2: Stochastic completeness (General case).** Let $G$ be the associated 1-resolvent operator to $T_t$; that is,

$$Gu = \int_0^\infty e^{-t} T_t u\, dt, \quad \text{for } u \in L^p \text{ with any } 1 \le p \le \infty.$$

Similar to $T_t$, $G$ is a bounded operator in any $L^p$ with $1 \le p \le \infty$. Since $T_t$ is an analytic semigroup, $G(L^2) \subset D(\Delta_D)$.

Let $e_n \in C_0^\infty(M)$ satisfy that $0 \le e_n \le e_{n+1} \le 1$ for every $n > 1$, and $e_n \uparrow 1$ $\mu$-a.e. as $n \to \infty$. The next lemma follows immediately from the definition.

**Lemma 3.1.** *The following three conditions are equivalent.*

1. *$M$ is stochastically complete.*
2. *$Ge_n \uparrow 1, \quad \mu$-a.e. as $n \to \infty$.*
3. *$\Delta(Ge_n) = Ge_n - e_n \to 0, \quad \mu$-a.e. as $n \to \infty$.*

*Proof of Theorem 1.2.* First, we assume that $M$ is stochastically complete. Let $u \in D(\Delta_D) \cap L^1$ be such that $\Delta u \in L^1$. By Lemma 3.1,

$$\int \Delta u\, d\mu = \lim_{n\to\infty} (\Delta u, Ge_n). \tag{14}$$

If $u \in D(\Delta_D)$, then, again by Lemma 3.1, (14) is

$$\lim_{n\to\infty} (u, \Delta(Ge_n)) = \lim_{n\to\infty} (u, Ge_n - e_n) = 0.$$

We prove the opposite implication. Assume

$$\int \Delta v = 0, \quad \text{for every } v \in L^1 \cap D(\Delta_D) \text{ such that } \Delta v \in L^1. \tag{15}$$

Let $\phi \in C_0^\infty(M)$ be a non-trivial and non-negative function. Set $u = T_t\phi$ and $v = Gu$ with arbitrary $t > 0$. Note that $v$ satisfies the assumption of (15). Since $G$ is self-adjoint,

$$(u, G(e_n) - e_n) = (u, \Delta G e_n) = (u, G\Delta e_n) = (v, \Delta e_n) = (\Delta v, e_n).$$

The most right-hand side of this equation tends to 0 by (15). Since $u > 0$, this implies that $G(e_n) - e_n \to 0$; that is, the stochastic completeness of $M$ by Lemma 2.2. □

Theorem 1.2 can be extended as follows.

**Proposition 3.2.** *(1) If $M$ is stochastically complete, then*

$$\int \Delta u = 0 \quad \textit{for all } u \in D(\Delta_D) \cup D \cap L^1 \textit{ such that } \Delta u \in L^1. \tag{16}$$

*(2) If*

$$\int \Delta u = 0 \quad \textit{for all } u \in D\prime \cap L \textit{ such that } \Delta u \in L,$$

*then $M$ is stochastically complete.*

*Proof.* (1) If $u \in D(\Delta_D)$, then the statement was proved in Theorem 1.2. If $u \in D$, then there exists $u_k \in C_0^\infty$ such that $u_k \to u$ and $\Delta u_k \to \Delta u$ in $L^p$ with some $p \in [1, \infty]$. Taking into account that $Ge_n \in L$, Lemma 3.1 yields that

$$\begin{aligned}
\int \Delta u \, d\mu &= \lim_{n\to\infty} \lim_{k\to\infty} (\Delta u_k, (Ge_n)) \\
&= \lim_{n\to\infty} \lim_{k\to\infty} (u_k, \Delta(Ge_n)) \\
&= \lim_{n\to\infty} (u, \Delta(Ge_n)) = \lim_{n\to\infty} (u, Ge_n - e_n) = 0.
\end{aligned}$$

(2) can be proven as in the proof of Theorem 1.2] since $G \circ T_t(C_0^\infty) \subset D\prime \cap L$ and $G$, $T_t$, and $\Delta$ commute with each other on $C_0^\infty$. □

3.3. **Proof of Theorem 1.4: Stochastic completeness (Geodesically complete case).** In this subsection, we assume that $M$ is geodesically complete. Fix an arbitrary point $x_o \in M$ and set for $k \geq 1$ that

$$\chi_k(x) = 1 \wedge (k^{-1}(2k - d(x, x_0))_+. \tag{17}$$

This sequence of functions enjoys the property: $\chi_k \in W_0^{1,2}$ (due to the geodesic completeness), $\chi_k(x) \uparrow 1$ and $\nabla\chi_k(x) \to 0$ as $k \to \infty$ $\mu$-a.e.

*Proof of Theorem 1.4.* First we assume that $M$ is stochastically complete. Let $u \in L^1 \cap L^2$ be such that $\nabla u \in L^2$ and $\Delta u$ is integrable. Let $e_n \in L^2$ be the function which appeared above. By Lemma 2.2 $Ge_n \uparrow 1$ $\mu$-a.e. as $n \to \infty$. Since $(\Delta u)(Ge_n)$ is integrable and $\chi_k \uparrow 1$ $\mu$-a.e. as $k \to \infty$,

$$\int \Delta u = \lim_{n\to\infty} \int (\Delta u)(Ge_n) = \lim_{n\to\infty} \lim_{k\to\infty} \int (\Delta u)(Ge_n)\chi_k.$$

Because $\chi_k$ has compact support (due to the geodesic completeness), the last expression of the above equation is

$$\begin{aligned}
&\lim_{n\to\infty} \lim_{k\to\infty} - \int (\nabla u, \nabla(\chi_k(Ge_n))) \\
= \quad &\lim_{n\to\infty} \lim_{k\to\infty} - \left[ \int \chi_k(\nabla u, \nabla(Ge_n)) + (Ge_n)(\nabla u, \nabla\chi_k) \right].
\end{aligned} \tag{18}$$

Since $(Ge_n)\nabla u \in L^1$ and $\nabla\chi_k \to 0$ $\mu$-a.e. as $k \to \infty$, the second term in (18) tends to 0 as $k \to \infty$. Due to the fact $u \in L^2$ and $Ge_n \in D(\Delta) \subset W_0^{1,2}$, it follows that $u\nabla(Ge_n) \in L^1$ and

$$\lim_{k\to\infty} \int (u\nabla\chi_k, \nabla(Ge_n)) = 0.$$

Hence, (18) is

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{k\to\infty} -\left[\int (\chi_k\nabla u, \nabla(Ge_n)) + \int (u\nabla\chi_k, \nabla(Ge_n))u\right] \\
=\ &\lim_{n\to\infty}\lim_{k\to\infty} -\int (\nabla(\chi_k u), \nabla(Ge_n)) \\
=\ &\lim_{n\to\infty}\lim_{k\to\infty} \int \chi_k u\Delta(Ge_n) \\
=\ &\lim_{n\to\infty} \int u(\Delta(Ge_n)),
\end{aligned}$$

where the last expression is 0 because $u \in L^1$ and $\Delta(Ge_n) \to 0$ $\mu$-a.e. as $n \to \infty$ by Lemma 2.2.

The opposite implication follows from Theorem 1.2. □

### 3.4. **Proof of Theorem 1.5: Sobolev spaces.**

*Proof of Theorem 1.5* $(a)$. Let $A(k)$ be a subset of $M$ as in Introduction, and $u \in L^1_{\mathrm{loc}}$ be a function such that

$$\begin{cases} \nabla u \in L^2(B) \cap L^1(A) \cap L^2_{\mathrm{loc}} \text{ for some open } B \supset \partial_C M \\ \Delta u \in L^1 \end{cases}$$

for a sequence $k(n) \to \infty$ as $n \to \infty$ and for $A = \cup_{n\geq 1} A(k(n))$.

Since $W_0^{1,2} = W^{1,2}$ and $V(r) < \infty$ for every $r > 0$, $\partial_C M$ is polar by Lemma 2.2. Hence we find a sequence of open sets $O_l$ of $\overline{M}$ satisfying $\partial_C M \subset O_l \subset B$, and a sequence of functions $e_l \in \mathcal{L}(O_l)$ such that $\|e_l\|_{W^{1,2}} \to 0$ as $l \to \infty$.

Employing $\chi_k$ defined in (17), set $\chi_k^l = (1 - e_l)\chi_k$ with $k, l \geq 1$. Then $\chi_k^l \in W_0^{1,2}$. Taking into account that $\chi_k$ and $\nabla\chi_k$ are supported in $B(x_0, 2k)$ and $A$, respectively,

$$\begin{aligned}
0 &= \int \mathrm{div}_\mu \left(\chi_{k(n)}^l \nabla u\right) \\
&= \int_{B(x_0,2k)} \chi_{k(n)}(\nabla e_l, \nabla u) + \int_A (1 - e_l)(\nabla\chi_{k(n)}, \nabla u) + \int \chi_{k(n)}^l \Delta u. \qquad (19)
\end{aligned}$$

Since $B(x_0, 2k) = (B(x_0, 2k) \setminus B) \cup B$, $\nabla u \in L^2(B(x_0, 2k))$. Therefore the first term in (19) tends to 0 as $l \to \infty$ because $\nabla e_l \to 0$ in $L^2$. The second term of (19) tends to 0 as $n \to \infty$ because $\nabla u$ is integrable on $A$ and $\nabla\chi_{k(n)} \to 0$ as $n \to \infty$. The third term of (19) clearly converges to $\int \Delta u$. □

If $\partial_C M$ is finite, then let $B$ be a pre-compact open set of $\overline{M}$ such that $\partial_C M \subset B$. We denote by $\overline{B}$ the closure (not the completion) of $B$ in $M$. In order to prove Theorem 1.5 $(b)$, we need

**Lemma 3.3.** *Assume that $\partial_C M$ is finite and has finite capacity. If $W_0^{1,2}(M) \neq W^{1,2}(M)$, then $W_0^{1,2}(\overline{B}) \neq W^{1,2}(\overline{B})$.*

*Proof.* Since $\partial_C M$ has a finite capacity, there exists an open set $B'$ of $\overline{M}$ such that $\partial_C M \subset B'$ and $\mu(B') < \infty$. Because $B \setminus \overline{B'}$ is a pre-compact subset of $M$, $\mu(B) \le \mu(B \setminus B') + \mu(B') < \infty$. Thus, $1 \in W^{1,2}(\overline{B})$, and

$$\mathrm{Cap}(\partial_C \overline{B}) < \infty.$$

Since if $u \in W^{1,2}(M)$ then $u|_{\overline{B}} \in W^{1,2}(\overline{B})$, it follows that $\mathrm{Cap}(\partial_C \overline{B}) \ge \mathrm{Cap}(\partial_C M) > 0$. The assertion follows from Lemma 2.2. □

*Proof of Theorem 1.5 (b).* Since $\partial_C M$ is finite and $V(r) < \infty$ for every $r > 0$, we find a pre-compact open set $B$ of $\overline{M}$ such that $\partial_C M \subset B$ and $\mu(B) < \infty$. Let $\nabla_{DN}$ be the restriction of $\nabla_N$ to $W_0^{1,2}(\overline{B})$. The associated Laplacian $\Delta_{DN}$ is

$$\Delta_{DN} = -\nabla_{DN}^* \nabla_{DN}.$$

Due to the Von Neumann theorem, $\Delta_{DN}$ is self-adjoint. Moreover, since $\nabla_{DN}$ is a restriction of $\nabla_N$, $\Delta_{DN}$ is Markovian by Lemma 3.5. Any function $u \in D(\Delta_{DN})$ satisfies the Dirichlet boundary condition on $\partial_C B (= \partial_C M)$ and the Neumann boundary condition on $\partial B$.

If we suppose $W_0^{1,2}(M) \ne W^{1,2}(M)$, then $W_0^{1,2}(\overline{B}) \ne W^{1,2}(\overline{B})$ by Lemma 3.3. Since $1 \in W^{1,2}(\overline{B})$ and $e^{t\Delta_N} 1 = 1$, where $\Delta_N$ is the Neumann Laplacian of $\overline{B}$, this implies that $e^{t\Delta_{DN}} 1 < 1$, and we may apply the argument of the proof of Theorem 1.2 and the following remark to find a function $u \in L^p(B) \cap D(\Delta_{DN}^k)$ such that $\Delta u \in L^p(B)$ with all $1 \le p \le \infty$ and $k \ge 1$ and

$$\int_B \Delta u \ne 0.$$

For an open set $B' \subset \overline{M}$ such that $\partial_C M \subset B'$ and $\overline{B'} \subset B$, let $\chi \in C^\infty(M)$ satisfy that $\chi|_{B'} = 1$ and $\mathrm{supp}[\chi] \subset B$. Since $\nabla\chi$ and $\Delta\chi$ are supported in $B \setminus B'$ and $B \setminus B'$ is pre-compact in $M$, both $\nabla(\chi u)$ and $\Delta(\chi u)$ are in $L^2$. Because $(1-\chi)u$ satisfies the Neumann boundary condition on $\partial(B \setminus B')$,

$$\int_B \Delta u = \int_B \Delta(\chi u) + \int_B \Delta(1-\chi)u = \int_B \Delta(\chi u).$$

Thus, we may assume that $\mathrm{supp}[u] \subset B$ without loss of generality. We extend $u$ to $M$ by defining its value to be 0 on $M \setminus B$ and denote it by the same symbol. Clearly $u \in L \cap D\prime$ and $\Delta u \in L$. Finally, since $\mu(B) < \infty$, $\nabla u$ is integrable, which completes the proof. □

3.5. **Proof of Theorem 1.7: Markov uniqueness.** Denote by $\mathcal{A}$ the set of non-positive definite self-adjoint extensions $A$ of the minimal Laplacian $\Delta_M$, i.e,

$$\mathcal{A} = \{A \subset \Delta_M^* : A = A^* \text{ and } (Au, u) \le 0 \text{ for } u \in D(A)\}.$$

We say that $S \in \mathcal{A}$ is Markovian if the semigroup generated by $S$ in $L^2$ is Markovian. A subset $\mathcal{A}_M$ of $\mathcal{A}$ is

$$\mathcal{A}_M = \{S \in \mathcal{A} : S \text{ is Markovian}\}.$$

For $A \in \mathcal{A}_M$, consider the closure $\mathcal{E}_A$ of the quadratic form $(-Au, v)$ with $u, v \in D(A)$. We denote the domain of $\mathcal{E}_A$ by $\mathcal{F}_A$. The pair $(\mathcal{E}_A, \mathcal{F}_A)$ is called the Dirichlet form associated with $A$. A Dirichlet form $(\mathcal{E}, \mathcal{F})$ defines a complete metric $\mathcal{E}$-1 on $\mathcal{F}$:

$$\mathcal{E}\text{-}1[u] = \|u\|^2 + \mathcal{E}[u], \quad u \in \mathcal{F},$$

where $\mathcal{E}[\cdot] = \mathcal{E}(\cdot, \cdot)$. A semi-order $\prec$ in $\mathcal{A}$ is defined by $A_1 \prec A_2$ if and only if

$$D(\mathcal{E}_{A_1}) \subset D(\mathcal{E}_{A_2}) \text{ and } \mathcal{E}_{A_1}[u] \ge \mathcal{E}_{A_2}[u], \text{ for all } u \in D(\mathcal{E}_{A_1}).$$

The following fact is well known (see for e.g., [24]), but we give an alternative proof for the sake of the completeness.

**Lemma 3.4** ([24])**.** *The Laplacians $\Delta_D$ and $\Delta_N$ are Markovian on an arbitrary weighted manifold.*

*Proof.* Let $\psi_\epsilon \in C^\infty(\mathbb{R})$ with $\epsilon > 0$ satisfy $-\epsilon \le \psi_\epsilon \le 1+\epsilon$, $\psi_\epsilon(t) = t$ if $t \in [0,1]$, and $0 \le \psi' - \epsilon \le 1$. For $u \in W^{1,2} \cap C^\infty$ and $u_\epsilon = \psi_\epsilon(u)$,

$$\mathcal{E}[u_\epsilon] = \int |\nabla u_\epsilon|^2 = \int |\psi_\epsilon'(u)\nabla u|^2 \le \int |\nabla u|^2 = \mathcal{E}[u],$$

where $\mathcal{E}[u] = \int |\nabla u|^2$. Hence $(\mathcal{E}, W^{1,2} \cap C^\infty)$ is a Markovian form [11]. The generator of the closure of this form is $\Delta_N$, and since the generator associated to the closure of a Markovian form is Markovian [11], $\Delta_N$ is Markovian.

We can prove that $\Delta_D$ is Markovian in the same way. □

The following is (i) in Theorem 1.7.

**Lemma 3.5.** *The Dirichlet Laplacian and Neumann Laplacian are the minimum and maximum elements in $\mathcal{A}_M$, respectively.*

*Proof.* First, we show that $(\mathcal{E}, W_0^{1,2})$ is the minimum element. Let $A \in \mathcal{A}_M$. Since $W_0^{1,2}$ is the closure of $C_0^\infty$ with respect to $\mathcal{E}$-1 norm, $W_0^{1,2} \subset \mathcal{F}_A (= D(\mathcal{E}_A))$. For any $u \in W_0^{1,2}$, let $u_n \in C_0^\infty$ such that $u_n \to u$ in $W_0^{1,2}$ as $n \to \infty$. By the equation $Au_n = \Delta u_n$ and the lower-semicontinuity of $\mathcal{E}_A$,

$$\mathcal{E}[u] = \lim_{n\to\infty} (-Au_n, u_n) = \lim_{n\to\infty} \mathcal{E}_A[u_n] \ge \mathcal{E}_A[u].$$

Next, we show that $\Delta_N$ is the maximum element. The associated form is $(\mathcal{E}, W^{1,2})$. Let $\phi \in C_0^\infty$ such that $0 \le \phi \le 1$. Let $v \in \mathcal{F}_A$ be a solution of $Av = \lambda v$ with $\lambda > 0$. By the hypo-ellipticity of $A$, $v$ is smooth. Set $v_n = (v \vee (-n)) \wedge n$ with $n = 1, 2, \cdots$. Since $v_n^2 \in \mathcal{F}_A \cap L^\infty$ for any $n$, it follows by (3.2.13) in [11] that

$$\mathcal{E}_A[v] = \lim_{n\to\infty} \mathcal{E}_A[v_n] \ge \lim_{n\to\infty} \left[\mathcal{E}_A(v_n\phi, v_n) - \frac{1}{2}\mathcal{E}_A(v_n^2, \phi)\right].$$

Since $v_n\phi = v\phi$ for large $n$ and $v\phi \in C_0^\infty$, on which $A$ and $\Delta$ agree point wise, the most right-hand side in the above equation is

$$\begin{aligned}
\lim_{n\to\infty} \left[\mathcal{E}_A(v\phi, v_n) + \frac{1}{2}(v_n^2, A\phi)\right] &= \lim_{n\to\infty} \left[(-A(v\phi), v_n) + \frac{1}{2}(v_n^2, \Delta\phi)\right] \\
&= (-A(v\phi), v) + \frac{1}{2}(v^2, \Delta\phi) \\
&= (-\Delta(v\phi), v) + \frac{1}{2}(\Delta(v^2), \phi) \\
&= (-v\phi, \Delta v) + (|\nabla v|^2 + v\Delta v, \phi) \\
&= (-v\phi, \lambda v) + (|\nabla v|^2 + \lambda v^2, \phi) \\
&= \int \phi |\nabla v|^2.
\end{aligned}$$

By letting $\phi \uparrow 1$, $\mathcal{E}_A[u] \ge \mathcal{E}[u]$. Since

$$\mathcal{F}_A = W_0^{1,2} \oplus \{u \in \mathcal{F}_A : \Delta u = \lambda u\}, \quad \lambda > 0,$$

any $w \in \mathcal{F}_A$ can be decomposed as $w = \eta + u$, where $\eta \in W_0^{1,2}$ and $u \in \mathcal{F}_A$ satisfies $\Delta u = \lambda u$. Now,

$$\mathcal{E}[w] \le \mathcal{E}[\eta] + \mathcal{E}[u] \le \mathcal{E}[\eta] + \mathcal{E}_A[u] = \mathcal{E}_A[w] < \infty,$$

and $w \in W^{1,2}$. Thus $\mathcal{F}_A \subset W^{1,2}$ and we arrived at the conclusion. □

The following, which is (ii) of Theorem 1.7, is easy

**Lemma 3.6.** *The following conditions are equivalent.*
(*i*) $\Delta_D = \Delta_N$.
(*ii*) $\Delta_G$ *is self-adjoint.*
(*iii*) $\Delta_M$ *has a unique Markov extension.*

*Proof.* The implication (i) $\Rightarrow$ (ii) follows from the definition of $\Delta_D$, $\Delta_N$, and $\Delta_G$. Since $\Delta_D \subset \Delta_G$ and $\Delta_N \subset \Delta_G$, if $\Delta_G$ is self-adjoint, then $\Delta_D = \Delta_G = \Delta_N$, which is (ii) $\Rightarrow$ (i). The equivalence between (iii) and (i) follows from the fact that $\Delta_D$ and $\Delta_N$ are Markovian, and they are the minimum and maximum elements of Markovian operators by Proposition 3.5. □

The next is (iii) of Theorem 1.7.

**Lemma 3.7.** *If $M$ is stochastically complete or geodesically complete, or $\partial_C M$ is polar, then $W_0^{1,2} = W^{1,2}$.*

*Proof.* Suppose $W_0^{1,2} \neq W^{1,2}$. Then two operators $\Delta_D$ and $\Delta_N$ are not identical, and hence $e^{t\Delta_D} \neq e^{t\Delta_N}$. Taking into account that the kernel of $e^{t\Delta_D}$ is the smallest positive fundamental solution to the heat equation,

$$u(t) := \left(e^{t\Delta_N} - e^{t\Delta_D}\right) u > 0$$

for a non-trivial $u \in C_0$ with $u \geq 0$. Since $u(t)$ is a bounded solution to the Cauchy problem with initial data 0, $M$ is stochastically incomplete by Theorem 6.2 in [22].

We already showed that the polarity of $\partial_C M$ implies $W_0^{1,2} = W^{1,2}$ in Proposition 2.2. In particular, since the Cauchy boundary of a geodesically complete manifold is empty and it is polar, $W_0^{1,2} = W^{1,2}$. (The last fact is well known. See for e.g., [12], [1]). □

## 4. Polarity of the Cauchy boundary

This section consists of two subsections. In Subsection 4.1, we present some examples of manifolds which demonstrate that we can not drop certain conditions from main theorems. In Subsection 4.2 we will mainly study the Cauchy boundary. We will present an example of $\partial_C M$ which has co-dim($\partial_C M$)=2 but not polar, and an example of a non-parabolic geodesically complete manifold with $V(r) \sim r^2$ for large $r > 0$ (Proposition 4.4). We also present an example which demonstrates that if $\partial_C M$ has infinite capacity, then both $W^{1,2} = W_0^{1,2}$ and $W^{1,2} \neq W_0^{1,2}$ may occur (Proposition 4.5).

Our examples are warp-prodcuts or model manifolds. Let us recall the definitions and their Laplacians and the Green functions. For further properties of a model manifold, see [22]. The product $N = (0, \infty) \times \mathbb{S}^n$ with the Riemannian metric

$$dr^2 + \sigma^2(r) g_\theta,$$

where $g_\theta$ is the Riemannian metric of $\mathbb{S}^n$ and $\sigma = \sigma(r)$ is a positive smooth function, is called the *warp product* of $(0, \infty)$ and $\mathbb{S}^n$. The condition

$$\sigma(0) = 0 \text{ and } \lim_{r \to 0} \sigma'(r) = 0 \tag{20}$$

is the necessary and sufficient condition such that the Riemannian metric extends to $o := \{0\} \times \mathbb{S}^n$. The point $o$ is called the *pole* of $N$ and it is the Cauchy boundary $\partial_C N$. If (20) is satisfied, then the manifold $M = N \cup \{o\}$ is called a *model manifold.* Clearly, a model manifold is geodesically complete.

The volume element is

$$\omega_n \sigma^n dr,$$

where $\omega_n$ is the volume of $\mathbb{S}^n$. The surface area $S = S(r)$ of the boundary $\partial B(o, r)$ of $B(o, r)$ is

$$S(r) = \omega_n \sigma^n(r)$$

and the volume $V(r)$ of $B(o, r)$ is

$$V(r) = \int_0^r S(\xi)\, d\xi = \omega_n \int_0^r \sigma^n(\xi)\, d\xi.$$

The associated Laplacian is

$$\Delta u = u'' + \frac{\sigma'}{\sigma} u' + \frac{1}{\sigma^2} \Delta_\theta u = u'' + \frac{S'}{S} u' + \frac{1}{\sigma^2} \Delta_\theta u, \tag{21}$$

where the prime stands for the derivative with $r > 0$ and $\Delta_\theta$ is the Laplacian on $\mathbb{S}^n$. By (21), the positive function[4]:

$$g(x) = \int_{r(x)}^\infty \frac{dr}{S(r)} \tag{22}$$

solves Laplace's equation. If $M$ is a model manifold, the function (22) is Green's function $g$ with pole at $o$.

4.1. **Examples.** The following is the example to Theorem 1.2.

**Proposition 4.1.** *Let $M$ be a model manifold and $g$ be Green's function. Then*

$$\textit{non-parabolic} \quad \Longleftrightarrow \quad \int^\infty \frac{dr}{S(r)} < \infty \quad \Longleftrightarrow \quad \textit{finiteness of } g.$$

*If $M$ is non-parabolic, then*

$$\textit{stochastic incomplete} \quad \Longleftrightarrow \quad \int^\infty \frac{V(r)}{S(r)} dr < \infty \quad \Longleftrightarrow \quad g \in L^1 \ \textit{(outside a compact)}.$$

*In particular, if $M$ is stochastically complete, there exists a positive super-harmonic function $u$ such that $u \in L^1 \cap L^\infty$, $\Delta u \in L^2$, and*

$$\int \Delta u < 0.$$

*Proof.* The implications for parabolicity follow from (22) and the definition. The first equivalence for the stochastic completeness can be found in [22]. For the second equivalence, observe

$$\begin{aligned}
\int_{B^c(o,1)} g\, d\mu &= \int_1^\infty g(r)\, S(r) dr \\
&= \int_1^\infty S(r) \int_r^\infty \frac{dt}{S(t)} dr \\
&= \iint_{1 \le r \le t < \infty} \frac{g(r)}{S(t)}\, dt dr \\
&= \int_1^\infty \frac{1}{S(t)} \int_1^t S(r) dr dt \\
&= \int_1^\infty \frac{V(t) - V(1)}{S(t)} dt.
\end{aligned}$$

[4] It may be identically $+\infty$.

Next assume that $M$ is stochastically incomplete and let $\phi \in C^\infty(0,\infty)$ be a super-harmonic function satisfying that

$$\psi(t) = \begin{cases} t, & 1 < t, \\ 2, & t > 3. \end{cases}$$

Set $u = \psi(g)$. Clearly $u \in L^1 \cap L^\infty$ and $\Delta u$ has compact support. Moreover, $u$ is super-harmonic and $\int \Delta u < 0$. □

The next proposition shows that we may not drop the conditions $u \in L^1$ in (5) and $\nabla u \in L^1$ in (6), respectively.

**Proposition 4.2.** *Let $M$ be an $(n+1)$-dimensional model manifold. We assume that $\sigma(r) = r^s$ with $s > 0$ at infinity. If $sn > 3$, then $M$ is stochastically complete and there exists a measurable function $u$ such that*

$$\int \Delta u \neq 0, \quad u \in D(\Delta_D) \setminus L^1,\ \nabla u \notin L^1,\ \textit{and}\ \Delta u \in L^1.$$

*Proof.* Since

$$V(r) = \omega_n \int_0^r \xi^{sn}\, d\xi = \frac{\omega_n}{sn+1} r^{sn+1},$$

$M$ is stochastically complete by the volume test (2).

Let $g$ be Green's function with pole at $o$:

$$g(x) = \int_{r(x)}^\infty \frac{dr}{S(r)} = \frac{1}{\omega_n} \int_{r(x)}^\infty r^{-sn}\, dr = \frac{1}{(sn-1)\omega_n} r(x)^{1-sn}.$$

Let $\psi \in C^\infty(0,\infty)$ be super-harmonic such that

$$\psi(t) = \begin{cases} t, & \text{if } t < 1 \\ 2, & \text{if } t > 3, \end{cases}$$

and $\psi''(t) < 0$ if $t \in (1,3)$. Set

$$u(x) = \begin{cases} \psi(g(x)), & \text{if } x \neq 0 \\ 2, & \text{if } x = 0. \end{cases}$$

Then

$$\int u \geq C(n,s) \int_{g \leq 3} r^{1-sn} r^{sn}\, dr = \infty$$

and

$$\begin{aligned} \int u^2 &\leq \int_{\{g>1\}} 2^2 + \int_{\{g<1\}} g^2 \leq 4\mu(g>1) + C(n,s) \int_{\{g<1\}} r^{2(1-sn)} r^{sn}\, dr \\ &= 4\mu(g>1) + C(n,s) \int_{\{g<1\}} r^{2-sn}\, dr, \end{aligned}$$

which is finite since $sn > 3$. Since $\Delta g = 0$ if $r > 0$, it follows that

$$\begin{aligned} \Delta u(x) &= \left(\psi''(g)|\nabla g|^2 + \psi'(g)\Delta g\right)(x) \\ &= \psi''(g)|\nabla g|^2(x) \begin{cases} < 0, & \text{if } 1 < g(x) < 3 \\ = 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Thus,

$$\Delta u \in L^1 \cap L^2 \text{ and } \int_M \Delta u < 0.$$

Since $u$ and $\Delta u$ are in $L^2$ and $M$ is geodesically complete, $u \in D(\Delta)$. Since $\nabla u = \phi'(g)\nabla g$,

$$\int |\nabla u| \geq \int_{g<1} |\nabla g| = \int_{g<1} r^{-sn} r^{sn} dr = \infty.$$

$\square$

Any Euclidean space $\mathbb{R}^n$ with $n \geq 5$ satisfies the condition of Proposition 4.2 since $\sigma(r) = r$ and $s = 1$.

The next proposition shows that we may not drop the condition $\nabla u \in L^2$ from (4) and (6).

**Proposition 4.3.** *Let $N$ be a 3-dimensional non-parabolic model manifold and $M = N \setminus \{o\}$. There exists a measurable function $u$ on $N$ such that*

$$\int \Delta u \neq 0, \qquad u \in L^1 \cap L^2,\ \nabla u \in L^1 \setminus L^2,\ \textit{and}\ \Delta u \in L^1 \cap L^2.$$

*Note that the Cauchy boundary $\partial_C M = \{o\}$ is polar and hence $W^{1,2}(N) = W_0^{1,2}(N)$. The manifold can be stochastically complete.*

*Proof.* Let $g(\cdot) = g_N(x_o, \cdot)$, where $g_N$ is Green's function of $N$. Let $\psi \in C^\infty(\mathbf{R})$ be a convex function defined as

$$\psi(t) = \begin{cases} 0, & t \leq 1, \\ t-2, & t \geq 3, \end{cases}$$

and $\psi'' > 0$ if $t \in (1,3)$. Set $u = \psi(g) \in C^\infty(M)$. Since the magnitude of the singularity of $g$ is $r^{-1}$, where $r$ is the distance from $x_o$,

$$\int_M u^k \asymp \int_{g>1} (\psi(g(r)))^k r^2 dr \asymp \int_{g>1} (r^{-1}-2)^k r^2\, dr < \infty \text{ if } k = 1, 2,$$

thus $u \in L^1 \cap L^2$. Since $\Delta g = 0$ and $\nabla g \neq 0$, it follows that

$$\Delta u(x) = \left(\psi''(g)|\nabla g|^2 + \psi'(g)\Delta g\right)(x) \begin{cases} > 0, & \text{if } 1 < g(x) < 3 \\ = 0, & \text{otherwise.} \end{cases}$$

Thus, $\Delta u \in L^1 \cap L^2$. On the other hand, since for small $r$

$$|\nabla u(x)| = |\psi'(g)\nabla g(x)| \asymp r(x)^{-2},$$

we obtain

$$\int |\nabla u| = \int_{g>1} |\nabla u| \asymp \int_{g>1} r^{-2} r^2 dr < \infty,$$

and $\nabla u \notin L^2$. $\square$

The Euclidean space $\mathbb{R}^3$ is parabolic and satisfies the condition of Proposition 4.3.

4.2. **Some notes about the Cauchy boundary.** Let us recall some known sufficient conditions for $\partial_C M$ to be polar which is closely related with our examples.

A very general criteria is the following: If $\partial_C M$ is compact and

$$\int_0 \frac{\rho\, d\rho}{V(\rho)} = \infty, \tag{23}$$

then $\partial_C M$ is polar. This statement can be proven in a similar way of the proof of Theorem 7.1 [22]. This condition is satisfied; for example, if $V(\rho) \leq \rho^2$, or $V(\rho) \leq \rho^2(\ln(1/\rho))$, or if there exists $\rho_k \to 0$ such that

$$V(\rho_k) \leq \text{const}\, \rho_k^2.$$

Concrete examples, whose $\partial_C M$'s polarity can be stated in terms of a certain "co-dimension", are the following:

- Let $M$ be a manifold with polar Cauchy boundary and $\Sigma$ be its compact submanifold. If $\Sigma$ has co-dimension equal or greater than 2, then $M \setminus \Sigma$ has polar Cauchy boundary.
- If $M$ is an algebraic variety in $\mathbb{CP}^n$ or an Riemannian orbifold, then the singular set $\Sigma \subset M$ is the Cauchy boundary of its regular part, $M \setminus \Sigma$. If $\Sigma$ has a (real) co-dimension equal or greater than 2, then it is polar (see for e.g., M. Nagase [34] and P. Li and G. Tian [30] for algebraic varieties and T. Shioya [36] for Riemannian orbifolds.).
- The lower-Minkowski codimension $\underline{\mathrm{codim}}_M(\partial_C M)$ of $\partial_C M$ is

$$\liminf_{\rho \to 0} \left( \frac{\ln V(\rho)}{\ln \rho} \right).$$

If $\underline{\mathrm{codim}}_M(\partial_C M) \geq 2 + \epsilon$ with some $\epsilon > 0$ then (23) is satisfied [31].

It is easy to show that (23) implies $\underline{\mathrm{codim}}_M(\partial_C M) \geq 2$ and all the Cauchy boundaries of the examples above satisfy this estimate. However, the opposite implication does not need to be true. Namely,

**Proposition 4.4.** (*a*) *There exists a Cauchy boundary $\partial_C M$, which is a manifold, the Minkowski co-dimension is* 2, *namely,*

$$\lim_{r \to 0} \frac{\ln V(r)}{\ln r} = 2,$$

*not polar, and $W_0^{1,2}(M) \neq W^{1,2}(M)$.*

(*b*) *There exists a non-parabolic model manifold $M$ such that*

$$\lim_{r \to \infty} \frac{\ln V(r)}{\ln r} = 2.$$

*Proof of (a).* Let $M = (0, \infty) \times \mathbb{S}^n$ with $n \geq 1$ be the warp product of $(0, \infty)$ and $\mathbb{S}^n$. Let

$$\sigma(r) = \left( \frac{r^f (f \ln r)'}{\omega_n} \right)^{1/n}, \quad r \in (0, 1/2),$$

where

$$f(r) = 2 + (1 + \epsilon) \frac{\ln |\ln r|}{\ln r}, \quad \epsilon > 0, \ r > 0.$$

A direct calculation shows that $\lim_{r \to 0} \sigma(r) = 0$. The Cauchy boundary is the point $\{0\} \times \mathbb{S}^n$. We also find $\lim_{r \to 0} \sigma'(r) \neq 0$. (Thus the Riemannian metric does not extends to the Cauchy boundary.)

Then $V(r) = r^{f(r)}$ for $r \in (0, 1/2)$ and

$$\lim_{r \to 0} \frac{\ln V(r)}{\ln r} = \lim_{r \to 0} f(r) = 2.$$

We claim that $V$ is convex for small $r > 0$. Defining a function $f$ from the identity $\ln V = f \ln r$, we obtain $V'/V = (f \ln r)'$, $V' = V (f \ln r)'$, and

$$\begin{aligned} V'' &= V'(f \ln r)' + V (f \ln r)'' = V((f \ln r)')^2 + V (f \ln r)'' \\ &= V[((f \ln r)')^2 + (f \ln r)'']. \end{aligned} \tag{24}$$

Obviously, we have

$$f' = (1+\epsilon)\left[\frac{\frac{\ln r}{r\ln r} - \frac{\ln|\ln r|}{r}}{(\ln r)^2}\right] = (1+\epsilon)\frac{(1-\ln|\ln r|)}{r(\ln r)^2}$$

$$\begin{aligned} f'' &= (1+\epsilon)\frac{-\frac{r(\ln r)^2}{r\ln r} - (1-\ln|\ln r|)(r(\ln r)^2)'}{(r(\ln r)^2)^2} \\ &= (1+\epsilon)\frac{-\ln r - (1-\ln|\ln r|)(\ln r)(\ln r + 2)}{(r(\ln r)^2)^2} \\ &= -(1+\epsilon)\frac{\ln r}{(r(\ln r)^2)^2}[1 + (1-(\ln|\ln r|))(\ln r + 2)], \end{aligned}$$

whence it follows that

$$\begin{aligned} (f\ln r)'' &= f''\ln r + 2f'(\ln r)' + f(\ln r)'' = f''\ln r + 2f'r^{-1} - fr^{-2} \\ &= -(1+\epsilon)\frac{[1+(1-\ln|\ln r|)(\ln r+2)]}{(r\ln r)^2} + 2(1+\epsilon)\frac{(1-\ln|\ln r|)}{(r\ln r)^2} \\ &\quad - r^{-2}\Big[2 + (1+\epsilon)\frac{\ln|\ln r|}{\ln r}\Big] \\ &= \frac{(1+\epsilon)}{(r\ln r)^2}[-1 + \frac{\ln|\ln r|}{\ln r} - \ln r] - r^{-2}\Big[2 + (1+\epsilon)\frac{\ln|\ln r|}{\ln r}\Big] \\ &\sim -\frac{2}{r^2} \text{ as } r \to 0 \end{aligned}$$

On the other hand, we have,

$$\begin{aligned} ((f\ln r)')^2 = (f'\ln r + f/r)^2 &= \left[\frac{(1+\epsilon)(1-\ln|\ln r|)}{r\ln r} + \frac{2 + (1+\epsilon)\frac{\ln|\ln r|}{\ln r}}{r}\right]^2 \\ &= r^{-2}\left[\frac{(1+\epsilon)(1-\ln|\ln r|)}{\ln r} + 2 + (1+\epsilon)\frac{\ln|\ln r|}{\ln r}\right]^2 \\ &\sim \frac{4}{r^2} \text{ as } \ r \to 0 \end{aligned}$$

Thus, comparing this with the above estimate of $(f\ln r)''$ and taking into account (24), we see that there exists $R \in (0, 1/2)$ such that

$$V''(r) \geq 0 \ \text{ for } \ r \in (0, R).$$

Therefore,

$$V'(r) \geq \frac{(V(r) - V(0))}{r}$$

and

$$\int_r^R \frac{dr}{V'(r)} \leq \int_r^R \frac{rdr}{V(r)} = \int_r^R \frac{dr}{r^{1+(1+\epsilon)\ln|\ln r|/\ln r}} = \int_r^R \frac{dr}{r|\ln r|^{1+\epsilon}} = [|\ln r|^{-\epsilon}]_r^R. \qquad (25)$$

Let us extend $V(r)$ for $r \geq R$ to satisfy

$$\int_R^\infty \frac{dr}{V'(r)} = C < \infty.$$

Let $\phi$ be the solution to the following boundary problem in $\Omega(r, r') := B(r') \setminus \overline{B(r)}$ with $r < r'$:

$$\Delta\phi = 0, \ \phi|_{\partial B(r)} = 1, \ \phi|_{\partial B(r')} = 0.$$

By (21), we have

$$\phi(s) = \left(\int_r^{r'} \frac{dr}{V'(r)}\right)^{-1} \int_s^{r'} \frac{d\rho}{V'(\rho)}.$$

Hence,

$$\begin{aligned}
\mathrm{Cap}(B(r)) &\geq \lim_{r'\to\infty} \int_{\Omega(r,r')} |\nabla \phi|^2 \, d\mu = \int_{\partial B(r)} \frac{\partial \phi}{\partial \nu} \, d\sigma \\
&= \left(\int_r^{\infty} \frac{dr}{V'(r)}\right)^{-1} \\
&= \left(\int_r^{R} \frac{dr}{V'(r)} + \int_R^{\infty} \frac{dr}{V'(r)}\right)^{-1}.
\end{aligned}$$

By (25), the last expression is bounded from below by

$$\left(\int_r^R \frac{rdr}{V(r)} + C\right)^{-1} = \left([|\ln r|^{-\epsilon}]_r^R + C\right)^{-1} \to \left(|\ln R|^{-\epsilon} + C\right)^{-1} > 0, \quad \text{as } r \to 0.$$

This shows that $\partial_C M$ is not polar. Because $\mathrm{Cap}(\partial_C M) < \infty$, we conclude by Proposition 2.2 that $W_0^{1,2}(M) \neq W^{1,2}(M)$. □

*Proof of (b).* Let $f$ be the same function as in the previous example. Assume that the Riemannian metric extends to $\partial_C M$ and $\overline{M}$ is a geodesically complete Riemannian manifold; namely, a model manifold. If $V(r) = r^{f(r)}$ for large $r > 0$, then

$$\lim_{r\to\infty} \frac{V(r)}{r^2} = 2$$

and

$$\int^{\infty} \frac{rdr}{V(r)} < \infty.$$

Furthermore, in the similar way as above, we find $V''(r) \geq 0$ for large $r > 0$ which implies that $M$ is not parabolic. □

**Proposition 4.5.** (*a*) *There exists a stochastically complete manifold $M$ such that* $\mathrm{Cap}(\partial_C M) = \infty$. *In particular,* $W^{1,2} = W_0^{1,2}$.

(*b*) *There exists a stochastically incomplete manifold $M$ such that* $\mathrm{Cap}(\partial_C M) = \infty$ *and* $W^{1,2} = W_0^{1,2}$.

(*c*) *There exists a manifold $M$ such that* $\mathrm{Cap}(\partial_C M) = \infty$ *and* $W^{1,2} \neq W_0^{1,2}$*; for instance,* $M = \mathbb{R}^2 \setminus \mathbb{R}$ *with standard Euclidean measure.*

*Proof of (a).* Consider $M = ((0,1];\mu)$, where $d\mu(x) = dx/x$. We impose the Neumann boundary condition at $x = 1$. The Cauchy boundary is the point $\partial_C M = \{0\}$ and the volume of $B(0,r)$ is $\int_0^r dx/x = \infty$ for any $r \in (0,1)$. Therefore, the $L^2$-norm of any function which is 1 on a neighborhood of $\partial_C M$ is infinite, thus, $\mathrm{Cap}(\partial_C M) = \infty$.

Next we show the stochastic completeness. Let $r$ be the distance from $x = 1$ and

$$V(r) = \int_0^r \frac{dx}{1-x}.$$

Then

$$\int^{\rho} \frac{V(r)}{V'(r)} dr = \int^{\rho} \frac{-\ln(1-r)}{1-r} dr = \frac{1}{2}\left(\ln(1-\rho)\right)^2 \to \infty, \quad \rho \to 1.$$

This implies the stochastic completeness (see Section 6 [22]). In particular, $W^{1,2} = W_0^{1,2}$. □

*Proof of (b).* Consider $M = ((0,\infty);\mu)$, where the measure $\mu$ and the volume $V(r)$ satisfy: $d\mu(x) = dx/x$ for $x$ close to 0 and

$$\int^{\infty} \frac{V(r)}{V'(r)} dr < \infty.$$

Then $M$ is stochastically incomplete (Section 6 [22]). We have showed $\mathrm{Cap}(\partial_C M) = \infty$ above.

We show that $W^{1,2} = W_0^{1,2}$. For $u \in W^{1,2}$, we construct a sequence $u_\epsilon \in W_0^{1,2}$ such that $u_\epsilon \to u$ in $W^{1,2}$ as $\epsilon \to 0$. Since $W^{1,2} \cap C^\infty$ is dense in $W^{1,2}$, we may assume that $u$ is smooth without loss of generality. Then $u$ should satisfy:

$$u(0) = \lim_{x\to 0} u(x) = 0 \quad \text{ and } \quad u'(0) = \lim_{x\to 0} u'(x) = 0.$$

Set

$$\psi_\epsilon(t) = \begin{cases} (t-\epsilon)_+, & \text{if } t \geq 0, \\ (t+\epsilon)\wedge 0, & \text{if } t < 0, \end{cases}$$

with $\epsilon > 0$ and $u_\epsilon = \psi_\epsilon(u)$. If $u(x) < \epsilon$ then $u_\epsilon(x) = \psi_\epsilon(u_\epsilon(x)) = 0$. Since $u(0) = 0$, $u_\epsilon \in W_0^{1,2}$ with any $\epsilon > 0$. Set

$$O_\epsilon = \{x \in M : |u(x)| < \epsilon\} \quad \text{ and } \quad C_\epsilon = M \setminus O_\epsilon.$$

If $v_\epsilon = u - u_\epsilon$, then

$$v_\epsilon(x) = \begin{cases} u(x), & \text{if } x \in O_\epsilon, \\ \pm\epsilon, & \text{if } u(x) \in C_\epsilon. \end{cases}$$

$$\int_0^1 v_\epsilon^2\, d\mu = \int_{O_\epsilon} v_\epsilon^2\, d\mu + \int_{C_\epsilon} v_\epsilon^2\, d\mu = \int_{O_\epsilon} u^2\, d\mu + \int_{C_\epsilon} \epsilon^2\, d\mu =: (\mathrm{I}) + (\mathrm{II}).$$

Set $O_u = \{x \in M : u(x) = 0\}$. Since $\mathbf{1}_{O_\epsilon \setminus O_u} \to 0$ $\mu$-a.e. as $\epsilon \to 0$,

$$(\mathrm{I}) = \int_{O_\epsilon} u^2 \frac{dx}{x} = \int_{O_\epsilon \setminus O_u} u^2 \frac{dx}{x} \to 0, \quad \epsilon \to 0.$$

Since $u'(0) = 0$, we may assume that $|u'(x)| \leq 1$ for any $x \in (0,\epsilon)$ for sufficiently small $\epsilon > 0$, and

$$|u(x)| \leq \int_0^x |u'(\xi)|\, d\xi \leq x.$$

If $x_\epsilon = \min\{x > 0 : u(x) = \epsilon\}$, then $x_\epsilon \geq \epsilon$. Thus,

$$(\mathrm{II}) \leq \int_{x_\epsilon}^1 \epsilon^2 \frac{dx}{x} = -\epsilon^2 \ln(x_\epsilon) \leq -\epsilon^2 \ln(\epsilon) \to 0, \quad \epsilon \to 0.$$

Since

$$\int_0^1 (v_\epsilon')^2 \frac{dx}{x} = \int_{O_\epsilon} (u')^2 \frac{dx}{x} = \int_{O_\epsilon \setminus O_u} (u')^2 \frac{dx}{x} \to 0, \quad \epsilon \to 0,$$

$u_\epsilon \to u$ in $W^{1,2}$ as $\epsilon \to 0$ and $W^{1,2} = W_0^{1,2}$. $\square$

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