# RANGE OF FLUCTUATION OF BROWNIAN MOTION ON A COMPLETE RIEMANNIAN MANIFOLD

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Contents



Abstract. We investigate the escape rate of the Brownian motion $W_x(t)$ on a complete non-compact Riemannian manifold. Assuming that the manifold has at most polynomial volume growth and that its Ricci curvature is bounded below, we prove that

$$\operatorname{dist}(W_x(t), x) \leq \sqrt{Ct \log t}$$

for all large $t$ with probability 1. On the other hand, if the Ricci curvature is nonnegative and the volume growth is at least polynomial of the order $n > 2$, then

$$\operatorname{dist}(W_x(t), x) \geq \frac{\sqrt{Ct}}{\log^{\frac{1}{n-2}} t \log \log^{\frac{2+\varepsilon}{n-2}} t}$$

again for all large $t$ with probability 1 (where $\varepsilon > 0$).

## 1. Introduction and main results

Let $M$ be a smooth connected Riemannian manifold and let $\Delta$ be the Laplace operator of the Riemannian metric of $M$. We consider the minimal diffusion $W_x(t)$ on $M$ (starting at the point $x \in M$) generated by the operator $\frac{1}{2}\Delta$. Let us denote by $\mathbb{P}_x$ the corresponding probability measure on the paths emanating from $x$.

This paper is primarily concerned with the distance $\operatorname{dist}(W_x(t), x)$ the process $W_x(t)$ moves from the initial point $x$ over time $t$. The *escape rate* of the Brownian motion is measured by estimates of this distance for large $t$. A core objective of the paper is to relate the escape rate to the appropriate geometric properties of the manifolds.

In order to avoid trivial situations, we assume henceforth that the manifold in question is non-compact and geodesically complete. Moreover, we will deal only with stochastically complete

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manifolds. A manifold is *stochastically complete* if for all $x \in M$ and $t > 0$, $\mathbb{P}_x \{W_x(t) \in M\} = 1$, which prevents the Brownian particle from reaching infinity in a finite time. In what follows, we either explicitly assume stochastical completeness or it will be consequence of other hypotheses.

The movement of the Brownian particle can be described in terms of *an upper radius* and *a lower radius.* Let us denote by $B(x, r)$ the geodesic ball of radius $r$ centred at the point $x \in M$.

**Definition 1.1.** *For a fixed a point $x \in M$, a non-negative function $R(t)$ is called the upper radius of the process $W_x(t)$ if*

$$\mathbb{P}_x \{\exists T > 0 \, s.t. \; \mathrm{dist}\,(W_x(t), x) \le R(t) \; for\ all\ t > T\} = 1\,.$$

*A non-negative function $r(t)$ is called the lower radius of the process $W_x(t)$ if*

$$\mathbb{P}_x \{\exists T > 0 \;\; s.t.\, \mathrm{dist}\,(W_x(t), x) > r(t)\, for\ all\ t > T\} = 1\,.$$

In other words, the process $W_x(t)$ stays a.s. within the annulus $\overline{B(x, R(t))} \backslash B(x, r(t))$ for large enough $t$.

It follows from the definition that if $R_1(t) \ge R(t)$ for large $t$ and if $R(t)$ is an upper radius, then $R_1(t)$ is also an upper radius, and the same is true (with the opposite inequality) for a lower radius. The spheres $\partial B(x, R(t))$ and $\partial B(x, r(t))$ can be regarded as a forefront and a rear front, respectively, of the diffusion as $t \to \infty$.

It is obvious that if $R(t)$ and $r(t)$ are upper and lower radii, respectively, then with probability 1

$$\limsup_{t \to \infty} \frac{\mathrm{dist}\,(W_x(t), x)}{R(t)} \le 1 \tag{1.1}$$

and

$$\liminf_{t \to \infty} \frac{\mathrm{dist}\,(W_x(t), x)}{r(t)} \ge 1. \tag{1.2}$$

In $\mathbb{R}^n$, as a consequence of the law of the iterated logarithm, equality holds in (1.1) for $R(t) = \sqrt{2t \log \log t}$ (see [11] and[14], and also [1] for the modern account of the law of the iterated logarithm and related topics). The function

$$R(t) = \sqrt{(2 + \varepsilon) t \log \log t}$$

is an upper radius for any $\varepsilon > 0$, and it is not if $\varepsilon \le 0$.

The lower radius case is different. If the manifold $M$ is parabolic (which means that the Brownian motion on $M$ is recurrent), then

$$\liminf_{t \to \infty} \mathrm{dist}\,(W_x(t), x) = 0\,,$$

and a lower radius $r(t)$ cannot be bounded away from 0, so that this case is not particularly interesting. On the contrary, if the manifold $M$ is nonparabolic, then the lower radius $r(t)$ can be regarded as a "measure" of transience.

As was shown by Dvoretzky and Erdös [5], in $\mathbb{R}^n$ with $n > 2$ the following function is a lower radius:

$$r(t) = \frac{C\sqrt{t}}{\log^{\frac{1}{n-2}} t \, (\log \log t)^{\frac{1+\varepsilon}{n-2}}} \tag{1.3}$$

for any $\varepsilon > 0$ and $C > 0$, and it is not a lower radius for $\varepsilon = 0$ irrespective of $C$. It seems that there is no sharp lower radius for which the limit (1.2) would be equal to 1: at least if the function $r(t)/\sqrt{t}$ is decreasing, then this limit is either $\infty$ or 0 as follows from the theorem of Dvoretzky and Erdös [5] (see also [11], Section 4.12). In other words, the rear front of the Brownian motion is not as distinct as the forefront.

We shall construct upper and lower radii in the setting of manifolds of polynomial volume growth, under some additional geometric assumption. The case of superpolynomial volume growth will be addressed elsewhere.

Let us first introduce the necessary definitions. We say that the ball $B(x,R) \subset M$ possesses $(a,\nu)$-*isoperimetric inequality* if for any region $\Omega \subset\subset B(x,R)$ we have

$$\lambda_1(\Omega) \geq \frac{a}{R^2}\left(\frac{\mathrm{Vol}B(x,R)}{\mathrm{Vol}\Omega}\right)^{\nu}, \tag{1.4}$$

where $a$ and $\nu$ are positive constants and $\lambda_1(\Omega)$ denotes the first eigenvalue of the Laplace operator in $\Omega$ with the Dirichlet boundary condition.

For example, in $\mathbb{R}^n$ any ball possesses $(a,\nu)$-isoperimetric inequality with $\nu = \frac{2}{n}$ and $a = a_n$. Indeed, since $\mathrm{Vol}B(x,R) \sim R^n$ then (1.4) amounts to

$$\lambda_1(\Omega) \geq a\,(\mathrm{Vol}\Omega)^{-2/n}$$

which is true by the Faber-Krahn theorem with the constant $a = a_n$ depending only on $n$.

By the compactness argument, this implies that *any* ball on any geodesically complete Riemannian manifold possesses $(a,\nu)$-isoperimetric inequality with *some* positive $a$ and $\nu = 2/n$, but the number $a$ will in general depend on the ball.

As was shown in [8], all balls on a manifold of nonnegative Ricci curvature possess $(a_n, 2/n)$-isoperimetric inequality with the same constant $a_n$ where $n$=dim$\,M$. If the manifold has a (possibly negative) bounded-below Ricci curvature then the same is true for all balls of the bounded radius $R < \rho$, and $a$ depends on $n, \rho$ and the lower bound on the Ricci curvature (see Appendix C for more details).

We say that $M$ is a manifold with *a weak bounded geometry* if there are positive numbers $a, \rho, \nu$ such that any ball of radius smaller than $\rho$ on $M$ possesses an $(a,\nu)$-isoperimetric inequality. Normally, one has $\nu = 2/n$. The number $\rho$ is referred to as *a bounded geometry radius.* For example, any manifold of a Ricci curvature bounded from below, possesses a weak bounded geometry as was explained above.

Let us state our main results:

**Theorem 1.1.** *Let $M$ be a complete non-compact Riemannian manifold with a weak bounded geometry, and assume that, for some $x \in M$ and for all $R > R_0$,*

$$\mathrm{Vol}B(x,R) \leq A\,R^N \tag{1.5}$$

(*where $A > 0$ and $N > 0$ do not depend on $R$*). *Then the function*

$$\mathcal{R}(t) = \sqrt{(2N+4)t\log t}$$

*is an upper radius for $W_x(t)$.*

The constant $2N+4$ is not claimed to be sharp. The function $\sqrt{t\log t}$ is the same as that in the old theorem of Hardy and Littlewood for sums of independent Bernoulli random variables (which was later improved by Khinchin to $\sqrt{2t\log\log t}$). It is likely that, for certain manifolds satisfying the hypotheses of Theorem 1.1, the function $\sqrt{Ct\log\log t}$ is not an upper radius.

**Theorem 1.2.** *Let $M$ be a complete noncompact Riemannian manifold and suppose that the following are true*:

(U) (*uniform isoperimetric inequality*) *there are constants $a > 0$ and $\nu > 0$ such that $(a, 2/\nu)$-isoperimetric inequality holds in every geodesic ball of positive radius*;

(V) (*volume comparison condition*) *for some $n > 2$, for some $x \in M$ and for all sufficiently large $R$ and $r$ such that $R > r$,*

$$\frac{\mathrm{Vol}B(x,R)}{\mathrm{Vol}B(x,r)} \geq c\left(\frac{R}{r}\right)^n \tag{1.6}$$

*with a positive constant $c$.*

*Then for any $\varepsilon > 0$ and $C > 0$ the function*

$$r(t) = \frac{C\sqrt{t}}{\log^{\frac{1}{n-2}} t\,(\log\log t)^{\frac{2+\varepsilon}{n-2}}} \tag{1.7}$$

*is a lower radius for* $W_x(t)$.

In the Euclidean case Dvoretzky and Erdös [5] obtained a better power of $\log\log t$ in (1.3) than that in (1.7). In view of this, it seems likely that our result is not the sharpest possible, but we have not yet succeeded in replacing the exponent $\frac{2+\varepsilon}{n-2}$ by $\frac{1+\varepsilon}{n-2}$.

Let us note that hypothesis (U) implies that, for any $R > r > 0$,

$$\frac{\mathrm{Vol}B(x,R)}{\mathrm{Vol}B(x,r)} \leq \mathrm{const}_{a,\nu}\left(\frac{R}{r}\right)^{\nu}$$

(see [9], Proposition 5.2). In particular, the volume growth in this setting is polynomial.

By the theorem of Cheng and Yau [3], if $V(x,R) \leq \mathrm{const}R^2$ for $R \to \infty$, then the manifold is parabolic. Therefore, we cannot drop the condition (1.6) with $n > 2$ which is partly intended to exclude parabolicity. Moreover, under hypothesis (U), nonparabolicity of the manifold is equivalent to

$$\int^{\infty} \frac{dt}{\mathrm{Vol}B(x,\sqrt{t})} < \infty\,, \tag{1.8}$$

which follows from the estimates of the heat kernel

$$\frac{C_1}{V\left(x,\sqrt{t}\right)} \leq p(t,x,x) \leq \frac{C_2}{V\left(x,\sqrt{t}\right)}$$

(see [4], Corollary 7.3). Thus, condition (1.6) with $n > 2$ guarantees that $M$ is nonparabolic.

However, hypothesis (V) is somewhat excessive, and one may wonder if it can be replaced by the exact condition (1.8). It should be possible to show that this is the case using the same methods but it will be technically more involved.

Under the assumptions of Theorem 1.2, one can say more about the upper radius:

**Theorem 1.3.** *Let a complete noncompact manifold* $M$ *possess condition (U) as above. Then for any* $x \in M$ *and* $\varepsilon > 0$ *the function*

$$\mathcal{R}(t) = \sqrt{(2+\varepsilon)t\log\log t} \tag{1.9}$$

*is an upper radius for* $W_x(t)$.

Under a more restrictive hypothesis, we prove a full analogue of the law of the iterated logarithm:

**Theorem 1.4.** *Let a complete noncompact manifold* $M$ *have a non-negative Ricci curvature. Then the function (1.9) is an upper radius for* $\varepsilon > 0$*, whereas it is not an upper radius for* $\varepsilon < 0$*. In particular, we have, for any* $x \in M$,

$$\limsup_{t\to\infty} \frac{\mathrm{dist}(W_x(t),x)}{\sqrt{2t\log\log t}} = 1\,.$$

The hypotheses of the above theorems are related in the following way:

$$\begin{array}{ccccc} \textit{Ricci} \geq 0 & \Longrightarrow & \textit{condition}\,(U) & \Longrightarrow & \textit{weak bounded geometry} \\ & & \Downarrow & & \\ & & \textit{condition}\,(1.5) & & \end{array}$$

[see [8] for the implication *Ricci* $\geq 0 \Longrightarrow$ *(U)*]. It follows that the hypotheses get stronger from Theorem 1.1 through Theorem 1.4, with the exception of condition (V) in Theorem 1.2. [However, the condition (V) is also implied by (U) with *some* $n > 0$ rather than with $n > 2$ (see Appendix C).]

The proof of Theorems 1.1–1.4 splits naturally into two parts. In the first part (Sections 2 and 3) we use the probabilistic argument based on the lemmas of Borel and Cantelli and on Kolmogorov's inequality, to reduce the question of constructing upper and lower radii to certain estimates of the heat kernel. Let us recall that the heat kernel $p(t,x,y)$ is the density of the transition probability $\mathbb{P}_x$, that is, for any Borel set $E \subset M$ and for any $t > 0$, we have

$$\mathbb{P}_x\left\{W_x(t) \in E\right\} = \int_E p(t,x,y)dy.$$

In this part of the proof, no a priori geometric assumption is required. In the second part of the proof (Sections 4–8), we obtain the necessary estimates of the heat kernel by analytic methods under appropriate geometric assumptions, and finish the proof.

The logical relationships between the theorems are presented in the diagram in Figure 1.

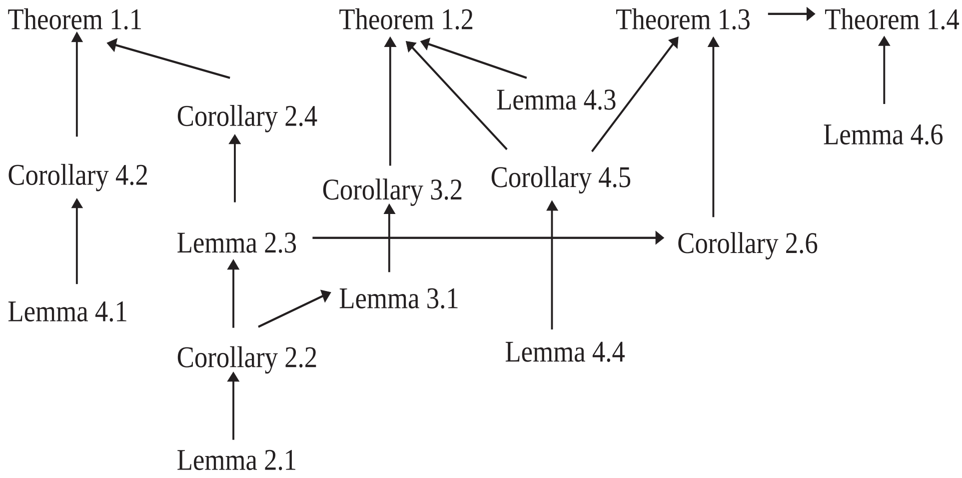


*Figure* 1.

NOTATION. We denote by $\mathrm{const}_{x,y,...}$ any positive constant depending on the variables $x, y, ...$ . Different occurrences of this notation may refer to different constants even within the same relation. For example, $a + \mathrm{const}_a = \mathrm{const}_a$.

Other notation:

(a) $M$ - a non-compact geodesically complete Riemannian manifold;
(b) $W_x(t)$ - the standard Brownian motion on $M$ starting at the point $x \in M$;
(c) $p(t,x,y)$ - the transition density of $W_x(t)$ or, in other words, the minimal positive fundamental solution of the diffusion equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u$;
(d) $\mathrm{dist}(x,y)$ - the geodesic distance between the points $x, y \in M$;
(e) $\mathrm{Vol}E$ - the Riemannian volume of a set $E \subset M$ (all integrations over $M$ are done against the Riemannian volume);
(f) $B(x,R)$ - the (open) geodesic ball of radius $R$ centred at the point $x \in M$.
(g) $V(x,R) \equiv \mathrm{Vol}B(x,R)$.

## 2. Construction of the upper radius

Let $p(t,x,y)$ be the heat kernel of the diffusion $W_x(t)$. The following lemma is a manifold version of Kolmogorov's inequality. For any set $\Omega \subset M$, we denote by $\Omega^r$ the open $r$-neighbourhood of $\Omega$.

**Lemma 2.1.** *Let $\eta, \delta$ be positive numbers and let $\Omega$ be a region on $M$. We assume for some $t > 0$ that*

$$\inf_{s\in(0,t]} \inf_{z\in\partial\Omega} \int_{B(z,\eta)} p(s,z,y)dy \geq \delta. \tag{2.1}$$

*Then*

$$\mathbb{P}_x\left\{\, W_x(s) \in \Omega \text{ for some } s < t\right\} \leq \delta^{-1}\mathbb{P}_x\left\{W_x(t) \in \Omega^\eta\right\}. \tag{2.2}$$

*Proof.* Let $\tau$ denote the first time the process $W_x(t)$ enters $\Omega$. Then

$$\begin{aligned}\mathbb{P}_x\{\exists\, s<t:\, W_x(s)\in\Omega\} &= \mathbb{P}_x\{\tau<t\}\\ &= \mathbb{P}_x\{\tau<t \text{ and } W_x(t)\in\Omega^\eta\}\\ &\quad +\mathbb{P}_x\{\tau<t \text{ and } W_x(t)\in M\backslash\Omega^\eta\}\\ &\le \mathbb{P}_x\{W_x(t)\in\Omega^\eta\}\\ &\quad +\mathbb{P}_x\{\tau<t \text{ and } W_x(t)\in M\backslash\Omega^\eta\}\,.\end{aligned}$$

Let $\mu$ be the probability measure on $\partial\Omega$ equal to the distribution of the random point $z = W_x(\tau)$. We estimate the second term above by using the strong Markov property of Brownian motion:

$$\begin{aligned}&\mathbb{P}_x\{\tau<t \text{ and } W_x(t)\in M\backslash\Omega^\eta\}\\ \le\;& \mathbb{P}_x\{\tau<t \text{ and } \operatorname{dist}(W_x(\tau),W_x(t))\ge\eta\}\\ =\;& \int_{\partial\Omega}\int_0^t \mathbb{P}_z\{\operatorname{dist}(z,W_z(t-s))\ge\eta\}\,d_s\mathbb{P}_x(\tau<s)d_z\mu\\ \le\;& \sup_{z\in\partial\Omega}\sup_{s\in(0,t)}\mathbb{P}_z\{\operatorname{dist}(z,W_z(t-s))\ge\eta\}\int_0^t d_s\mathbb{P}_x(\tau<s)\,.\end{aligned}$$

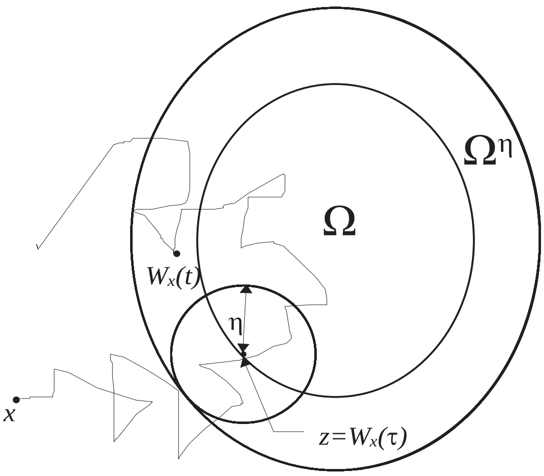


*Figure* 2.

On the other hand, for all $s\in(0,t)$ and any $z\in\partial\Omega$,

$$\mathbb{P}_z\{\operatorname{dist}(z,W_z(t-s))\ge\eta\} = \int_{M\backslash B(z,\eta)} p(t-s,z,y)ds \le 1-\delta$$

by (2.1). Therefore

$$\mathbb{P}_x\{\exists\, s<t:\, W_x(s)\in\Omega\} \le \mathbb{P}_x\{W_x(t)\in\Omega^\eta\} + (1-\delta)\int_0^t d_s\mathbb{P}_x(\tau<s).$$

Since

$$\int_0^t d_s\mathbb{P}_x(\tau<s) = \mathbb{P}_x(\tau<t) = \mathbb{P}_x\{\exists s<t:\, W_x(s)\in\Omega\},$$

we obtain (2.2), which concludes the proof. ∎

Let us denote

$$\mathcal{M}_x(t) = \sup_{s\le t}\operatorname{dist}(W_x(s),x)\;; \tag{2.3}$$

that is, let $\mathcal{M}_x(t)$ be the maximum distance which the process moves from the origin $x$ over time $t$.

**Corollary 2.2.** *Let $\zeta, \eta, \delta$ be positive numbers and, for some $x \in M$ and for some $t > 0$,*

$$\inf_{s\in(0,t]} \inf_{z\in\partial B(x,\zeta)} \int_{B(z,\eta)} p(s,z,y)dy \geq \delta.$$

*Then*

$$\mathbb{P}_x\left\{\mathcal{M}_x(t) > \zeta\right\} \leq \delta^{-1}\mathbb{P}_x\left\{\operatorname{dist}\left(W_x(t),x\right) > \zeta-\eta\right\}.$$

Indeed, it follows immediately from Corollary 2.2 if we set $\Omega = M\backslash\overline{B(x,\zeta)}$.

The following lemma provides a general method of constructing of an upper radius assuming the existence of certain estimates of the heat kernel.

**Lemma 2.3.** *Let $\{t_k\}, \{R_k\}, \{h_k\}$, $k = 1,2,3,\dots$, be increasing sequences of positive numbers going to $\infty$ as $t \to \infty$. Let us assume that for some point $x \in M$ the following holds:*

$$\sum_{k=1}^{\infty} \int_{M\backslash B(x,R_k)} p(t_k,x,y)dy < \infty\,; \tag{2.4}$$

*for all $k$ large enough,*

$$\inf_{t\in(0,t_k]} \inf_{z\in\partial B(x,R_k+h_k)} \int_{B(z,h_k)} p(t,z,y)dy \geq \delta \tag{2.5}$$

*with some $\delta > 0$ which does not depend on $k$.*
*Define the function $R(t)$ as follows:*

$$R(t) \equiv R_k + h_k \quad \text{if } \ t \in (t_{k-1},t_k],\ k = 1,2,\dots$$

(*where $t_0 \equiv 0$*). *Then the function $R(t)$ is an upper radius for $W_x(t)$.*

*Proof.* Use the maximum process $\mathcal{M}(t)$ defined by (2.3). By Corollary 2.2, we have for large $k$ and $t = t_k,\ \eta = h_k,\ \zeta = R_k + h_k$,

$$\begin{aligned}\mathbb{P}_x\left\{\mathcal{M}_x(t_k) > R_k + h_k\right\} &\leq \delta^{-1}\mathbb{P}_x\left\{W_x(t_k) > R_k)\right\} \\ &\leq \delta^{-1} \int_{M\backslash B(x,R_k)} p(t_k,x,y)dy\,.\end{aligned}$$

By (2.4) it follows that

$$\sum_{k=1}^{\infty} \mathbb{P}_x\left\{\mathcal{M}_x(t_k) > R_k + h_k\right\} < \infty,$$

and by the Borel–Cantelli lemma the inequality $\mathcal{M}_x(t_k) \leq R_k + h_k$ holds almost surely for all $k$ large enough. Let $t \in (t_k, t_{k+1}]$. Then, by monotonicity of $\mathcal{M}_x(t)$,

$$\mathcal{M}_x(t) \leq \mathcal{M}_x(t_{k+1}) \leq R_{k+1} + h_{k+1} = R(t),$$

which was to be proved. ∎

**Corollary 2.4.** *Let $\mathcal{R}(t)$ and $h(t)$ be increasing positive functions of $t$, and assume that for some $x \in M$ the following holds:*

$$\int^{\infty} \left( \int_{M\backslash B(x,\mathcal{R}(t)-h(t))} p(t,x,y)dy \right) dt < \infty\,; \tag{2.6}$$

*and for all $t$ large enough,*

$$\inf_{s\in(0,t]}\inf_{z\in\partial B(x,\mathcal{R}(t))}\int\limits_{B(z,h(t))} p(s,z,y)dy \geq \delta \tag{2.7}$$

*where $\delta > 0$.*
*Then the function $\mathcal{R}(t+\varepsilon)$ is an upper radius for the process $W_x(t)$ for any $\varepsilon > 0$.*

*Proof.* We take advantage of the following elementary fact (see the proof in Appendix A):

**Lemma 2.5.** *If $f(t)$ is a nonnegative measurable function on $[0,+\infty)$ and*

$$\int_0^\infty f(t)dt < \infty,$$

*then, for almost all $\xi > 0$,*

$$\sum_{k=1}^\infty f(k\xi) < \infty. \tag{2.8}$$

Let us apply this lemma to the following function

$$f(t) = \int_{M\setminus B(x,\mathcal{R}(t)-h(t))} p(t,x,y)dy$$

and find $\xi \in (0,\varepsilon)$ such that for $t_k = k\xi,\ k = 1,2,3,...$, hypothesis (2.4) of Lemma 2.3 holds with $R_k = \mathcal{R}(t_k) - h(t_k)$. Hypothesis (2.5) follows from (2.7) for $h_k = h(t_k)$. Therefore, by Lemma 2.3, the function $R(t)$ defined at $t \in (t_k, t_{k+1}]$ as

$$R(t) \equiv R_{k+1} + h_{k+1} = \mathcal{R}(t_{k+1})$$

is the upper radius. So is the function $\mathcal{R}(t+\varepsilon)$ because $t+\varepsilon \geq t_k+\xi = t_{k+1}$, and by monotonicity of $\mathcal{R}(t)$ we have

$$\mathcal{R}(t+\varepsilon) \geq \mathcal{R}(t_{k+1}) = R(t)\,.$$

**Corollary 2.6.** *Let $\{t_k\},\ k = 1,2,3,...,$ be an increasing sequence such that $t_k \to \infty$, and $\mathcal{R}(t), h(t)$ be increasing positive functions on $(0,+\infty)$ such that, for some $\varepsilon > 0$,*

$$\mathcal{R}(t_{k+1}) \leq (1+\varepsilon)\mathcal{R}(t_k) \tag{2.9}$$

*and, for some $x \in M$,*

$$\sum_{k=1}^\infty \int\limits_{M\setminus B(x,\mathcal{R}(t_k)-h(t_k))} p(t_k,x,y)dy < \infty\,. \tag{2.10}$$

*Let us also suppose that for all $t$ large enough*

$$\inf_{s\in(0,t]}\inf_{z\in\partial B(x,\mathcal{R}(t))}\int\limits_{B(z,h(t))} p(s,z,y)dy \geq \delta$$

*for some $\delta > 0$. Then the function $(1+\varepsilon)\mathcal{R}(t)$ is an upper radius for Brownian motion.*

*Proof.* Indeed, let us take $R_k = \mathcal{R}(t_k) - h(t_k),\ h_k = h(t_k)$. Then we obtain by Lemma 2.3 that the function $R(t)$ which is defined for $t \in (t_k, t_{k+1}]$ as

$$R(t) = R_{k+1} + h_{k+1} = \mathcal{R}(t_{k+1})$$

is the upper radius for $W_x(t)$. Since for $t \in (t_k, t_{k+1}]$ we have

$$R(t) = \mathcal{R}(t_{k+1}) \leq (1+\varepsilon)\mathcal{R}(t_k) \leq (1+\varepsilon)\mathcal{R}(t),$$

then the function $(1+\varepsilon)\mathcal{R}(t)$ is the upper radius as well.

## 3. Construction of a lower radius

**Lemma 3.1.** *Let $x \in M$ be a fixed point, let $\{t_k\}$ be an increasing sequence of times and $\{R_k^*\}, \{R_k\}, \{r_k\}, \{h_k\}$ be sequences of positive numbers such that we have the following:*

$$\sum_{k=1}^{\infty} \int_{B(x,R_k)} p(t_k, x, y)dy < \infty \tag{3.1}$$

*and*

$$\sum_{k=1}^{\infty} \int_{M \setminus B(x,R_k^*)} p(t_k, x, y)dy < \infty; \tag{3.2}$$

*and, for all $k$ large enough*

$$\inf_{s \in (0, t_{k+1}-t_k]} \inf_{z \in B(x, R_k^* + r_k + h_k)} \int_{B(z,h_k)} p(s, z, y)dy \geq \delta, \tag{3.3}$$

*where $\delta > 0$ does not depend on $k$;*
*and, finally,*

$$\sum_{k=1}^{\infty} \sup_{z \in B\left(x,R_k^*\right)} \int_{M \setminus B(z,r_k)} p(t_{k+1} - t_k, z, y)dy < \infty. \tag{3.4}$$

*Then the function $r(t)$ defined as*

$$r(t) \equiv R_k - r_k - h_k \quad \text{if } \; t \in [t_k, t_{k+1}),\ k = 1, 2, ... \tag{3.5}$$

*is the lower radius for the process $W_x(t)$.*

The proof of Lemma 3.1 will be given at the end of this section.

**Corollary 3.2.** *Suppose that we have an increasing sequence $\{t_k\}$ of times, $t_k \to \infty$, and the sequences of positive numbers $\{r_k\}$ and $\{h_k\}$. Let $\mathcal{R}(t)$ be an increasing function on $(0, +\infty)$, and suppose that the sequence $R_k \equiv \mathcal{R}(t_k)$ satisfies for some $\varepsilon > 0$ and for all large $k$ the conditions*

$$R_{k+1} \leq (1 + \varepsilon)R_k \tag{3.6}$$

*and*

$$r_k + h_k \leq \varepsilon R_k\ . \tag{3.7}$$

*Let the following hypotheses hold on $M$:*

(a) *for some $x \in M$,*

$$\sum_{k=1}^{\infty} \int_{B(x,R_k)} p(t_k, x, y)dy < \infty; \tag{3.8}$$

(b) *for all $k$ large enough,*

$$\inf_{s \in (0, t_{k+1}-t_k]} \inf_{z \in M} \int_{B(z,h_k)} p(s, z, y)dy \geq \delta, \tag{3.9}$$

*where $\delta > 0$ does not depend on $k$;*

(c) *also*

$$\sum_{k=1}^{\infty} \sup_{z \in M} \int_{M \setminus B(z,r_k)} p(t_{k+1} - t_k, z, y)dy < \infty. \tag{3.10}$$

*Then the function $(1 - 2\varepsilon)\mathcal{R}(t)$ is the lower radius.*

*Proof of Corollary 3.2.* Let us verify the hypotheses of Lemma 3.1.

(i) Hypothesis (3.1) of Lemma 3.1 coincides with (3.8).

(ii) There is no $\{R_k^*\}$ so far to state (3.2). Let us take $\{R_k^*\}$ to be an arbitrary quickly growing sequence so that the sum in (3.2) is finite. It always exists because

$$\int_M p(t,z,y)dy \le 1\,.$$

(iii) Hypotheses (2.7) and (3.4) follow from (3.9) and (3.10), respectively, because inf and sup in (3.9) and (3.10) are taken over the entire manifold $M$, which makes these assumptions independent of the choice of $R_k^*$.

Thus, by Lemma 3.1 the function $r(t)$ defined by (3.5) is the lower radius. Finally, if $t \in [t_k, t_{k+1})$ and $k$ is large enough, then by (3.7) and (3.6) we have

$$r(t) = R_k - r_k - h_k \ge R_k - \varepsilon R_k \ge \frac{1-\varepsilon}{1+\varepsilon} R_{k+1} \ge (1-2\varepsilon)\mathcal{R}(t_{k+1}) \ge (1-2\varepsilon)\mathcal{R}(t),$$

whence it follows that $(1-2\varepsilon)\mathcal{R}(t)$ is also the lower radius. ∎

*Proof of Lemma 3.1.* Hypothesis (3.1) implies that

$$\sum_{k=1}^{\infty} \mathbb{P}_x \{W_x(t_k) \in B(x, R_k)\} < \infty$$

or, by the Borel–Cantelli lemma, we have that with probability 1 for all large $k$,

$$W_x(t_k) \notin B(x, R_k)\ . \tag{3.11}$$

Since for $t = t_k$ we have $R_k > r(t)$, then (3.11) implies that, for all large $k$ and $t = t_k$,

$$W_x(t) \in M\backslash B(x, r(t))\,. \tag{3.12}$$

If we prove (3.12) for all large $t$ (not only for $t = t_k$) then $r(t)$ is indeed the lower radius. The main technical difficulty is to handle the values of $t$ when $t \in (t_k, t_{k+1})$. To that end, we will estimate the deviation $\operatorname{dist}(W_x(t), W_x(t_k))$ :

$$\begin{aligned}
&\mathbb{P}_x \left\{ \sup_{t \in (t_k, t_{k+1})} \operatorname{dist}(W_x(t), W_x(t_k)) > r_k + h_k \right\} \\
=\ & \mathbb{P}_x \left\{ \sup_{t \in (t_k, t_{k+1})} \operatorname{dist}(W_x(t), W_x(t_k)) > r_k + h_k \text{ and } W_x(t_k) \in M\backslash B(x, R_k^*) \right\} \\
&+\mathbb{P}_x \left\{ \sup_{t \in (t_k, t_{k+1})} \operatorname{dist}(W_x(t), W_x(t_k)) > r_k + h_k \text{ and } W_x(t_k) \in B(x, R_k^*) \right\} \\
\le\ & \mathbb{P}_x \{\ W_x(t_k) \in M\backslash B(x, R_k^*)\} \\
&+\mathbb{P}_x \left\{ \sup_{t \in (t_k, t_{k+1})} \operatorname{dist}(W_x(t), W_x(t_k)) > r_k + h_k \text{ and } W_x(t_k) \in B(x, R_k^*) \right\}.
\end{aligned} \tag{3.13}$$

For the first term on the right-hand side of (3.13) we have

$$\mathbb{P}_x \{W_x(t_k) \in M\backslash B(x, R_k^*)\} = \int_{M\backslash B(x, R_k^*)} p(x, u, t_k)du,$$

and we will use (3.2) to ensure that it is small.

To estimate the second term, we apply Corollary 2.2. It says that, for any point $y \in M$,

$$\begin{aligned}
&\mathbb{P}_y \left\{ \sup_{t \in (0, t_{k+1}-t_k)} \operatorname{dist}(W_y(t-t_k), y) > r_k + h_k \right\} \\
\le\ & \delta_{k,y}^{-1} \mathbb{P}_y \{\operatorname{dist}(W_y(t_{k+1}-t_k), y) > r_k\}\,,
\end{aligned}$$

where

$$\delta_{k,y} \equiv \inf_{s\in(0,t_{k+1}-t_k)} \inf_{z\in\partial B(y,r_k+h_k)} \int_{B(z,h_k)} p(s,z,u)du\,.$$

By taking $y = W_x(t_k)$ and by using the strong Markov property of Brownian motion, we obtain

$$\begin{aligned}&\mathbb{P}_x\left\{\sup_{t\in(t_k,t_{k+1})} \operatorname{dist}\left(W_x(t),W_x(t_k)\right) > r_k+h_k \text{ and } W_x(t_k)\in B(x,R_k^*)\right\}\\ \le\ & \delta_k^{-1} \sup_{y\in B(x,R_k^*)} \mathbb{P}_y\left\{\operatorname{dist}\left(W_y(t_{k+1}-t_k),y\right) > r_k\right\},\end{aligned}$$

where

$$\delta_k \equiv \inf_{y\in B(x,R_k^*)} \delta_{k,y} = \inf_{s\in(0,t_{k+1}-t_k)} \inf_{z\in B(x,R_k^*+r_k+h_k)} \int_{B(z,h_k)} p(s,z,u)du\,.$$

As follows from (3.3), we have $\delta_k \ge \delta$, whence

$$\begin{aligned}&\mathbb{P}_x\left\{\sup_{t\in(t_k,t_{k+1})} \operatorname{dist}\left(W_x(t),W_x(t_k)\right) > r_k+h_k \text{ and } W_x(t_k)\in B(x,R_k^*)\right\}\\ \le\ & \delta^{-1} \sup_{y\in B(x,R_k^*)} \int_{B(y,r_k)} p(t_{k+1}-t_k,y,u)du\,.\end{aligned}$$

Finally, (3.13) implies

$$\begin{aligned}&\mathbb{P}_x\left\{\sup_{t\in(t_k,t_{k+1})} \operatorname{dist}\left(W_x(t),W_x(t_k)\right) > r_k+h_k\right\}\\ \le\ & \int\limits_{M\setminus B(x,R_k^*)} p(x,u,t_k)du + \delta^{-1} \sup_{y\in B(x,R_k^*)} \int\limits_{B(y,r_k)} p(t_{k+1}-t_k,y,u)du\end{aligned}$$

and, by hypotheses (3.2) and (3.4),

$$\sum_{k=1}^{\infty} \mathbb{P}_x\left\{\sup_{t\in(t_k,t_{k+1})} \operatorname{dist}\left(W_x(t),W_x(t_k)\right) > r_k+h_k\right\} < \infty.$$

By the Borel–Cantelli lemma of , we conclude that, with probability 1 for all large $k$,

$$\sup_{t\in(t_k,t_{k+1})} \operatorname{dist}\left(W_x(t),W_x(t_k)\right) \le r_k+h_k.$$

Combining it with (3.12) we obtain that a.s., for all large $k$ and all $t\in[t_k,t_{k+1})$,

$$W_x(t)\in M\setminus B\left(x,R_k-r_k-h_k\right) = M\setminus B(x,r(t)),$$

which was required. ∎

## 4. Estimates of the integrals of the heat kernel

We denote by $V(x,R)$ the Riemannian volume of the ball $B(x,R)$.

**Lemma 4.1.** *Let a complete noncompact manifold $M$ satisfy for some $x\in M$ and for some $R_0>0$ the following volume growth condition:*

$$\operatorname{Vol}B(x,R) \le AR^N$$

*for all $R>R_0$. For some $\rho>0$, let the ball $B(z,\rho)$ possess $(a,2/n)$-isoperimetric inequality where $z$ is another point on $M$. Then for any $R>R_0$, $t>0$ and $D>1$ we have*

$$\int\limits_{M\setminus B(z,R)} p(t,z,y)dy \le \mathrm{const}_{A,N,a,n,D} \frac{t^{N/4}+R^{N/2}+d^{N/2}}{\min\left\{t^{n/4},\rho^{n/2}\right\}} \exp\left(-\frac{R^2}{2Dt}\right),$$

*where* $d = \operatorname{dist}(x,z)$.

**Corollary 4.2.** *If, under the hypotheses of Lemma 4.1,* $z = x$, *then*

$$\int\limits_{M\setminus B(x,R)} p(t,x,y)dy \leq \mathrm{const}_{A,N,a,n,D} \frac{t^{N/4}+R^{N/2}}{\min\left\{t^{n/4},\rho^{n/2}\right\}} \exp\left(-\frac{R^2}{2Dt}\right).$$

*Proof of Lemma 4.1.* Let us denote $r = r(y) = \operatorname{dist}(z,y)$. We have, by the Cauchy–Schwarz inequality,

$$\int\limits_{M\setminus B(z,R)} p(t,z,y)dy \leq \left\{\int\limits_M p^2(t,z,y)e^{\frac{r^2}{Dt}}dy\right\}^{\frac{1}{2}} \left\{\int\limits_{M\setminus B(z,R)} e^{-\frac{r^2}{Dt}}dy\right\}^{\frac{1}{2}}. \tag{4.1}$$

To treat the first integral on the right-hand side, we use the following results of [9] (see also [7]). The integral

$$\int\limits_M p^2(t,z,y)\exp\left(\frac{r^2}{Dt}\right)dy$$

is known to be finite provided $D > 1$, and to be decreasing in $t$. Moreover, it admits the following upper bound for any $t < \rho^2$ (see [9, Corollary 4.2]):

$$\int\limits_M p^2(t,z,y)\exp\left(\frac{r^2}{Dt}\right)dy \leq \frac{\mathrm{const}_{\,a,n}}{t^{n/2}}.$$

Therefore, we can estimate it for all $t > 0$ as

$$\int\limits_M p^2(t,z,y)\exp\left(\frac{r^2}{Dt}\right)dy \leq \frac{\mathrm{const}_{\,a,n}}{\min\left\{t^{n/2},\rho^n\right\}}. \tag{4.2}$$

We estimate the second integral in (4.1) as follows, assuming $R > R_0$ :

$$\begin{aligned}
\int\limits_{M\setminus B(z,R)} \exp\left(-\frac{r^2}{Dt}\right)dy &= \int\limits_R^\infty \exp\left(-\frac{r^2}{Dt}\right)d_rV(z,r) \\
&= V(z,r)\exp\left(-\frac{r^2}{Dt}\right)\Big|_R^\infty + \int\limits_R^\infty \frac{2r}{Dt}\exp\left(-\frac{r^2}{Dt}\right)V(z,r)dr \\
&\leq \int\limits_R^\infty \frac{2r}{Dt}\exp\left(-\frac{r^2}{Dt}\right)V(z,r)dr.
\end{aligned}$$

Let us note that

$$V(z,r) \leq V(x,r+d) \leq A(r+d)^N \leq A2^N r^N + A2^N d^N$$

whence

$$\begin{aligned}
\int\limits_{M\setminus B(z,R)} \exp\left(-\frac{r^2}{Dt}\right)dy \ &\leq\ 2^{N+1}A\int\limits_R^\infty \frac{r^{N+1}}{Dt}\exp\left(-\frac{r^2}{Dt}\right)dr \\
&\quad +2^N A\int\limits_R^\infty \frac{2rd^N}{Dt}\exp\left(-\frac{r^2}{Dt}\right)dr\,.
\end{aligned}$$

The second integral is equal to

$$2^N A d^N \exp\left(-\frac{R^2}{Dt}\right),$$

whereas, to estimate the first integral, we apply the following inequality

$$\int_q^\infty s^{N+1} e^{-s^2} ds \le \text{const}_N \left(1 + q^N\right) e^{-q^2},$$

where $q > 0$ is arbitrary (see Appendix B for the proof).

Hence, we proceed as follows

$$\begin{aligned}
\int_{M\setminus B(z,R)} \exp\left(-\frac{r^2}{Dt}\right) dy &\le 2^{N+1} A \int_{R/\sqrt{Dt}}^\infty D^{\frac{N}{2}} t^{\frac{N}{2}} s^{N+1} e^{-s^2} ds + 2^N A d^N \exp\left(-\frac{R^2}{Dt}\right) \\
&\le \text{const}_{A,D,N} t^{\frac{N}{2}} \left(1 + \left(\frac{R}{\sqrt{Dt}}\right)^N\right) \exp\left(-\frac{R^2}{Dt}\right) \\
&\quad + 2^N A d^N \exp\left(-\frac{R^2}{Dt}\right) \\
&\le \text{const}_{A,N,D} \left(t^{\frac{N}{2}} + R^N + d^N\right) \exp\left(-\frac{R^2}{Dt}\right). \qquad (4.3)
\end{aligned}$$

Finally, we substitute inequalities (4.3) and (4.2) into (4.1). ∎

**Lemma 4.3.** *If a manifold $M$ satisfies hypothesis (U) of Theorem 1.2, then, for any $x \in M$ and any $R > 0$,*

$$\int_{B(x,R)} p(t,x,y) dy \le \text{const}_{a,\nu} \frac{V(x,R)}{V(x,\sqrt{t})}.$$

*Proof.* We apply the result of [9] (see [9, Proposition 5.2]) which says, in particular, that the uniform isoperimetric inequality (U) implies the following:

(i) the heat kernel upper bound for all $x, y \in M$ and all $t > 0$,

$$p(t,x,y) \le \frac{\text{const}_{a,\nu}}{V\left(x,\sqrt{t}\right)}; \qquad (4.4)$$

(ii) the volume comparison condition: for any two balls $B(y,r) \subset B(x,R)$,

$$\frac{V(x,R)}{V(y,r)} \le \text{const}_{a,\nu} \left(\frac{R}{r}\right)^\nu. \qquad (4.5)$$

By using (4.4) we obtain

$$\int_{B(x,R)} p(t,x,y) dy \le \frac{\text{const}_{a,\nu}}{V\left(x,\sqrt{t}\right)} V(x,R),$$

which was to be proved. ∎

**Remark 4.1.** *We have not used (4.5) in the proof but we have mentioned it for further applications.*

**Lemma 4.4.** *Let a complete non-compact manifold $M$ satisfy, for some $x \in M$ and all $R > 0$, the volume growth condition*

$$\text{Vol} B(x,R) \le v(R) \qquad (4.6)$$

*where the function $v(\cdot)$ is increasing on $(0,+\infty)$ and, for all $R>0$,*

$$\frac{v(2R)}{v(R)} \leq A \tag{4.7}$$

*with a (large) constant $A$. Also let the following inequality hold for any $t>0$ and some $C$:*

$$p(t,x,x) \leq \frac{C}{v\left(\sqrt{t}\right)}\,. \tag{4.8}$$

*Then, for all $R>0$, $t>0$ and $D>1$,*

$$\int\limits_{M\setminus B(x,R)} p(t,x,y)dy \leq \operatorname{const}_{A,C,D} \exp\left(-\frac{R^2}{2Dt}\right). \tag{4.9}$$

*Proof.* We start again with inequality (4.1):

$$\int\limits_{M\setminus B(x,R)} p(t,x,y)dy \leq \left\{\int\limits_M p^2(t,x,y)e^{\frac{r^2}{Dt}}dy\right\}^{\frac{1}{2}}\left\{\int\limits_{M\setminus B(x,R)} e^{-\frac{r^2}{Dt}}dy\right\}^{\frac{1}{2}} \tag{4.10}$$

where $r=r(y)=\operatorname{dist}(x,y)$. To estimate the first integral on the right-hand side, we apply the result of [10] which says that the heat kernel on-diagonal bound (4.8) known to be true at a point $x$ and for all $t>0$ implies, for any $D>1$,

$$\int\limits_M p^2(t,x,y)\exp\left(\frac{r^2}{Dt}\right)dy \leq \frac{\operatorname{const}_{A,C,D}}{v(\sqrt{t})} \tag{4.11}$$

provided the function $v(\cdot)$ satisfies the doubling property (4.7) that holds now.

The second integral in (4.10) can be estimated by using the upper bound of the volume (4.6). As in the previous proof, we have

$$\begin{aligned}\int\limits_{M\setminus B(x,R)} \exp\left(-\frac{r^2}{Dt}\right)dy &\leq \int\limits_R^\infty \frac{2r}{Dt}\exp\left(-\frac{r^2}{Dt}\right)V(x,r)dr\\ &\leq \frac{2}{Dt}\int\limits_R^\infty rv(r)\exp\left(-\frac{r^2}{Dt}\right)dr\,.\end{aligned}$$

By changing a variable $s=\frac{r}{\sqrt{Dt}}$ , we get

$$\frac{2}{Dt}\int\limits_R^\infty rv(r)\exp\left(-\frac{r^2}{Dt}\right)dr = 2\int\limits_{R/\sqrt{Dt}}^\infty sv(s\sqrt{Dt})e^{-s^2}ds\,.$$

Let us denote $w(s)=sv(s\sqrt{Dt})$ and note that $w(s)$ is an increasing function and, for any $s>0$,

$$w(2s)\leq 2Aw(s)\,.$$

As will be proved in Appendix B, for any positive $q$,

$$\int\limits_q^\infty w(s)e^{-s^2}ds \leq \left(K+2A\frac{w(q)}{q}\right)e^{-q^2}$$

where $K\equiv w(q_0)\left(q_0+\frac{2A}{q_0}\right)$ and $q_0=\sqrt{\frac{1}{3}\log 2A}$.

Let us notice that (4.7) implies the following inequality for all positive $r_1, r_2$:

$$\frac{v(r_2)}{v(r_1)} \leq A\left(1+\frac{r_2}{r_1}\right)^{\log_2 A}. \tag{4.12}$$

Indeed, if $r_2 \leq r_1$, then this is true simply by monotonicity of $v(\cdot)$. If $r_2 > r_1$, then (4.12) is obtained by successive application of (4.7) to the balls of radii $r_1, 2r_1, 4r_1, \ldots$ at most $1+\log_2 \frac{r_2}{r_1}$ times (see Appendix C for details).

Returning to the constant $K$, we have obviously

$$\begin{aligned} K &= q_0 v\left(q_0\sqrt{Dt}\right)\left(q_0+\frac{2A}{q_0}\right) \\ &\leq \text{const}_{A,D}\left(1+q_0\sqrt{D}\right)^{\log_2 A} v\left(\sqrt{t}\right) \\ &= \text{const}_{A,D} v\left(\sqrt{t}\right) \end{aligned}$$

whence

$$\frac{2}{Dt}\int_R^\infty r v(r)\exp\left(-\frac{r^2}{Dt}\right)dr \leq \text{const}_{A,D}\left(v\left(\sqrt{t}\right)+v(R)\right)\exp\left(-\frac{R^2}{Dt}\right).$$

By combining this inequality with (4.11) and (4.10) we arrive at

$$\int_{M\setminus B(x,R)} p(t,x,y)dy \leq \text{const}_{A,C,D}\left(1+\frac{v(R)}{v\left(\sqrt{t}\right)}\right)^{\frac{1}{2}}\exp\left(-\frac{R^2}{2Dt}\right)$$

Since, by (4.12) ,

$$\frac{v(R)}{v\left(\sqrt{t}\right)} \leq A\left(1+\frac{R}{\sqrt{t}}\right)^{\log_2 A},$$

and the polynomial of $\frac{R}{\sqrt{t}}$ can be absorbed by the exponential $\exp\left(-\frac{R^2}{2Dt}\right)$ at the cost of slightly increasing $D$, then we obtain (4.9). ∎

**Corollary 4.5.** *If the manifold $M$ satisfies hypothesis (U) of Theorem 1.2, then, for all $x \in M$, $t > 0$, $R > 0$ and $D > 1$,*

$$\int_{M\setminus B(x,R)} p(t,x,y)dy \leq \text{const}_{a,\nu,D}\exp\left(-\frac{R^2}{2Dt}\right). \tag{4.13}$$

Indeed, let us fix $x$ and take $v(r) \equiv V(x,r)$. As was mentioned in the course of the proof of Lemma 4.3, condition (U) implies both (4.7) and (4.8) so that Lemma 4.4 is applicable and yields (4.13).

**Lemma 4.6.** *If the manifold $M$ has a nonnegative Ricci curvature, then, for all $x \in M$, $t > 0$, $R > 0$ and $D \in (0,1)$,*

$$\int_{M\setminus B(x,R)} p(t,x,y)dy \geq \text{const}_{n,D}\,\theta\left(\frac{R}{\sqrt{t}}\right)\exp\left(-\frac{R^2}{2Dt}\right), \tag{4.14}$$

*where*

$$\theta(s) \equiv \text{const}_n \begin{cases} s^n, & \text{if } s \leq 1, \\ s^\lambda, & \text{if } s > 1, \end{cases} \tag{4.15}$$

*$n = \dim M$ and $\lambda = \lambda(n) > 0$.*

*Proof.* By the theorem of Li and Yau [12], on a manifold with nonnegative Ricci curvature, there is the following uniform lower bound of the heat kernel:

$$p(t,x,y) \geq \frac{\mathrm{const}_{n,D}}{V(x,\sqrt{t})} \exp\left(-\frac{r^2}{2Dt}\right), \tag{4.16}$$

where $r = \mathrm{dist}(x,y)$ and $D \in (0,1)$ is arbitrary.

Therefore, by taking any $\varepsilon > 0$, we have

$$\begin{aligned}
\int\limits_{M\setminus B(x,R)} p(t,x,y)dy &\geq \frac{\mathrm{const}_{n,D}}{V(\sqrt{t})} \int\limits_R^\infty \exp\left(-\frac{r^2}{2Dt}\right) d_r V(x,r) \\
&\geq \frac{\mathrm{const}_{n,D}}{V(\sqrt{t})} \int\limits_R^{(1+\varepsilon)R} \exp\left(-\frac{r^2}{2Dt}\right) d_r V(x,r) \\
&\geq \frac{\mathrm{const}_{n,D}}{V(\sqrt{t})} \exp\left(-\frac{(1+\varepsilon)^2 R^2}{2Dt}\right) \left(V\left(x,(1+\varepsilon)R\right) - V\left(x,R\right)\right).
\end{aligned}$$

On a manifold of nonnegative Ricci curvature one has, for any $R_2 > R_1 > 0$ and $z \in M$,

$$\frac{V(z,R_2)}{V(z,R_1)} \leq \left(\frac{R_2}{R_1}\right)^n. \tag{4.17}$$

The property (4.17) (and, more generally, the doubling volume property) implies on any non-compact complete manifold that, for any $\varepsilon > 0$ ,

$$\frac{V\left(x,(1+\varepsilon)R\right)}{V\left(x,R\right)} \geq 1+\delta,$$

where $\delta = \delta(n,\varepsilon) > 0$ (see Appendix C for the proof), whence

$$V\left(x,(1+\varepsilon)R\right) - V\left(x,R\right) \geq \delta V(x,R)$$

and

$$\int\limits_{M\setminus B(x,R)} p(t,x,y)dy \geq \mathrm{const}_{n,\varepsilon,D} \frac{V(x,R)}{V(x,\sqrt{t})} \exp\left(-\frac{(1+\varepsilon)^2 R^2}{2Dt}\right). \tag{4.18}$$

Next, let us show that, for any positive $R_1$, $R_2$,

$$\frac{V(x,R_1)}{V(x,R_2)} \geq \theta\left(\frac{R_1}{R_2}\right), \tag{4.19}$$

where $\theta(\cdot)$ is defined by (4.15). Indeed, if $R_1 \leq R_2$, then this follows from (4.17). If $R_1 > R_2$, then we apply another consequence of (4.17) which says (see Appendix C) that

$$\frac{V(x,R_1)}{V(x,R_2)} \geq c\left(\frac{R_1}{R_2}\right)^\lambda$$

where $c$ and $\lambda$ are positive and depend on $n$ only.

Applying (4.19) in (4.18), we obtain

$$\int\limits_{M\setminus B(x,R)} p(t,x,y)dy \geq \mathrm{const}_{n,\varepsilon,D}\, \theta\left(\frac{R}{\sqrt{t}}\right) \exp\left(-(1+\varepsilon)^2 \frac{R^2}{2Dt}\right)$$

whence we get (4.14) by taking $\varepsilon$ to be small enough and by absorbing $(1+\varepsilon)^2$ into $D$. ∎

## 5. Proof of Theorem 1.1 (upper radius $\sqrt{t\log t}$)

Let $\rho$ be the weak bounded geometry radius of $M$ and let $(a,2/n)$-isoperimetric inequality hold in any ball of radius $\rho$. In order to construct the upper radius, we will apply Corollary 2.4. Let us define the function $h(t)$ in the following way:

$$h(t)=\begin{cases}\sqrt{ct} & \text{for } t<t_0,\\ \sqrt{bt\log t} & \text{for } t\geq t_0,\end{cases}$$

where the constant $b$ is so far any positive number, $c$ will be chosen below to be large enough and $t_0$ is taken from $b\log t_0=c$ to ensure continuity and monotonicity of $h(t)$,

$$t_0=\exp\left(\frac{c}{b}\right). \tag{5.1}$$

Let us also define the function $\mathcal{R}(t)$ in a similar way:

$$\mathcal{R}(t)=h(t)+\begin{cases}\sqrt{\widetilde{c}t} & \text{for } t<t_0,\\ \sqrt{\widetilde{b}t\log t} & \text{for } t\geq t_0,\end{cases}$$

where $\widetilde{b}>0$ and $\widetilde{c}$ are determined from the condition

$$\widetilde{c}=\frac{\widetilde{b}c}{b}$$

to ensure continuity and monotonicity of $\mathcal{R}(t)$.

We will see later that $b$ should be taken greater than $\frac{N}{2}$ but arbitrarily close to $\frac{N}{2}$, and $\widetilde{b}$ should be greater than $N/2+2$ but arbitrarily close to $N/2+2$. For such $b$ and $\widetilde{b}$ we will have

$$\widetilde{c}\leq \mathrm{const}_N\ c,$$

which will be used below.

First we verify hypothesis (2.7) of Corollary 2.4. Given a large enough $t$, namely,

$$t>\max\left\{t_0,\rho^2,h^{-1}(R_0)\right\}, \tag{5.2}$$

any $z\in\partial B(x,\mathcal{R}(t))$ and any $s\in(0,t]$, we will show that

$$\int_{B(z,h(t))}p(s,z,y)dy>\frac{1}{2} \tag{5.3}$$

provided $c$ is chosen large enough.

Lemma 4.1 yields the following for $R=h(t)>R_0$, $d\equiv\operatorname{dist}(x,z)=\mathcal{R}(t)$ and any $D>1$:

$$\int_{M\setminus B(z,h(t))}p(s,z,y)dy\leq \mathrm{const}_{A,N,a,n,D}\frac{s^{N/4}+h(t)^{\frac{N}{2}}+\mathcal{R}(t)^{N/2}}{\min\left\{s^{n/4},\rho^{n/2}\right\}}\exp\left(-\frac{h(t)^2}{2Ds}\right).$$

First of all, we show that $s$ can be replaced by $t$ at its any occurrence on the right-hand side. To that end, it suffices to verify that the function

$$\frac{1}{\min\left\{s^{n/4},\rho^{n/2}\right\}}\exp\left(-\frac{h(t)^2}{2Ds}\right)$$

is increasing in $s$ on $(0,t)$. If $s\geq\rho^2$, then this is obvious. If $s<\rho^2$, then the logarithmic derivative of this function is equal to

$$-\frac{n}{4}\frac{1}{s}+\frac{h(t)^2}{2D}\frac{1}{s^2},$$

which is positive provided

$$s<\frac{2h(t)^2}{nD}. \tag{5.4}$$

Since $s \le t$ and for all $t > 0$ we have $h(t) \ge \sqrt{ct}$, then (5.4) will follow from

$$\frac{2c}{nD} > 1,$$

which can be ensured by taking $c$ large enough. Hence, we obtain

$$\int\limits_{M\setminus B(z,h(t))} p(s,z,y)dy \le \text{const}_{A,N,a,n,D} \frac{t^{N/4} + h(t)^{\frac{N}{2}} + \mathcal{R}(t)^{N/2}}{\min\left\{t^{n/4}, \rho^{n/2}\right\}} \exp\left(-\frac{h(t)^2}{2Ds}\right).$$

By (5.2) we have $\mathcal{R}(t) > h(t) = \sqrt{bt\log t} \ge \sqrt{ct} \ge \sqrt{t}$, and the inequality above amounts to

$$\begin{aligned}
\int\limits_{M\setminus B(z,h(t))} p(s,z,y)dy &\le \text{const}_{A,N,a,n,D,\rho} \mathcal{R}(t)^{N/2} \exp\left(-\frac{h(t)^2}{2Ds}\right) \\
&= \text{const}_{A,N,a,n,D,\rho} (t\log t)^{\frac{N}{4}} \exp\left(-\frac{b\log t}{2D}\right) \\
&= \text{const}_{A,N,a,n,D,\rho} (\log t)^{\frac{N}{4}} t^{-\left(\frac{b}{2D}-\frac{N}{4}\right)}. \qquad (5.5)
\end{aligned}$$

It is easy to prove that the function

$$t^{-\beta}\log^{\alpha} t$$

is decreasing when $\log t > \frac{\alpha}{\beta}$ (assuming that $\alpha$ and $\beta$ are positive). Whatever is $b > \frac{N}{2}$, there exists $D > 1$ (may be very close to 1) such that

$$\frac{b}{2D} - \frac{N}{4} > 0\,.$$

Therefore, if

$$\log t > \frac{N/4}{\frac{b}{2D} - \frac{N}{4}}$$

then the right-hand side of (5.5) is decreasing in $t$. We would like to have this property for all $t \ge t_0$. To that end it suffices to have

$$\log t_0 > \frac{N/4}{\frac{b}{2D} - \frac{N}{4}}$$

or by, (5.1),

$$c > \frac{bN/4}{\frac{b}{2D} - \frac{N}{4}}\,,$$

which may be assumed to be true by the choice of $c$.

Therefore, we proceed with (5.5) as follows:

$$\begin{aligned}
\int\limits_{M\setminus B(z,h(t))} p(s,z,y)dy &\le \text{const}_{A,N,a,n,D,\rho} (\log t_0)^{\frac{N}{4}} t_0^{-\left(\frac{b}{2D}-\frac{N}{4}\right)} \\
&= \text{const}_{A,N,a,n,D,\rho} \left(\frac{c}{b}\right)^{\frac{N}{4}} \exp\left\{-\frac{c}{b}\left(\frac{b}{2D} - \frac{N}{4}\right)\right\}.
\end{aligned}$$

Again by choosing $c$ to be large enough, we obtain that the right-hand side is smaller than $\frac{1}{2}$.

Thus, we conclude, that for a proper choice of $c$ we have

$$\int\limits_{M\setminus B(z,h(t))} p(s,z,y)dy < \frac{1}{2}$$

for all large $t$. Since the polynomial volume growth (1.5) implies stochastical completeness of the manifold (see [6]), then, for any $s > 0$ and $z \in M$,

$$\int_M p(s,z,y)dy \equiv 1\,,$$

and we deduce that, for all $z \in \partial B\left(x, \mathcal{R}(t)\right)$ and $t$ large enough

$$\int_{B(z,h(t))} p(s,z,y)dy > \frac{1}{2}\,. \tag{5.6}$$

Hence, we have verified hypothesis (2.7) of Corollary 2.4.

Now let us check hypothesis (2.6). By Corollary 4.2 for $R = \mathcal{R}(t) - h(t) = \sqrt{\widetilde{b}t\log t}$ and by $R > \sqrt{t}$ (which can be assumed for large enough $t$) we have, for large $t$,

$$\int_{M\setminus B(x,R)} p(t,x,y)dy \le \text{const}\,\frac{R^{N/2}}{\rho^{n/2}}\exp\left(-\frac{R^2}{2Dt}\right)\,.$$

Therefore

$$\begin{aligned}
\int^{\infty}\left(\int_{M\setminus B(x,\mathcal{R}(t)-h(t))} p(t,x,y)dy\right)dt &\le \text{const}\int^{\infty}\frac{R^{N/2}}{\rho^{n/2}}\exp\left(-\frac{R^2}{2Dt}\right)dt\\
&\le \text{const}\int^{\infty}\left(\widetilde{b}t\log t\right)^{N/4}\exp\left(-\frac{\widetilde{b}\log t}{2D}\right)dt\\
&= \text{const}\int^{\infty} t^{-\left(\frac{\widetilde{b}}{2D}-\frac{N}{4}\right)}\log^{N/4} t\,dt\,,
\end{aligned}$$

which is finite whenever $\frac{\widetilde{b}}{2D} - \frac{N}{4} > 1$ or $\widetilde{b} > \frac{N}{2} + 2$ (since $D$ can be taken arbitrarily close to 1). By Corollary 2.4 the function $\mathcal{R}(t+\varepsilon)$ is the upper radius of $W_x(t)$ with any $\varepsilon > 0$.

To finish the proof we are left to notice that, for large $t$ ,

$$\mathcal{R}(t+\varepsilon) = \left(\sqrt{b}+\sqrt{\widetilde{b}}\right)\sqrt{(t+\varepsilon)\log(t+\varepsilon)} < \sqrt{(2N+4)t\log t}$$

since

$$\sqrt{\frac{N}{2}} + \sqrt{\frac{N}{2}+2} < 2\sqrt{\frac{N}{2}+1} = \sqrt{2N+4}\,,$$

and $b$ and $\widetilde{b}$ can be taken arbitrarily close to $N/2$ and $N/2+2$, respectively. ∎

## 6. Proof of Theorem 1.2 (lower radius)

We will use Corollary 3.2 to construct the lower radius, and Lemma 4.3 and Corollary 4.5 to obtain the necessary heat kernel estimates. Let us note that the latter two results are applicable because they require only condition (U) which holds by the hypothesis of Theorem 1.2.

Let $\{t_k\}, \{R_k\}, \{r_k\}, \{h_k\}$ be so far arbitrary positive increasing sequences. Let us impose on these sequences conditions strong enough to make them satisfy all the assumptions of Corollary 3.2. By Lemma 4.3, hypothesis (3.8) will follow from

$$\sum_{k=1}^{\infty}\frac{V(x,R_k)}{V(x,\sqrt{t_k})} < \infty\,. \tag{6.1}$$

We assume in the sequel that $R_k < \sqrt{t_k}$; then, by condition (V) of Theorem 1.2, (6.1) is implied by

$$\sum_{k=1}^{\infty} \left( \frac{R_k}{\sqrt{t_k}} \right)^n < \infty. \tag{6.2}$$

Hypothesis (3.9) will follow with $\delta = \frac{1}{2}$ from

$$\sup_{s \in (0, t_{k+1} - t_k]} \sup_{z \in M} \int_{M \setminus B(z, h_k)} p(s, z, y) dy \leq \frac{1}{2}. \tag{6.3}$$

Indeed, as we know from the proof of Lemma 4.3, hypothesis (U) implies a polynomial volume growth which, in turn, ensures stochastical completeness of the manifold [6], that is

$$\int_M p(s, z, y) dy \equiv 1 \, .$$

To get (6.3), we apply Corollary 4.5, which yields for any $D > 1$,

$$\sup_{s \in (0, t_{k+1} - t_k]} \sup_{z \in M} \int_{M \setminus B(z, h_k)} p(s, z, y) dy \leq \operatorname{const}_{a, \nu, D} \exp \left( - \frac{h_k^2}{2D \left( t_{k+1} - t_k \right)} \right)$$

so that (3.9) will follow from

$$h_k^2 \geq H \left( t_{k+1} - t_k \right), \tag{6.4}$$

where $H$ is a big enough constant which depends only on $a, \nu, D$ (we will choose $D > 1$ later on).

Similarly, hypothesis (3.4) will be implied by

$$\sum_{k=1}^{\infty} \exp \left( - \frac{r_k^2}{2D \left( t_{k+1} - t_k \right)} \right) < \infty \, . \tag{6.5}$$

Of course, we also have to satisfy (3.6) and (3.7): they will follow from

$$\lim_{k \to \infty} \frac{R_{k+1}}{R_k} = 1 \tag{6.6}$$

and

$$h_k + r_k = o(R_k) \qquad \text{as } k \to \infty \, . \tag{6.7}$$

Now let us show how to choose the sequences so that all the conditions (6.2), (6.4), (6.5), (6.6) and (6.7) hold. Assuming that $\{t_k\}$ has been chosen, let us define $\{R_k\}$, $\{h_k\}$ and $\{r_k\}$ to satisfy (6.2), (6.4) and (6.5), respectively. Namely, let us take, for some $\lambda > 1$,

$$R_k = \sqrt{F t_k} \left( \frac{1}{k \log^{\lambda} k} \right)^{\frac{1}{n}},$$

$$r_k = \sqrt{G \left( t_{k+1} - t_k \right) \log k}$$

and

$$h_k = \sqrt{H \left( t_{k+1} - t_k \right)},$$

where $F$ is an arbitrarily large constant, $H$ is as above, and $G > 2D$ so that (6.5) holds.

Obviously, $h_k = o\left(r_k\right)$ as $k \to \infty$ so we need to compare only $R_k$ and $r_k$ to ensure (6.7). We have

$$\left( \frac{r_k}{R_k} \right)^2 = \operatorname{const} \frac{t_{k+1} - t_k}{t_k} k^{\frac{2}{n}} \left( \log k \right)^{1 + \frac{2\lambda}{n}} ,$$

whence we see that (6.7) will follow from

$$\frac{t_{k+1}-t_k}{t_k}=O\left(k^{-\frac{2}{n}}\log^{-\omega}k\right) \tag{6.8}$$

provided

$$\omega>1+\frac{2\lambda}{n}\,. \tag{6.9}$$

The following sequence $\{t_k\}$ satisfies (6.8):

$$\log t_k=\frac{k^{1-\frac{2}{n}}}{\log^{\omega}k} \tag{6.10}$$

(note that $n>2$ so that $t_k\to\infty$ as $k\to\infty$). Since for this sequence

$$\lim_{k\to\infty}\frac{t_{k+1}}{t_k}=1,$$

then we have (6.6) as well.

We are left to find $R_k$ as a function of $t_k$. We have from (6.10), for large $k$,

$$\log\log t_k>\operatorname{const}_{n,\omega}\log k$$

and

$$k=\left(\log t_k\log^{\omega}k\right)^{\frac{n}{n-2}}<\operatorname{const}_{n,\omega}\left(\log t_k\left(\log\log t_k\right)^{\omega}\right)^{\frac{n}{n-2}}\ ,$$

whence

$$R_k\geq\frac{\operatorname{const}_{n,\omega,\lambda}\sqrt{Ft_k}}{\log^{\frac{1}{n-2}}t_k\left(\log\log t_k\right)^{\frac{\omega}{n-2}+\frac{\lambda}{n}}}\,.$$

If we take $\lambda$ close enough to 1 and choose $\omega$ to be sufficiently close to $1+\frac{2\lambda}{n}$, then $\frac{\omega}{n-2}+\frac{\lambda}{n}$ can be made arbitrarily close to

$$\frac{1+\frac{2}{n}}{n-2}+\frac{1}{n}=\frac{2}{n-2}\,.$$

Since $F$ is arbitrary, then by Corollary 3.2 the function

$$\mathcal{R}(t)=\frac{C\sqrt{t}}{\log^{\frac{1}{n-2}}t\left(\log\log t\right)^{\frac{2+\varepsilon}{n-2}}}$$

is the lower radius for $W_x(t)$ for any $C>0$ and $\varepsilon>0$. ∎

The proof can be obviously modified to improve the function $\mathcal{R}(t)$ slightly to

$$\mathcal{R}(t)=\frac{C\sqrt{t}}{\log^{\frac{1}{n-2}}t\left(\log\log t\right)^{\frac{2}{n-2}}\left(\log\log\log t\right)^{\frac{1+\varepsilon}{n-2}}}\,.$$

To that end, one chooses the sequences as follows:

$$R_k=\sqrt{Ft_k}\left(k^{-1}\log^{-1}k\ \left(\log\log k\right)^{-\lambda}\right)^{\frac{1}{n}}$$

and

$$\frac{t_{k+1}-t_k}{t_k}=O\left(k^{-\frac{2}{n}}\log^{-(1+2/n)}k\left(\log\log k\right)^{-\omega}\right),$$

where $\lambda>1$ and $\omega>\frac{2\lambda}{n}$. The sequence $t_k$ satisfying this condition is defined from

$$\log t_k=k^{1-\frac{2}{n}}\log^{-(1+2/n)}k\left(\log\log k\right)^{-\omega}$$

whence we get

$$R_k \geq \frac{\mathrm{const}_{n,\omega,\lambda}\sqrt{Ft_k}}{\log^{\frac{1}{n-2}} t_k \, (\log\log t_k)^{\frac{2}{n-2}} \, (\log\log\log t_k)^{\frac{\omega}{n-2}+\frac{\lambda}{n}}}.$$

Finally, we notice that $\frac{\omega}{n-2}+\frac{\lambda}{n}$ can be made arbitrarily close to $\frac{2/n}{n-2}+\frac{1}{n}=\frac{1}{n-2}$.

## 7. Proof of Theorem 1.3 (upper radius $\sqrt{t\log\log t}$)

We will apply Corollary 2.6 to construct the upper function. A slightly modified argument from the previous proof shows that for the function

$$h(t)=\sqrt{Ht}$$

with a large constant $H=H(a,\nu)$, and for all $x\in M$, $t>0$ we have

$$\int\limits_{M\setminus B(x,h(t))} p(t,x,y)dy > \frac{1}{2}\,.$$

Let us take $\mathcal{R}(t)=\sqrt{\alpha t\log\log t}$ with arbitrary $\alpha>2$ and show that it is the upper function. To that end, we choose $\varepsilon>0$ to be so small that

$$\alpha(1-\varepsilon)^2>2$$

and take $t_k=(1+\varepsilon)^k$. Then hypothesis (2.9) of Corollary 2.6 is obviously true, and we are left to verify (2.10).

Since $h(t)<\varepsilon\mathcal{R}(t)$ for large enough $t$, then by Corollary 4.5 we have

$$\begin{aligned}
\sum_{k=1}^{\infty}\int\limits_{M\setminus B(x,\mathcal{R}(t_k)-h(t_k))} p(t_k,x,y)dy &\leq \sum_{k=1}^{\infty}\int\limits_{M\setminus B(x,(1-\varepsilon)\mathcal{R}(t_k))} p(t_k,x,y)dy \\
&\leq \mathrm{const}_{\,a,\nu,D}\sum_{k=1}^{\infty}\exp\left(-(1-\varepsilon)^2\frac{\mathcal{R}^2(t_k)}{2Dt_k}\right) \\
&= \mathrm{const}_{a,\nu,D}\sum_{k=1}^{\infty}\exp\left(-\frac{(1-\varepsilon)^2\alpha}{2D}\log\log t_k\right) \\
&= \mathrm{const}_{a,\nu,D}\sum_{k=1}^{\infty}\left(k\log(1+\varepsilon)\right)^{-\frac{(1-\varepsilon)^2\alpha}{2D}},
\end{aligned}$$

which is finite provided $D>1$ is chosen close enough to 1 to ensure $\frac{(1-\varepsilon)^2\alpha}{2D}>1$. ∎

## 8. Proof of Theorem 1.4 (lower bound of the upper radius)

Let us set $\mathcal{R}(t)=\sqrt{2t\log\log t}$ and prove that on a manifold of non-negative Ricci curvature the function $(1-\varepsilon)\mathcal{R}(t)$ is not an upper radius for any $\varepsilon>0$ . Let us fix a point $x\in M$ , numbers $T>1$, $\alpha\in(0,1)$, set $t_k=T^k$, $k=1,2,...$, and introduce the events

$$A_k=\left\{\mathrm{dist}\left(W_x(t_{k+1}),W_x(t_k)\right)>\alpha\mathcal{R}(t_{k+1}-t_k)\right\}\,.$$

Our purpose is to show that

$$\sum_{k=1}^{\infty}\mathbb{P}_x(A_k)=\infty\,. \tag{8.1}$$

Indeed, suppose that we have (8.1) already. Since the events $A_k$ are independent, we may conclude by the Borel–Cantelli lemma that infinitely many events of $A_k$ occur with probability 1.

On the other hand, by Theorem 1.3 we have that, with probability 1 for large $k$,

$$W_x(t_k) \in B(x, \beta \mathcal{R}(t_k)), \tag{8.2}$$

where $\beta > 1$. Therefore, there exist a.s. an infinite number of $k$'s such that $A_k$ and (8.2) occur simultaneously. For those $k$, we have

$$\begin{aligned} \operatorname{dist}(W_x(t_{k+1}), x) &\geq \operatorname{dist}(W_x(t_{k+1}), W_x(t_k)) - \operatorname{dist}(W_x(t), x) \\ &\geq \alpha \mathcal{R}(t_{k+1} - t_k) - \beta \mathcal{R}(t_k). \end{aligned}$$

Thus, we are left to choose $\alpha$, $\beta$ and $T$ so that (8.1) holds and

$$\alpha \mathcal{R}(t_{k+1} - t_k) - \beta \mathcal{R}(t_k) \geq (1-\varepsilon) \mathcal{R}(t_{k+1}), \tag{8.3}$$

which would imply almost surely

$$W_x(t_{k+1}) \in M \backslash B\left(x, (1-\varepsilon)\mathcal{R}(t_{k+1})\right)$$

for infinitely many $k$'s and, thus, the function $(1-\varepsilon)\mathcal{R}(t_{k+1})$ is no upper radius.

By Lemma 4.6 we have, for any $z \in M,\ t > 0,\ D \in (0,1)$ and $R > \sqrt{t}$,

$$\mathbb{P}_z \left\{ \operatorname{dist}(W_z(t), z) \geq R \right\} = \int\limits_{M \backslash B(z,R)} p(t,z,y)dy \geq \operatorname{const}_{n,D} \exp\left(-\frac{R^2}{2Dt}\right).$$

Since for large $t$ we have $\alpha \mathcal{R}(t) > t$, we may apply this inequality with $R = \alpha \mathcal{R}(t)$. We obtain that, for any $k$ large enough and for $T$ large enough

$$\begin{aligned} \mathbb{P}_x(A_k) &\geq \operatorname{const}_{n,D} \exp\left(-\frac{\alpha^2 \mathcal{R}^2(t_{k+1}-t_k)}{2D(t_{k+1}-t_k)}\right) \\ &= \operatorname{const}_{n,D} \exp\left(-\frac{\alpha^2}{D} \log\log(t_{k+1}-t_k)\right) \\ &\geq \frac{\operatorname{const}_{n,D,\alpha,T}}{k^{\alpha^2/D}}. \end{aligned}$$

We see that for any $\alpha < 1$ there exists $D < 1$ such that $\frac{\alpha^2}{D} < 1$ and, hence, (8.1) holds.

Let us verify (8.3). We first note that, for all $k \geq 1$,

$$\frac{\mathcal{R}(t_k)}{\mathcal{R}(t_{k+1})} \leq \frac{1}{\sqrt{T}}$$

and

$$\frac{\mathcal{R}(t_{k+1}-t_k)}{\mathcal{R}(t_{k+1})} = \sqrt{(1-T^{-1}) \frac{\log\log T^{k+1}(1-T^{-1})}{\log\log T^{k+1}}} \geq 1 - \delta(T),$$

where $\delta(T) \to 0$ as $T \to \infty$.

Therefore, for sufficiently large $T$ and for $\alpha$ close enough to 1,

$$\begin{aligned} \alpha \mathcal{R}(t_{k+1}-t_k) - \beta \mathcal{R}(t_k) &\geq \left(\alpha(1-\delta(T)) - \beta T^{-\frac{1}{2}}\right) \mathcal{R}(t_{k+1}) \\ &\geq (1-\varepsilon)\mathcal{R}(t_{k+1}), \end{aligned}$$

which was required. ∎

## 9. Appendix A. Proof of Lemma 2.5.

Of course, if $f(t)$ is monotone, then the statement of the lemma is true for any $\xi > 0$. However, we do not know a priori that the function, to which we apply the lemma, is monotone.

Let us take any numbers $a, b$ such that $0 < a < b < \varepsilon$ and prove that

$$\sum_{k=1}^{\infty} \int\limits_a^b f(k\xi)d\xi < \infty, \tag{9.1}$$

which would then imply (2.8). A change of variable in the integral (9.1) reduces it to

$$\sum_{k=1}^{\infty}\frac{1}{k}\int\limits_{ak}^{bk} f(\zeta)d\zeta\,. \tag{9.2}$$

Let us consider the family of all intervals $(ak,bk)$, $k=1,2,\ldots$. They overlap for large $k$ but it is possible to estimate from above the multiplicity of overlapping which will provide an upper bound of (9.2) in terms of the integral

$$\int_0^{\infty} f(t)dt\,.$$

To that end, let us enumerate in increasing order all numbers in the union $\{ak\}\cup\{bk\}$, $k=1,2,\ldots$ , and denote them $\{\alpha_i\}, i=1,2,\ldots$. The sum (9.1) is represented as

$$\sum_{i=1}^{\infty} w_i \int\limits_{\alpha_i}^{\alpha_{i+1}} f(\zeta)d\zeta$$

where $w_i$ is the sum of $\frac{1}{k}$ over all $k'$s such that

$$(\alpha_i,\alpha_{i+1})\subset(ak,bk)$$

which is equivalent to

$$\frac{\alpha_{i+1}}{b}\le k\le\frac{\alpha_i}{a}\,.$$

Therefore, the weight $w_i$ admits the estimate

$$w_i=\sum_{\frac{\alpha_{i+1}}{b}\le k\le\frac{\alpha_i}{a}}\frac{1}{k}\le\frac{\frac{\alpha_i}{a}-\frac{\alpha_{i+1}}{b}+1}{\frac{\alpha_{i+1}}{b}}<\frac{b}{a}\frac{\alpha_i}{\alpha_{i+1}}+\frac{b}{\alpha_{i+1}}<\frac{2b}{a}\,,$$

and we have

$$\sum_{k=1}^{\infty}\frac{1}{k}\int\limits_{ak}^{bk} f(\zeta)d\zeta\le\frac{2b}{a}\sum_{i=1}^{\infty}\int\limits_{\alpha_i}^{\alpha_{i+1}} f(\zeta)d\zeta\le\frac{2b}{a}\int\limits_{0}^{\infty} f(\zeta)d\zeta<\infty,$$

which was to be proved.

## 10. Appendix B. Some elementary integral estimates

**Lemma 10.1.** *Let $w(r)$ be a positive increasing function on $(0,+\infty)$ such that, for any $r>0$,*

$$\frac{w(2r)}{w(r)}\le A\,. \tag{10.1}$$

*Let $\rho=\sqrt{\frac{1}{3}\log A}$ and $K=w(\rho)\left(\rho+\frac{A}{\rho}\right)$ .Then, for any $r\ge\rho$,*

$$\int\limits_{r}^{\infty} w(s)e^{-s^2}ds\le A\frac{w(r)}{r}e^{-r^2} \tag{10.2}$$

*and, for any positive $r<\rho$,*

$$\int\limits_{r}^{\infty} w(s)e^{-s^2}ds\le Ke^{-r^2}\,.$$

*In particular, for any $r>0$,*

$$\int_r^\infty w(s)e^{-s^2}ds \le \left(K + A\frac{w(r)}{r}\right)e^{-r^2}. \tag{10.3}$$

*Proof.* Let us consider a sequence $r_k = r2^k,\ k = 0,1,2,...$, and split the integral (10.2) as follows

$$\begin{aligned}
\int_r^\infty w(s)e^{-s^2}ds &= \sum_{k=0}^\infty \int_{r_k}^{r_{k+1}} w(s)e^{-s^2}ds \\
&\le \sum_{k=0}^\infty w(r_{k+1}) \int_{r_k}^{r_{k+1}} e^{-s^2}ds \\
&\le \sum_{k=0}^\infty w(r_{k+1}) \int_{r_k}^{\infty} e^{-s^2}ds.
\end{aligned}$$

It is easy to show that, for any $r>0$,

$$\int_r^\infty e^{-s^2}ds \le \frac{1}{2r}e^{-r^2},$$

whence

$$\begin{aligned}
\int_r^\infty w(s)e^{-s^2}ds &\le \sum_{k=0}^\infty w(r_{k+1})\frac{1}{2r_k}e^{-r_k^2} \\
&\le \frac{A}{2}\sum_{k=0}^\infty \frac{w(r_k)}{r_k}e^{-r_k^2}.
\end{aligned}$$

Let us estimate the ratio of any two consecutive terms in the sum above:

$$\frac{w(r_{k+1})}{r_{k+1}}e^{-r_{k+1}^2} : \frac{w(r_k)}{r_k}e^{-r_k^2} \le \frac{A}{2}\exp\left(-3r_k^2\right).$$

We see that if $3r_k^2 \ge \log A$ then the $k$-th term is larger than the next one by at least a factor 2. Therefore, if $3r^2 \ge \log A$, which is equivalent to $r \ge \rho$, then

$$\sum_{k=0}^\infty \frac{w(r_k)}{r_k}e^{-r_k^2} \le \frac{w(r)}{r}e^{-r^2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + ...\right) \le 2\frac{w(r)}{r}e^{-r^2}$$

and

$$\int_r^\infty w(s)e^{-s^2}ds \le A\frac{w(r)}{r}e^{-r^2}.$$

For $r < \rho$ we have

$$\begin{aligned}
\int_r^\infty w(s)e^{-s^2}ds &= \int_r^\rho w(s)e^{-s^2}ds + \int_\rho^\infty w(s)e^{-s^2}ds \\
&\le w(\rho)\int_r^\rho e^{-s^2}ds + A\frac{w(\rho)}{\rho}e^{-\rho^2} \\
&\le w(\rho)\rho e^{-r^2} + A\frac{w(\rho)}{\rho}e^{-r^2} \\
&= Ke^{-r^2}\,.
\end{aligned}$$

**Corollary 10.2.** *For any $N \ge 0$ and $q > 0$,*

$$\int_q^\infty s^{N+1}e^{-s^2}ds \le \mathrm{const}_N\left(1+q^N\right)e^{-q^2}\,.$$

## 11. Appendix C. Some useful implications

Here we describe the relationships between such properties as $(a,\nu)$-isoperimetric inequality, doubling volume property, lower bound on Ricci curvature etc. Let us say that a manifold $M$ possesses *a restricted doubling volume property* if the following holds:

(D) for some $A > 0$ and $\rho > 0$, for any ball $B(x,R) \subset M$ of radius $R < \rho$,

$$\frac{V(x,R)}{V(x,R/2)} \le A\,.$$

We say that a manifold $M$ possesses *a restricted weak Poincaré inequality* if the following holds:

(P) for some $b > 0$ and $\rho > 0$, for any ball $B(x,R) \subset M$ of radius $R < \rho$ and for any function $f \in C^\infty(B(x,R))$,

$$\int_{B(x,R)} |\nabla f(y)|^2\,dy \ge \frac{b}{R^2}\int_{B(x,R/2)} \left(f(y)-\overline{f}\right)^2 dy\,,$$

where

$$\overline{f} \equiv \frac{1}{V(x,R/2)}\int_{B(x,R/2)} f(y)dy\,.$$

If $\rho = \infty$ in either of these properties, then we do not apply the adjective "restricted" to it.

The following diagram shows connections between the conditions which have been used in this paper:

$$\begin{array}{ccccc}
\textit{Ricci} \ge -K^2 & \overset{(i)}{\Longrightarrow} & (D)\,\textit{and}\,(P) & \overset{(vii)}{\Longleftrightarrow} & \textit{Harnack inequality} \\
 & & \Downarrow^{(ii)} & & \\
 & & \textit{weak bounded geometry} & & \\
 & & \Downarrow^{(iii)} & & \\
\begin{array}{c}\frac{V(x,R)}{V(x,r)} \ge c\left(\frac{R}{r}\right)^{\lambda} \\ \forall\, r<R<\rho\end{array} & \overset{(vi)}{\Longleftarrow} & \textit{condition}\,(D) & \overset{(v)}{\Longrightarrow} & \begin{array}{c}\frac{V(x,R)}{V(x,r)} \ge 1+\psi\left(\frac{R}{r}\right) \\ \forall\, r<R<\rho\end{array} \\
 & & \Downarrow^{(iv)} & & \\
 & & \begin{array}{c}\frac{V(x,R)}{V(x,r)} \le C\left(\frac{R}{r}\right)^{N} \\ \forall\, r<R<\rho\end{array} & &
\end{array}$$

When departing from the Ricci curvature assumption, $\rho$ can be taken to be arbitrary but finite for $K > 0$, and $\rho = \infty$ for $K = 0$. The radius $\rho$ is preserved by every implication. In particular, if one of the properties holds with $\rho = \infty$, then all its descendants also have $\rho = \infty$. For example, this is the case when $K = 0$.

Let us mention also that a weak bounded geometry with $\rho = \infty$ is nothing other than property (U) of Theorem 1.2, so that we have the following implications:

$$\textit{Ricci} \ge 0 \quad\Longrightarrow\quad (D)\,\textit{and}\,(P)\;\textit{with}\;\rho = \infty \quad\Longrightarrow\quad \textit{condition}\,(U)\quad .$$

Implication $(i)$ was proved in [2] (the case $K = 0$ was also considered in [8]), implication $(ii)$ was proved in [8, Theorem 1.4] and implication $(iii)$ was proved in [9, Proposition 5.2]. Although the latter two references dealt with the case $\rho = \infty$ only, the case $\rho < \infty$ can be treated by the same arguments.

In implications $(iv)$ and $(vi)$, the constants $C$, $c$, $N$ and $\lambda$ are positive and depend only on the constant $A$ from (D). The function $\psi$ from $(v)$ is the following:

$$\psi(s) = C^{-1}\left(\frac{s-1}{s+3}\right)^{N} .$$

Although this function is not optimal, it is sufficient for our purposes to know that $\psi > 0$.

The implications $(iv) - (vi)$ are well known (see, e.g. [8, Theorem 1.1 ]); nevertheless we present their proof here for convenience of the reader. The equivalence $(vii)$ is discussed in the next section.

*Proof of* $(iv)$. Let $k \ge 0$ be an integer such that

$$2^k \le \frac{R}{r} < 2^{k+1}. \tag{11.1}$$

Then by applying (D) at most $k+1$ times to the consecutive concentric balls of radii $r, 2r, 4r, ...$ we get

$$\frac{V(x,R)}{V(x,r)} \le A^{k+1} \le A^{1+\log_2 \frac{R}{r}} \le A\left(\frac{R}{r}\right)^{\log_2 A} .$$

*Proof of* $(v)$. Let $z$ be an arbitrary point on the sphere $\partial B(x, \frac{R+r}{2})$ (existence of $z$ follows from noncompactness and completeness of the manifold). Since the annulus $B(x,R)\backslash B(x,r)$

contains the ball $B(z, \frac{R-r}{2})$ and $B(x,r) \subset B(z, r+\frac{R+r}{2})$ then, by $(iv)$,

$$\begin{aligned} V(x,r) &\leq V(z, r+\frac{R+r}{2}) \\ &\leq C\left(\frac{R+3r}{R-r}\right)^N V(z, \frac{R-r}{2}) \\ &\leq C\left(\frac{R+3r}{R-r}\right)^N (V(x,R)-V(x,r)), \end{aligned}$$

whence

$$\frac{V(x,R)}{V(x,r)} \geq 1 + C^{-1}\left(\frac{R-r}{R+3r}\right)^N .$$

*Proof of* $(vi)$. If $R = 2r$, then we have, by $(v)$ ,

$$\frac{V(x,2r)}{V(x,r)} \geq 1+\delta ,$$

where $\delta = \delta(A) > 0$.

Let $k$ be as in (11.1). If $k \geq 1$, then

$$\frac{V(x,R)}{V(x,r)} \geq (1+\delta)^k \geq (1+\delta)^{\log_2 \frac{R}{r} - 1} = (1+\delta)^{-1}\left(\frac{R}{r}\right)^\lambda$$

where $\lambda = \log_2(1+\delta)$. Finally, if $k = 0$, then

$$\frac{V(x,R)}{V(x,r)} \geq 1 \geq 2^{-\lambda}\left(\frac{R}{r}\right)^\lambda .$$

## 12. Appendix D. A Harnack inequality

We say that a manifold possesses *a restricted parabolic Harnack inequality* if the following holds:

(H) there are positive numbers $\rho$ and $C$ such that for any $x \in M$, for any $R \in (0,\rho)$ and for any positive solution $u(x,t)$ to the diffusion equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u$ defined in the cylinder $B(x,R) \times (0,R^2)$, the following inequality holds:

$$\sup_{B(x,R)\times(R^2,2R^2)} u \leq C \inf_{B(x,R)\times(3R^2,4R^2)} u . \tag{12.1}$$

If $\rho = \infty$, then we refer to (H) as *a parabolic Harnack inequality.*

As was proved in [13] [and as was mentioned above as implication $(vii)$], (H) is equivalent to the conjunction of (D) and (P). (Implications (D) & (P) $\Longrightarrow$(H)$\Longrightarrow$(D) were proved also in [8].) We have the following theorem:

**Theorem 12.1.** *Let (H) hold with $\rho = \infty$ on a complete non-compact manifold $M$. Then the following hold:*

(a) *for any $\varepsilon > 0$ the function $\sqrt{(2+\varepsilon)t\log\log t}$ is an upper radius for the Brownian motion;*
(b) *for some sufficiently small $\varepsilon > 0$ the function $\sqrt{\varepsilon t \log\log t}$ is not an upper radius;*
(c) *if for a point $x \in M$ condition (V) from Theorem 1.2 holds, then for any $C > 0$ and $\varepsilon > 0$ the function*

$$\frac{C\sqrt{t}}{\log^{\frac{1}{n-2}} t\,(\log\log t)^{\frac{2+\varepsilon}{n-2}}}$$

*is a lower radius for $W_x(t)$.*

*Proof.* Indeed, as (H) implies condition (U) [via (P) and (D)] then Theorem 1.3 yields (a). The Harnack inequality implies also the lower bound of the heat kernel (4.16) from the proof of Theorem 1.4, so that the proof may be repeated again. The only difference from the case of Theorem 1.4 is that the constant $D$ is now only positive rather than close to 1, which yields the lower estimate of the upper radius as $\sqrt{\varepsilon t \log\log t}$ rather than $\sqrt{(2-\varepsilon)t\log\log t}$. Finally, Theorem 1.2 is applicable and gives (c). ∎

The Harnack inequality (H) with $\rho = \infty$ is true on a manifold of nonnegative Ricci curvature (see [12]). In this case Theorem 1.4 provides a better estimate than (b). However, if the manifold $M$ is quasiisometric to one with nonnegative Ricci curvature, then Theorem 1.4 may not be applicable, whereas Theorem 12.1 works because the Harnack inequality (H) is stable under quasiisometry (see [13]).

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