Heat kernel on manifolds with ends

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1 Introduction

1.1 Motivation

Because of its intrinsic interest and its many applications in various areas of mathematics, the heat diffusion equation on manifolds has been studied intensively. In particular, during the past 30 years many authors attacked the problem of describing the global behavior of the heat diffusion kernel $p(t, x, y)$ on various Euclidean domains and manifolds. See for instance [3, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 26, 30, 31, 32, 33, 34, 48, 52, 56, 55, 60, 61, 62, 63, 64].

Since $p(t, x, y)$ represents the temperature at point y at time t starting with a unit amount of heat concentrated at x at time 0, one of the most basic questions one might ask concerns the behavior of the functions $p(t, x, y)$, $\sup_{y'} p(t, x, y')$, and $\sup_{x', y'} p(t, x', y')$ as t tends to $+\infty$. Another fundamental question is to describe the location of the approximate hot spot, that is, of the set

$$\{y : p(t, x, y) \geq \varepsilon \sup_{y'} p(t, x, y')\},$$

where $\varepsilon \in (0, 1)$, a starting point x and temperature t are fixed. The latter question is rather difficult since it calls for precise global two sided bounds of the heat kernel.

The aim of this paper is to prove satisfactory estimates for the heat kernel on complete manifolds with finitely many ends. These estimates were announced in [37]. The proofs are quite involved and, in particular, make use of results from [38], [39], [40] and [41] (in fact, these works were largely motivated by the applications presented here). Our main result, Theorem 6.6, allows us to answer the questions mentioned above and applies to a large class of manifolds including the catenoid-like surface in Fig. 1, the three dimensional body (with the Neumann boundary condition) in Fig. 2, and all non-parabolic manifolds with non-negative sectional curvature outside a compact set. It seems likely that the techniques introduced here will be essential to make further progress in our understanding of the heat kernel on manifolds that contain parts with different geometric characteristics.

To the best of our knowledge, the large time behavior of the heat kernel on manifolds with ends has been considered only in a handful of papers where some very partial results were obtained. Among them are the papers by Benjamini, Chavel, Feldman [5], Chavel and Feldman [9], and Davies [25], which have been a great source of motivation and insight for us. More recently, Carron, Coulhon, and Hassell [8] obtained precise asymptotic results for manifolds with a finite number of Euclidean ends.

It is well established that the long time behavior of the heat kernel reflects, in some way, the large scale geometry of the manifold. Still, the number of situations for which satisfactory upper and lower global bounds are known is very limited. If one excepts a few specific cases of symmetric spaces (see [1]) and the case of fractal like manifolds (see [4]), all the known global
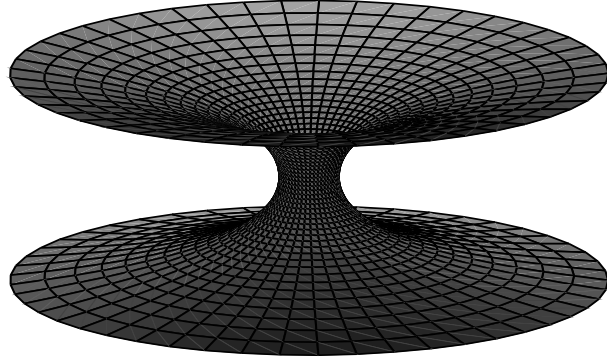
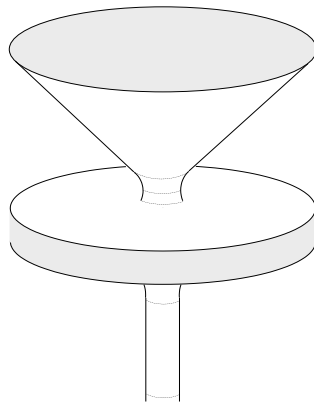Figure 1: Catenoid as a manifold with two ends

Figure 2: A domain in $\mathbb{R}^3$ with three ends: conical, planar, cylindrical

two-sided estimates of the heat diffusion kernel $p(t, x, y)$ have the form

$$\frac{c_1}{V(x, \sqrt{t})} \exp\left(-C_1 \frac{d^2(x, y)}{t}\right) \leq p(t, x, y) \leq \frac{C_2}{V(x, \sqrt{t})} \exp\left(-c_2 \frac{d^2(x, y)}{t}\right) \qquad (1.1)$$

where $V(x, r)$ is the volume of the ball of radius r around x and $d(x, y)$ is the distance between x and y. Such a two-sided bound indicates that the heat diffusion on M is controlled by the volume growth of balls and by a universal Gaussian factor that reflects a simple distance effect (see [29, 21, 34]). In terms of the hot spot problem, (1.1) indicates that the approximate hot spot at time t starting from x is roughly described by the ball of radius $\sqrt{t}$ centered at x.

Examples where (1.1) holds are complete manifolds having non-negative Ricci curvature [48], manifolds which are quasi-isometric to those with non-negative Ricci curvature [31, 56] and manifolds that cover a compact manifold with deck-transformation group having polynomial volume growth [55, 57]. In fact, the two-sided estimate (1.1) is rather well understood, since it is known to be equivalent to the conjunction of the following two properties:

(VD) the doubling volume property which asserts that there exists a finite constant C such that, for all $x \in M$ and $r > 0$,

$$V(x, 2r) \leq CV(x, r).$$

(PI) the Poincaré inequality on balls, which asserts that there exists a positive constant c such that, for any ball $B = B(x, r) \subset M$

$$\lambda_2^{(N)}(B) \geq \frac{c}{r^2}, \qquad (1.2)$$

where $\lambda_2^{(N)}(B)$ is the second Neumann eigenvalue of B (note that $\lambda_1^{(N)}(B) = 0$).

It is known that (1.1) is also equivalent to the validity of a uniform parabolic Harnack inequality for positive solutions of the heat equation in cylinder of the form $(s, s + r^2) \times B(x, r)$. See [31, 55] and Section 5.1 below.

Typically, manifolds with ends do not satisfy (1.1). An example that was first considered by Kuz'menko and Molchanov [46], is the connected sum of two copies of $\mathbb{R}^3$, that is, the manifold M obtained by gluing together two punctured three-dimensional Euclidean spaces through a small three dimensional cylinder. This manifold has two ends and its volume growth function is comparable to that of $\mathbb{R}^3$, that is, $V(x, r) \approx r^3$. However, as was shown in [5], the lower bound in (1.1) fails on M. Indeed, if x and y are in different ends and far enough from the compact cylinder, then $p(t, x, y)$ should be significantly smaller than predicted by (1.1), at least for some range of $t > 0$, because all paths from x to y must go through the cylinder[1]. In other words, there should be a bottleneck effect which must be accounted for if one wants to obtain precise heat kernel estimates on M.

The manifolds on Fig. 1 and 2 do not satisfy (1.1) either. This is easy to see for the Euclidean body in Fig 2 because the volume doubling property fails in this case. For the catenoid in Fig. 1, the volume doubling property is true but one can show that the Poincaré inequality (1.2)

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[1] Another way to see that the lower bound in (1.1) fails is to disprove the uniform Harnack inequality. Indeed, as was shown in [46], the connected sum of two copies of $\mathbb{R}^n$, $n > 2$, admits a non-constant bounded harmonic function, which contradicts the uniform Harnack inequality. The upper bound in (1.1) still holds on the manifold in question (see Section 4.1). Hence, the lower bound fails.

fails. Sharp two sided estimates for the heat kernel on the catenoid follow from Theorem 7.1 below.

The goal of this paper is to develop tools that lead, in some generality, to upper and lower bounds taking into account the bottleneck effect. In order to describe some of our results, let us introduce the following terminology. Let M be a complete non-compact Riemannian manifold. Let $K \subset M$ be a compact set with non-empty interior and smooth boundary such that $M \setminus K$ has k connected components $E_1, \ldots, E_k$ and each E_i is non-compact. We say in such a case that M has k ends E_i with respect to K. We will refer to K as the central part of M.

In many cases (in fact, in full generality if one admits as we will manifolds with boundary), each E_i is isometric to the exterior of a compact set in another manifold M_i. In such case, we write

$$M = M_1 \# M_2 \# ... \# M_k$$

and refer to M as a connected sum of the manifolds M_i. For instance, the example considered above can be described in this notation as $\mathbb{R}^3 \# \mathbb{R}^3$.

1.2 Description of the results in model cases

To obtain a rich class of elementary examples, fix a large integer N (which will be the topological dimension of M) and, for any integer $m \in [1, N]$, define the manifold $\mathcal{R}^m$ by

$$\mathcal{R}^1 = \mathbb{R}_+ \times \mathbb{S}^{N-1}, \quad \mathcal{R}^m = \mathbb{R}^m \times \mathbb{S}^{N-m}, \ m \geq 2. \tag{1.3}$$

The manifold $\mathcal{R}^m$ has topological dimension N but its "dimension at infinity" is m in the sense that $V(x, r) \approx r^m$ for $r \geq 1$. Thus, for different values of m, the manifolds $\mathcal{R}^m$ have different dimension at infinity but the same topological dimension N. This enables us to consider finite connected sums of the $\mathcal{R}^m$'s. In particular, for $n \neq m$, $\mathcal{R}^n \# \mathcal{R}^m$ is well-defined whereas $\mathbb{R}^n \# \mathbb{R}^m$ does not make sense in the category of manifolds.

Fix N and k integers $N_1, N_2, ..., N_k \in [1, N]$ and consider the manifold

$$M = \mathcal{R}^{N_1} \# \mathcal{R}^{N_2} \# ... \# \mathcal{R}^{N_k}. \tag{1.4}$$

From the viewpoint of this paper, this is already an interesting class of examples for which we would like to obtain global, two-sided, heat kernel estimates. This class of manifolds is also useful for testing the validity of various geometric and analytic properties.

We now describe how the results obtained in this paper apply to the manifold M at (1.4) when each N_i is larger than 2. This hypothesis means that all the ends of M are non-parabolic and we set

$$n := \min_{1 \leq i \leq k} N_i > 2. \tag{1.5}$$

Let K be the central part of M and $E_1, \ldots E_k$ be the ends of M so that E_i is isometric to the complement of a compact set in $\mathcal{R}^{N_i}$. With some abuse of notation, we write $E_i = \mathcal{R}^{N_i} \setminus K$. Thus $x \in \mathcal{R}^{N_i} \setminus K$ means that the point $x \in M$ belongs to the end associated with $\mathcal{R}^{N_i}$. For any point $x \in M$, set

$$|x| := \sup_{z \in K} d(x, z).$$

Observe that since K has non-empty interior, $|x|$ is separated from 0 on M and $|x| \approx 1 + d(x, K)$.

In the following estimates we always assume that $t \geq t_0$ (where $t_0 > 0$ is fixed), x, y are points on M and $d = d(x, y)$ is the geodesic distance in M. We follow the convention that $C, C_1, \ldots$ denote large finite positive constants whereas $c, c_1, \ldots$ are small positive constants

(these constants may depend on M but do not depend on the variables x, y, t). Given two non-negative functions f, g defined on a domain I, we write

$$f \approx g$$

to signify that there are constants $0 < c \leq C < \infty$ such that, on I, $cf \leq g \leq Cf$.

1. Let us first consider the simplest case $k = 2$, i.e., M at (1.4) has two ends. To simplify notation, set $M = \mathcal{R}^n \# \mathcal{R}^m$ where $2 < n \leq m$. Assume that $x \in \mathcal{R}^n \setminus K$ and $y \in \mathcal{R}^m \setminus K$. Then we claim that[2]

$$p(t, x, y) \leq C_1 \left(\frac{1}{t^{m/2} |x|^{n-2}} + \frac{1}{t^{n/2} |y|^{m-2}} \right) \exp\left(-c_1 \frac{d^2}{t} \right) \tag{1.6}$$

and

$$p(t, x, y) \geq c_2 \left(\frac{1}{t^{m/2} |x|^{n-2}} + \frac{1}{t^{n/2} |y|^{m-2}} \right) \exp\left(-C_2 \frac{d^2}{t} \right). \tag{1.7}$$
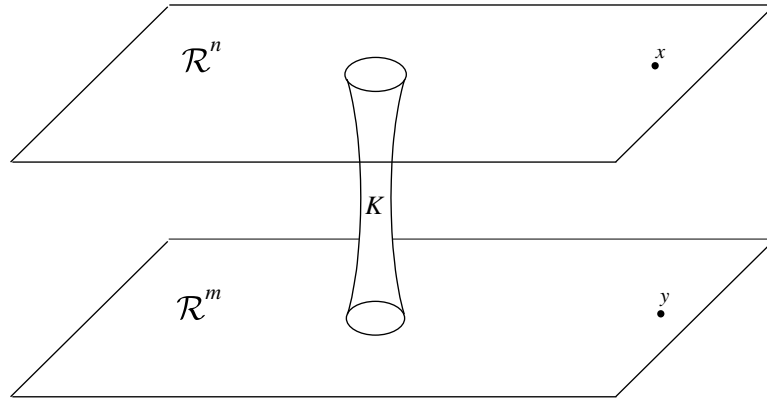
Figure 3: Points x and y on $\mathcal{R}^n \# \mathcal{R}^m$

In particular, if x and y are fixed and $t \to \infty$ then (1.6) and (1.7) yield

$$p(t, x, y) \approx \frac{1}{t^{n/2}}, \tag{1.8}$$

that is, the smallest end $\mathcal{R}^n$ determines the long term behavior of the heat kernel. This phenomenon was observed by E.B.Davies [25] for a weighted one-dimensional complex.

If we allow x, y, t to vary in the range — call it the long time asymptotic regime —

$$|x| \leq \eta(t), \quad |y| \leq \eta(t) \tag{1.9}$$

where η denotes a positive function going to infinity slower than any positive power of t then we obtain

$$p(t, x, y) \approx \frac{q(x, y)}{t^{n/2}}, \tag{1.10}$$

where

$$q(x, y) = \begin{cases} |y|^{2-m}, & m > n, \\ |x|^{2-n} + |y|^{2-n}, & m = n. \end{cases} \tag{1.11}$$

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[2]In fact, the upper bound (1.6) holds also when $n \in \{1, 2\}$, $m \geq n$. However, the lower bound (1.7) fails in this case.

If instead we consider the medium time asymptotic regime

$$|x| \approx |y| \approx \sqrt{t} \quad \text{and} \quad t \to \infty, \tag{1.12}$$

(1.6)-(1.7) implies

$$p(t, x, y) \approx \frac{1}{t^{(n+m)/2-1}} . \tag{1.13}$$

Clearly, the decay of the heat kernel given by (1.13) is much faster than that of (1.8). This is the bottleneck effect that was alluded to earlier. As far as we know, even the basic estimate (1.13) is new, not to mention the full inequalities (1.6) and (1.7). Benjamini, Chavel and Feldman [5] showed, for $n = m$ and assuming (1.12), that

$$p(t, x, y) \leq \frac{C_\epsilon}{t^{n-1-\varepsilon}} , \quad \forall \varepsilon > 0,$$

whereas (1.13) gives a better estimate

$$p(t, x, y) \approx \frac{1}{t^{n-1}} .$$

2. Let $k \geq 3$ be any integer. Assume that $x \in \mathcal{R}^{N_i} \setminus K$ and $y \in \mathcal{R}^{N_j} \setminus K$ where $i \neq j$. Then, for all $t \geq t_0$,

$$p(t, x, y) \leq C_1 \left(\frac{1}{t^{n/2} |x|^{N_i-2} |y|^{N_j-2}} + \frac{1}{t^{N_j/2} |x|^{N_i-2}} + \frac{1}{t^{N_i/2} |y|^{N_j-2}} \right) \exp\left(-c_1 \frac{d^2}{t} \right) \tag{1.14}$$

and

$$p(t, x, y) \geq c_2 \left(\frac{1}{t^{n/2} |x|^{N_i-2} |y|^{N_j-2}} + \frac{1}{t^{N_j/2} |x|^{N_i-2}} + \frac{1}{t^{N_i/2} |y|^{N_j-2}} \right) \exp\left(-C_2 \frac{d^2}{t} \right) . \tag{1.15}$$

The last two terms in (1.14) and (1.15) are the same as the terms in (1.6) and (1.7), respectively. There is also an additional effect due to the presence of at least three ends which is reflected in the first term

$$\frac{1}{t^{n/2} |x|^{N_i-2} |y|^{N_j-2}} . \tag{1.16}$$

Recall that n is the smallest of the numbers $N_1, N_2, ..., N_k$. If $n = N_i$ or $n = N_j$, then the term (1.16) is majorized by the other two terms in (1.14) and (1.15) (in particular, (1.14) and (1.15) formally hold also for $k = 2$ in which case they are equivalent to (1.6) and (1.7), respectively).

Assuming (1.9), (1.14) and (1.15) give (1.10) with

$$q(x, y) = \begin{cases} |y|^{2-N_j} , & n = N_j < N_i, \\ |x|^{2-N_i} , & n = N_i < N_j, \\ |x|^{2-n} + |y|^{2-n} , & n = N_i = N_j, \\ |x|^{2-N_i} |y|^{2-N_j} , & n < \min(N_i, N_j). \end{cases} \tag{1.17}$$

Note that the power of t in the long time asymptotic is again determined by the smallest end $\mathcal{R}^n$. In the last case in (1.17) when $n < \min(N_i, N_j)$, we see that the term (1.16) becomes the leading term provided t is large enough, and we have

$$p(t, x, y) \approx \frac{1}{t^{n/2} |x|^{N_i-2} |y|^{N_j-2}} . \tag{1.18}$$

Each factor in this asymptotic has a heuristic interpretation in terms of the Brownian motion $(X_t)_{t \geq 0}$ on the manifold M which we now explain.

a) $|x|^{-(N_i-2)}$ is roughly the probability that the process X_t started at x ever hits K;

b) $t^{-n/2}$ is roughly the probability of making a loop from K to K through the smallest end (i.e. the end $\mathcal{R}^n$) in a time of order t;

c) $|y|^{-(N_j-2)}$ is roughly the probability of getting from K to y.

In particular, (1.18) says that the most probable way of going from x to y in a very long time t involves visiting the smallest end.
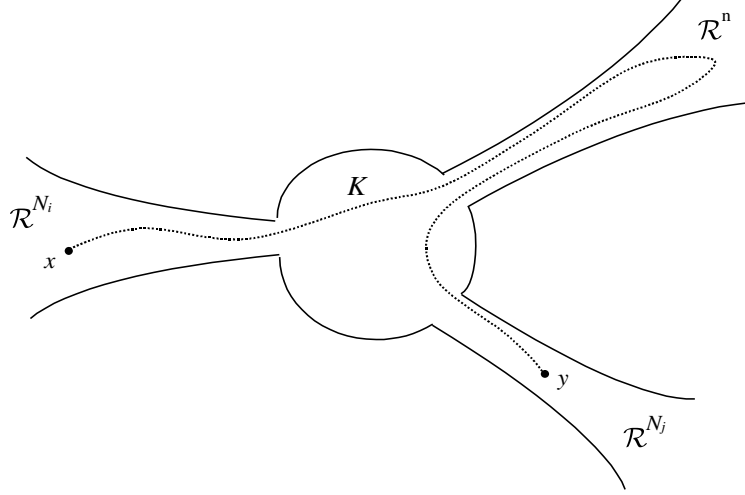

Figure 4: The most probable trajectories from x to y go through the smallest end $\mathcal{R}^n$

3. Finally, assume that both $x, y \in \mathcal{R}^{N_i} \setminus K$. Then, for all $t \geq t_0$,

$$p(t,x,y) \leq \frac{C_1}{t^{N_i/2}} e^{-c_1 d^2/t} + \frac{C_1}{t^{n/2} |x|^{N_i-2} |y|^{N_i-2}} e^{-c_1(|x|^2+|y|^2)/t} \tag{1.19}$$

and

$$p(t,x,y) \geq \frac{c_2}{t^{N_i/2}} e^{-C_2 d^2/t} + \frac{c_2}{t^{n/2} |x|^{N_i-2} |y|^{N_i-2}} e^{-C_2(|x|^2+|y|^2)/t}. \tag{1.20}$$

Assuming (1.9), we obtain (1.10) with

$$q(x,y) = \begin{cases} 1, & n = N_i, \\ |x|^{2-N_i} |y|^{2-N_i}, & n < N_i. \end{cases}$$

In particular, if $n < N_i$ then we obtain again (1.18), for t large enough.

Next, let us briefly discuss the mixed case where the restriction $n = \min_i N_i > 2$ is relaxed to $\max_i N_i > 2$. The word "mixed" refers to the fact that in this case M has both parabolic and non-parabolic ends. A detailed discussion is given in Section 6 where full two-sided bounds are obtained. Here we present selected results to give a flavour of what can occur:

1. Let $M = \mathcal{R}^1 \# \mathcal{R}^3$. Then, for large enough t, we have

$$p(t,x,y) \approx \frac{1}{t^{3/2}}, \quad \sup_y p(t,x,y) \approx \frac{1}{t}, \quad \sup_{x,y} p(t,x,y) \approx \frac{1}{t^{1/2}}.$$

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2. In the case $M = \mathcal{R}^2 \# \mathcal{R}^3$, we have

$$p(t,x,y) \approx \frac{1}{t \log^2 t}, \quad \sup_y p(t,x,y) \approx \frac{1}{t \log t}, \quad \sup_{x,y} p(t,x,y) \approx \frac{1}{t}$$

3. In the case $M = \mathcal{R}^1 \# \mathcal{R}^2 \# \mathcal{R}^3$, we have

$$p(t,x,y) \approx \frac{1}{t \log^2 t}, \quad \sup_y p(t,x,y) \approx \frac{1}{t}, \quad \sup_{x,y} p(t,x,y) \approx \frac{1}{t^{1/2}} \tag{1.21}$$

The estimates (1.21) also apply to the Euclidean body of Fig. 2.

Finally, heat kernel estimates for the manifold $M = \mathcal{R}^2 \# \mathcal{R}^2$ follow from the results of Section 7. We prove that, for large time,

$$p(t,x,y) \approx \sup_y p(t,x,y) \approx \sup_{x,y} p(t,x,y) \approx \frac{1}{t}$$

whereas in the medium time asymptotic regime (1.12),

$$p(t,x,y) \approx \begin{cases} t^{-1} & \text{if } x, y \text{ are in the same end} \\ (t \log t)^{-1} & \text{if } x, y \text{ are in different ends.} \end{cases}$$

The same estimates apply to the catenoid of Fig. 1.

The examples described above clearly show that the presence of more than one end brings in interesting and somewhat complex new phenomena as far as heat kernel bounds are concerned. The tools developed in this paper allows us to analyze much more general situations than (1.4). For instance, we obtain a complete generalization of the above results (i.e., global matching upper and lower heat kernel bounds) for the connected sum $M = M_1 \# M_2 \# ... \# M_k$ provided each M_i is a non-parabolic complete Riemannian manifold satisfying the hypotheses (VD) and (PI) stated at the beginning of this introduction (see Theorems 4.9 and 5.10). In particular, this result applies whenever each M_i has non-negative Ricci curvature.

1.3 Guide for the paper

The structure of the paper is as follows. Section 2 introduces notation and basic definitions.

Section 3 develops gluing techniques for which the key result is Theorem 3.5. These techniques enable us to obtain bounds on the heat kernel $p(t,x,y)$ on a manifold M with ends $E_1, \ldots, E_k$ and central part K in terms of:

(a) The size of $p(t,u,v)$ where $u, v \in K$. Roughly speaking, for large t, this can be thought of as a function of t alone but it depends on the global geometry of M and, in particular, of all the ends taken together.

(b) Quantities that depend only on the geometry of the ends taken separately. One such quantity is the Dirichlet heat kernel in E_i (i.e., the transition function of Brownian motion killed as it exits E_i). Another such quantity is the probability that Brownian motion started at $x \in E_i$ hits K before time t. In both cases, it is clear that these quantities involve only the end E_i.

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Section 4 is devoted to heat kernel upper bounds on manifolds with ends. It starts with background on various results that are used in a crucial way in this paper. Several of these results were in fact developed by the authors with the applications presented here in mind. Faber-Krahn inequalities on manifolds with ends are studied in [41] where a rough initial upper bound of the heat kernel on manifolds with ends is derived. Hitting probabilities are studied in detail in [39]. Using these ingredients and the gluing techniques of Section 3, we prove sharp heat kernel upper bounds on manifolds with ends under the basic assumption that each end satisfies a certain relative Faber-Krahn inequality (other situations, e.g. *flat* Faber-Krahn inequalities, can be treated by the same technique – see [42]). The main result of Section 4 is Theorem 4.9.

Section 5 is devoted to heat kernel lower bounds on $M = M_1 \# M_2 \# ... \# M_k$. These lower bounds match (in some sense) the upper bounds of Section 4 but they require stronger hypotheses. Namely, we assume that each M_i is a non-parabolic manifold satisfying (VD) and (PI). Here the key ingredients are a lower bound for hitting probabilities that is taken from [39] and a lower bound on the Dirichlet heat kernels of the different ends which is taken from [38]. Both the lower bound on hitting probability and the lower bound on the Dirichlet heat kernels depend crucially on the hypothesis that each end is non-parabolic. The main Theorem of Section 5 is Theorem 5.10.

Sections 4 and 5 both ends with examples illustrating Theorems 4.9 and 5.10 respectively. In particular, these examples cover the case of the manifolds $M = \mathcal{R}^{N_1} \# \ldots \# \mathcal{R}^{N_k}$ with $n = \min_{1 \le i \le k} N_i > 2$, discussed earlier in this introduction.

Section 6 treats the mixed case, that is, the case when at least one end is non-parabolic but parabolic ends are also allowed. The main result of this section (as well as that of the whole paper) is Theorem 6.6. In order to treat the mixed case, we use a Doob's transform technique that turns the original manifold into a weighted manifold all of whose ends are non-parabolic. The difficulty here is to verify that the ends of this weighted manifold still satisfy (VD) and (PI). This follows from a result of [40] provided the manifolds M_i satisfy (VD), (PI) and an additional property labeled by (RCA) (the relative connectedness of certain annuli in M_i). Theorem 6.6 applies to all manifolds of type (1.4) with $\max_i N_i > 2$. These examples are discussed in detail at the end of Section 6.

Section 7 deals with a restricted class of parabolic manifolds where all the ends have comparable volume growth. This allows us to treat the case of $\mathcal{R}^2 \# \mathcal{R}^2$. The general treatment of parabolic manifolds with ends including $\mathcal{R}^1 \# \mathcal{R}^2$ require different additional arguments and is postponed to the forthcoming paper [43].

Finally, Section 8 gives a perhaps surprising application of the main results (Theorems 4.9, 5.10) to the study of the *one-dimensional* Schrödinger operator with a positive potential of at least quadratic decay at ∞.

2 Preliminaries

2.1 Weighted manifolds

Let N be a positive integer and $M = (M, \mathbf{g})$ be an N-dimensional Riemannian manifold with boundary δM (which may be empty). Given a smooth positive function σ on M, we define a measure μ on M by $d\mu(x) = \sigma^2(x)dx$ where dx is the Riemannian measure. The pair (M, μ) is called *a weighted manifold* and it will be the main underlying space for our considerations. Let us recall some standard definition from Riemannian geometry.

For $x, y \in M$, denote by $d(x, y)$ the Riemannian distance induced by the metric $\mathbf{g}$. Let

$$B(x, r) = \{y \in M \mid d(x, y) < r\}$$

be the geodesic ball with center $x \in M$ and radius $r > 0$ and let

$$V(x, r) := \mu(B(x, r))$$

be its μ-volume. For any set $A \subset M$, denote by A_δ the δ-neighborhood of A.

The manifold M is called *complete* if the metric space (M, d) is complete. Equivalently, M is complete if all metric balls are precompact. If $\delta M = \emptyset$ then M is complete if and only if M is geodesically complete.

The Riemannian metric induces the notion of *gradient*. For any smooth enough function f and vector field X on M, the gradient ∇f is the unique vector field such that $\mathbf{g}(\nabla f, X) = df(X)$. In a coordinate chart $x_1, x_2, ... x_N$, the gradient ∇f is given by

$$(\nabla f)^i = \sum_{j=1}^{N} g^{ij} \frac{\partial f}{\partial x_j},$$

where g_{ij} are the matrix entries of the Riemannian metric $\mathbf{g}$ and g^{ij} are the entries of the inverse matrix $\|g_{ij}\|^{-1}$.

A weighted manifold possesses a *divergence* div_μ which is a differential operator acting on smooth vector fields and which is formally adjoint to ∇ with respect to μ. Namely, for any smooth enough vector field F, the divergence $\operatorname{div}_\mu F$ is a function which, in any coordinate chart, is given by

$$\operatorname{div}_\mu F := \frac{1}{\sigma^2 \sqrt{g}} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\sigma^2 \sqrt{g} F^i \right),$$

where $g := \det \|g_{ij}\|$. The *weighted Laplace operator* L on M is defined by

$$Lu := \operatorname{div}_\mu(\nabla u) = \sigma^{-2} \operatorname{div}(\sigma^2 \nabla u),$$

for any smooth function u on M. When $\sigma \equiv 1$, $\operatorname{div}_\mu F$ is the Riemannian divergence $\operatorname{div} F$ and L coincides with the Laplace-Beltrami operator $\Delta = \operatorname{div} \circ \nabla$.

Consider the Hilbert space $L^2(M, \mu)$ and the Dirichlet form

$$D(u, v) = \int_M (\nabla u, \nabla v) d\mu$$

defined for all $u, v \in C_0^\infty(M)$, where $C_0^\infty(M)$ is the set of smooth functions on M with compact support (note that functions in $C_0^\infty(M)$ do not have to vanish on δM). The integration-by-parts formula for the operator L implies

$$D(u, v) = -\int_M u L v \, d\mu - \int_{\delta M} u \frac{\partial v}{\partial \nu} d\mu', \tag{2.1}$$

where ν is the inward unit normal vector field on δM and μ' is the measure with density σ^2 with respect to the Riemannian measure of codimension 1 on any smooth hypersurface, in particular, on δM. Clearly, the operator L is symmetric on the subspace of $C_0^\infty(M)$ of functions with vanishing normal derivative on δM. It follows that the operator L initially defined on this

subspace, admits a Friedrichs extension $\overline{L}$ which is a self-adjoint non-positive definite operator on $L^2(M, \mu)$.

The associated heat semigroup $P_t = e^{t\overline{L}}$ has a smooth integral kernel $p(t, x, y)$ which is called the *heat kernel* of (M, μ). Alternatively, the heat kernel can be defined as the minimal positive solution $u(t, x) = p(t, x, y)$ of the Cauchy problem

$$\begin{cases} (\partial_t - L)u = 0 & \text{on} \quad (0, \infty) \times M \\ u(0, x) = \delta_y(x) \\ \left. \dfrac{\partial u}{\partial \nu} \right|_{\delta M} = 0. \end{cases} \tag{2.2}$$

(see [12], [28], [54]). Note that the heat kernel is symmetric in x, y, that is,

$$p(t, x, y) = p(t, y, x).$$

The operator $\overline{L}$ generates a diffusion process $(X_t)_{t \geq 0}$ on M. Denote by $\mathbb{P}_x$ the law of $(X_t)_{t > 0}$ given $X_0 = x \in M$ and by $\mathbb{E}_x$ the corresponding expectation. The heat kernel coincides with the transition density for X_t with respect to measure μ, that is, for any Borel set $A \subset M$,

$$\mathbb{P}_x(X_t \in A) = \int_A p(t, x, y)d\mu(y).$$

Note that the Neumann boundary condition corresponds to the fact that the process X_t is reflected on the boundary δM. A weighted manifold (M, μ) is called *parabolic* if

$$\int_1^\infty p(t, x, y)\, dt \equiv \infty$$

for some/all $x, y \in M$, and *non-parabolic* otherwise. It is known that the parabolicity of M is equivalent to the recurrence of the associated diffusion X_t (see, for example, [35]).

Any open set $\Omega \subset M$ (equipped with the restriction of μ to Ω) can be consider as a weighted manifold with boundary[3] $\delta\Omega = \Omega \cap \delta M$. The weighted Laplace operator $\overline{L}_\Omega$ on Ω generates a diffusion in Ω which is killed on $\partial\Omega$ and reflected on $\delta\Omega$. Let $p_\Omega(t, x, y)$ be the heat kernel in (Ω, μ). It is convenient to extend $p_\Omega(t, x, y)$ to M by setting $p_\Omega(t, x, y) = 0$ if one of the points x, y is outside Ω.

We say that an open set $\Omega \subset M$ has *smooth boundary* if the topological boundary $\partial\Omega$ is a smooth submanifold of M of dimension $N - 1$, which is transversal to δM (the latter condition being void if δM is empty). If Ω has smooth boundary then $p_\Omega(t, x, y)$ satisfies the Dirichlet boundary condition on $\partial\Omega \setminus \delta M$ and the Neumann boundary condition on $\delta\Omega$.

2.2 Connected sum of manifolds

Let $\{M_i\}_{i=1}^k$ be a finite family of non-compact Riemannian manifolds. We say that a Riemannian manifold M is a *connected sum* of the manifolds M_i and write

$$M = M_1 \# M_2 \# \cdots \# M_k \tag{2.3}$$

if, for some non-empty compact set $K \subset M$ (called *a central part* of M), the exterior $M \setminus K$ is a disjoint union of open sets $E_1, E_2, \ldots, E_k$, such that each E_i is isometric to $M_i \setminus K_i$, for some compact $K_i \subset M_i$; in fact, we will always identify E_i and $M_i \setminus K_i$ (see Fig. 5).

---

[3]Recall that, by the definition of a manifold with boundary, any point of δM is an *interior* point of M. For the same reason, any point of $\delta\Omega$ is an interior point of Ω. Hence, the boundary $\delta\Omega$ of Ω as a manifold with boundary is disjoint from the topological boundary $\partial\Omega$ of Ω as a subset of the topological space M.
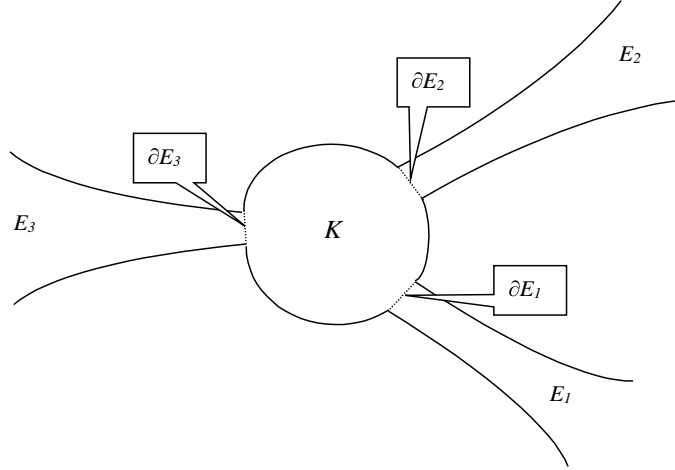
Figure 5:

If (M, μ) and (M_i, μ_i) are weighted manifolds then the isometry is understood in the sense of weighted manifolds, that is, it maps the measure μ to μ_i. Of course, taking connected sums is not a uniquely defined operation. Without loss of generality, we will always assume that K is the closure of an open set with smooth boundary.

Conversely, let M be a non-compact manifold and $K \subset M$ be a compact set with smooth boundary such that $M \setminus K$ is a disjoint union of a finitely many connected open sets $E_1, E_2, \ldots, E_k$ that are not precompact. We say that the E_i's are the *ends* of M with respect to K. Consider the closure $\overline{E}_i$ as manifold with boundary. Then by definition of a connected sum we have $M = \overline{E}_1 \# \overline{E}_2 \# \cdots \# \overline{E}_k$. Sometimes it will be convenient to choose a precompact open set $E_0 \subset M$ with smooth boundary containing K, so that M is covered by the open sets $E_0, E_1, ..., E_k$.

Example 2.1 Say that a complete non-compact Riemannian manifold M (without boundary) has *asymptotically non-negative sectional curvature* if there exists a point $o \in M$ and a continuous decreasing function $k : (0, \infty) \mapsto (0, \infty)$ satisfying

$$\int^\infty s k(s) ds < \infty$$

and such that the sectional curvature $\mathrm{Sect}(x)$ of M at $x \in M$ satisfies $\mathrm{Sect}(x) \geq -k(d(o, x))$. Such manifolds were studied in [47, 45] and include, of course, all manifolds with non-negative sectional curvature outside a compact set. The catenoid of Fig 1 is also a manifold with asymptotically non-negative curvature.

All such manifolds have a finite number of ends and thus can be written as a connected sum $M = M_1 \# \ldots \# M_k$ of complete manifolds; furthermore, each manifold M_i satisfies the properties (VD) and (PI) as well as the property (RCA)[4] (see [40, Sect. 7.5] and references therein). Hence, our main Theorem 6.6 applies to all non-parabolic manifolds with asymptotically non-negative sectional curvature.

Example 2.2 Let M be a complete non-compact Riemannian manifold (without boundary), and assume that M has non-negative Ricci curvature outside a compact set. Then M has

---

[4](RCA) stands for "relative connectedness of annuli" – see Section 6 for the definition.

finitely many ends ([6, 47]) and it can be written has has a connected sum $M = M_1 \# \ldots \# M_k$, where each M_i corresponds to an end of M. These M_i's should be thought of as manifolds with non-negative Ricci curvature outside a compact set having exactly one end (strictly speaking, even so M has no boundary, we may have to allow the M_i's to have a (compact) boundary). It is known that if an end M_i satisfies (RCA) then it satisfies also (VD) and (PI) (see [40, Propositions 7.6, 7.10]). Hence, our main Theorem 6.6 applies to all non-parabolic manifolds with non-negative Ricci curvature outside a compact set, provided each end satisfies (RCA).

3 Gluing techniques for heat kernels

We start with general inequalities which relate the heat kernel with hitting probabilities on an arbitrary weighted manifold (M, μ). These inequalities will be one of the main technical tools we introduce here to handle heat kernel estimates on manifolds with ends. However, in this section we do not make any a priori assumption about the manifold in question.

For any closed set $\Gamma \subset M$ define the first hitting time by

$$\tau_\Gamma = \inf\{t \geq 0 : X_t \in \Gamma\}.$$

Let us set

$$\psi_\Gamma(t, x) := \mathbb{P}_x(\tau_\Gamma \leq t). \tag{3.1}$$

In other words, $\psi_\Gamma(t, x)$ is the probability that the process hits Γ by time t. Observe that $\psi_\Gamma(t, x)$ is an increasing function in t, bounded by 1, and $\psi(x, t) = 1$ if $x \in \Gamma$. We will denote by ψ'_Γ the time derivative of $\psi_\Gamma(t, x)$.

Lemma 3.1 *Let $\Gamma \subset M$ be a closed set and $\Omega \subset M$ be an open set such that $\partial\Omega \subset \Gamma$. Then for all $x \in \Omega$, $y \in M$, and $t > 0$*

$$p(t, x, y) \leq p_\Omega(t, x, y) + \sup_{\substack{0 \leq s \leq t \\ z \in \Gamma}} p(s, z, y)\, \psi_\Gamma(t, x). \tag{3.2}$$

Furthermore, we have

$$p(t, x, y) \leq p_\Omega(t, x, y) + \sup_{\substack{t/2 \leq s \leq t \\ z \in \Gamma}} p(s, z, y)\, \psi_\Gamma(\frac{t}{2}, x) + \sup_{t/2 \leq s \leq t} \psi'_\Gamma(s, x) \int_0^{t/2} \sup_{z \in \Gamma} p(\theta, z, y) d\theta \tag{3.3}$$

and

$$p(t, x, y) \geq p_\Omega(t, x, y) + \inf_{\substack{t/2 \leq s \leq t \\ z \in \Gamma}} p(s, z, y)\, \psi_\Gamma(\frac{t}{2}, x) + \inf_{t/2 \leq s \leq t} \psi'_\Gamma(s, x) \int_0^{t/2} \inf_{z \in \Gamma} p(\theta, z, y) d\theta. \tag{3.4}$$

Remark 3.2 Inequality (3.2) will not be used in the main part of the paper. However, its proof is instructive since it contains the main idea of the proof of the more involved inequalities (3.3), (3.4) as well as other inequalities presented below.

Proof. By hypothesis any continuous path from x to y either intersects Γ or stays in Ω (the latter can happen only in the case $y \in \Omega$). Set $\tau = \tau_\Gamma$. The strong Markov property yields

$$p(t, x, y) = p_\Omega(t, x, y) + \mathbb{E}_x\left(\mathbf{1}_{\{0 \leq \tau \leq t\}} p(t - \tau, X_\tau, y)\right) \tag{3.5}$$

$$= p_\Omega(t, x, y) + \mathbb{E}_x\left(\mathbf{1}_{\{0 \leq \tau \leq \frac{t}{2}\}} p(t - \tau, X_\tau, y)\right) \tag{3.6}$$

$$+ \mathbb{E}_x\left(\mathbf{1}_{\{\frac{t}{2} < \tau \leq t\}} p(t - \tau, X_\tau, y)\right) \tag{3.7}$$
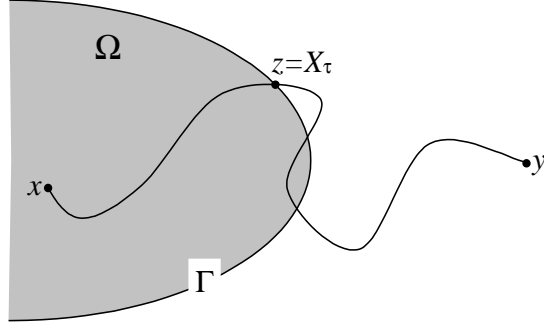
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(see Fig. 6).

Figure 6: A path between points x, y

The identity (3.5) implies

$$p(t, x, y) \leq p_\Omega(t, x, y) + \sup_{\substack{z \in \Gamma \\ 0 \leq \theta \leq t}} p(t - \theta, z, y) \mathbb{P}_x \{0 \leq \tau \leq t\}$$

which is exactly (3.2).

To prove (3.3) and (3.4) we will use (3.6)-(3.7). The second term in (3.6) can be estimated as above. This gives

$$\mathbb{E}_x \left(\mathbf{1}_{\{0 \leq \tau \leq \frac{t}{2}\}} p(t - \tau, X_\tau, y) \right) \leq \sup_{\substack{z \in \Gamma \\ 0 \leq \theta \leq \frac{t}{2}}} p(t - \theta, z, y) \psi_\Gamma(\frac{t}{2}, x). \tag{3.8}$$

To estimate the term in (3.7), let us denote by ν the joint distribution of (τ, X_τ) on $(0, \infty) \times \Gamma$. Then we have

$$
\begin{aligned}
\mathbb{E}_x \left(\mathbf{1}_{\{\frac{t}{2} < \tau \leq t\}} p(t - \tau, X_\tau, y) \right) &= \int_{t/2}^{t} \int_\Gamma p(t - s, z, y) d\nu(s, z) \\
&\leq \int_{t/2}^{t} \sup_{z \in \Gamma} p(t - s, z, y) \int_\Gamma d\nu(s, z) \\
&= \int_{t/2}^{t} \sup_{z \in \Gamma} p(t - s, z, y) \partial_s \psi_\Gamma(s, x) ds \\
&\leq \sup_{\frac{t}{2} \leq s \leq t} \psi_\Gamma'(s, x) \int_{t/2}^{t} \sup_{z \in \Gamma} p(t - s, z, y) ds
\end{aligned}
$$

whence (3.3) follows.

To prove (3.4) note that the second term in (3.6) is bounded below by

$$\inf_{\substack{z \in \Gamma \\ 0 \leq \theta \leq t/2}} p(t - \theta, z, y) \mathbb{P}_x \left(0 \leq \tau \leq t/2 \right) = \inf_{\substack{z \in \Gamma \\ t/2 \leq s \leq t}} p(s, z, y) \psi_\Gamma(t/2, x).$$

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Finally, the term in (3.7) is estimated from below by writing

$$
\mathbb{E}_x\left(\mathbf{1}_{\{\frac{t}{2}<\tau\leq t\}}p(t-\tau,X_\tau,y)\right) \;\geq\; \int_{t/2}^{t}\inf_{z\in\Gamma}p(t-s,z,y)\int_\Gamma d\nu(s,z)
$$

$$
=\;\int_{t/2}^{t}\inf_{z\in\Gamma}p(t-s,z,y)\partial_s\psi_\Gamma(s,x)ds
$$

$$
\geq\;\inf_{t/2\leq s\leq t}\psi_\Gamma'(s,x)\int_{t/2}^{t}\inf_{z\in\Gamma}p(t-s,z,y)ds.
$$

Inequality (3.4) follows. $\blacksquare$

Lemma 3.3 *Let Ω_1 and Ω_2 be two open sets in M with the topological boundaries Γ_1 and Γ_2 respectively. Assume that Γ_2 separates Ω_2 from Γ_1. Then for all $x\in\Omega_1$, $y\in\Omega_2$, and $t>0$ we have*

$$
2p(t,x,y)\geq p_{\Omega_1}(t,x,y)+\inf_{\substack{t/2\leq s\leq t\\ v\in\Gamma_1}}p(s,v,y)\,\psi_{\Gamma_1}(\tfrac{t}{2},x)+\inf_{\substack{t/2\leq s\leq t\\ w\in\Gamma_2}}p(s,w,x)\,\psi_{\Gamma_2}(\tfrac{t}{2},y) \tag{3.9}
$$

and

$$
p(t,x,y)\leq p_{\Omega_1}(t,x,y)+\sup_{\substack{t/2\leq s\leq t\\ v\in\Gamma_1}}p(s,v,y)\,\psi_{\Gamma_1}(\tfrac{t}{2},x)+\sup_{\substack{t/2\leq s\leq t\\ w\in\Gamma_2}}p(s,w,x)\,\psi_{\Gamma_2}(\tfrac{t}{2},y). \tag{3.10}
$$

Furthermore, the following refinement of (3.10) takes places:

$$
p(t,x,y)\leq p_{\Omega_1}(t,x,y)+\sup_{\substack{t/2\leq s\leq t\\ v\in\Gamma_1}}p(s,v,y)\,\psi_{\Gamma_1}(\tfrac{t}{2},x)+\sup_{\substack{t/2\leq s\leq t\\ w\in\Gamma_2}}\widehat{p}_{\Omega_1}(s,w,x)\,\psi_{\Gamma_2}(\tfrac{t}{2},y), \tag{3.11}
$$

where

$$
\widehat{p}_{\Omega_1}(s,w,x):=p(s,w,x)-p_{\Omega_1}(s,w,x). \tag{3.12}
$$

Remark 3.4 The hypothesis that Γ_2 separates Ω_2 from Γ_1 means that either Ω_1 and Ω_2 are disjoint or $\Omega_2\subset\Omega_1$ (see below Fig. 7 and 8 respectively). Note that in the former case the term $p_{\Omega_1}(t,x,y)$ vanishes.

    Proof. Applying (3.4) with $\Omega=\Omega_1$, $\Gamma=\Gamma_1$ we obtain

$$
p(t,x,y)\geq p_{\Omega_1}(t,x,y)+\inf_{\substack{t/2\leq s\leq t\\ v\in\Gamma_1}}p(s,v,y)\,\psi_{\Gamma_1}(\tfrac{t}{2},x)
$$

and similarly

$$
p(t,x,y)\geq p_{\Omega_2}(t,x,y)+\inf_{\substack{t/2\leq s\leq t\\ w\in\Gamma_2}}p(s,w,x)\,\psi_{\Gamma_2}(\tfrac{t}{2},y).
$$

Adding up these inequalities, we obtain (3.9).

For the upper bound (3.10) we need some preparation. Fix some $T > 0$ and consider $\mathbb{P}_x$ as a measure in the space $\mathbf{\Omega}_T$ of all continuous paths $\omega : [0, T] \to M$. Note that $\mathbb{P}_x$ sits in $\mathbf{\Omega}_{T,x} := \{\omega \in \mathbf{\Omega}_T : \omega(0) = x\}$. For any μ-measurable set $A \subset M$ with $\mu(A) < \infty$ define a measure $\mathbb{P}_A$ in $\mathbf{\Omega}_T$ by

$$\mathbb{P}_A(\mathcal{A}) = \int_A \mathbb{P}_x(\mathcal{A}) \, d\mu(x), \qquad (3.13)$$

where $\mathcal{A}$ is an event in $\mathbf{\Omega}_T$. For any two such sets $A, B \subset M$ define a probability measure $\mathbb{P}_{T,A,B}$ in $\mathbf{\Omega}_T$ by

$$\mathbb{P}_{T,A,B}(\mathcal{A}) := \frac{\mathbb{P}_A(\mathcal{A} \cap (X_T \in B))}{\mathbb{P}_A(X_T \in B)}.$$

For any paths $\omega \in \mathbf{\Omega}_T$ denote by ω^* the path obtained from ω by the time change $t \mapsto T - t$, that is $\omega^*(t) = \omega(T - t)$. Respectively, for any event $\mathcal{A} \subset \mathbf{\Omega}_T$ set $\mathcal{A}^* = \{\omega^* : \omega \in \mathcal{A}\}$. Then we claim that

$$\mathbb{P}_{T,A,B}(\mathcal{A}) = \mathbb{P}_{T,B,A}(\mathcal{A}^*). \qquad (3.14)$$

Indeed, observe that by the symmetry of the heat kernel

$$\mathbb{P}_A(X_T \in B) = \int_A \mathbb{P}_x(X_T \in B) \, d\mu(x) = \int_A \int_B p(T, x, y) \, d\mu(y) \, d\mu(x) = \mathbb{P}_B(X_T \in A). \quad (3.15)$$

Hence, (3.14) will follow if we show that

$$\mathbb{P}_A(\mathcal{A} \cap (X_T \in B)) = \mathbb{P}_B(\mathcal{A}^* \cap (X_T \in A)). \qquad (3.16)$$

It suffices to prove (3.16) for an elementary event $\mathcal{A}$, that is for

$$\mathcal{A} = (X_{t_1} \in E_1, X_{t_2} \in E_2, ..., X_{t_n} \in E_n)$$

where $0 < t_1 < t_2 < ... < t_n < T$ and E_k are measurable sets in M. For this $\mathcal{A}$, we have

$$\mathbb{P}_A(\mathcal{A} \cap (X_T \in B)) = \int_A \mathbb{P}_x(X_{t_1} \in E_1, X_{t_2} \in E_2, ..., X_{t_n} \in E_n, X_T \in B) \, d\mu(x),$$

where the right hand side is equal to

$$\int_B \int_{E_n} ... \int_{E_1} \int_A p(t_1, x, z_1) \, p(t_2 - t_1, z_1, z_2) ... p(T - t_n, z_n, y) \, d\mu(x) \, d\mu(z_1) ... d\mu(z_n) \, d\mu(y).$$

$$(3.17)$$

Similarly, we have

$$\mathbb{P}_B(\mathcal{A}^* \cap (X_T \in A)) = \int_B \mathbb{P}_y(X_{T-t_n} \in E_n, ..., X_{T-t_1} \in E_1, X_T \in A) \, d\mu(y),$$

where the right hand side is equal to

$$\int_A \int_{E_1} ... \int_{E_n} \int_B p(T - t_n, y, z_n) \, p(t_n - t_{n-1}, z_n, z_{n-1}) ... p(t_1, z_1, x) \, d\mu(y) \, d\mu(z_n) ... d\mu(z_1) \, d\mu(x).$$

$$(3.18)$$

Comparing (3.17) and (3.18) we obtain (3.16).

Now we are in position to prove (3.10). For any path $\omega \in \mathbf{\Omega}_T$, denote by $\tau_i(\omega)$ the first time the path ω hits Γ_i, provided ω does intersect Γ_i. Fix sets $A \subset \Omega_1$ and $B \subset \Omega_2$ and observe that the measure $\mathbb{P}_{T,A,B}$ sits on the set $\mathbf{\Omega}_{T,A,B}$ of paths ω such that $\omega(0) \in A$ and $\omega(T) \in B$. Clearly, for any $\omega \in \mathbf{\Omega}_{T,A,B}$, either ω stays in Ω_1 (which is only possible in the case $\Omega_2 \subset \Omega_1$) or both $\tau_1(\omega)$ and $\tau_2(\omega^*)$ are defined and $\tau_1(\omega) + \tau_2(\omega^*) \leq T$. Hence, in the latter case we have either $\tau_1(\omega) \leq T/2$ or $\tau_2(\omega^*) \leq T/2$ (see Fig. 7 and 8).
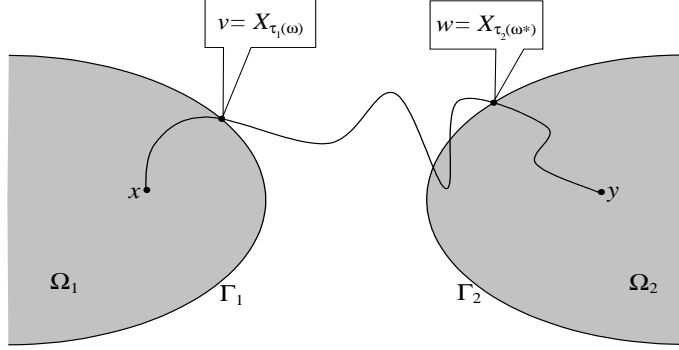
Figure 7: The case Ω_1 and Ω_2 are disjoint. Any path from x to y crosses Γ_1 and Γ_2.
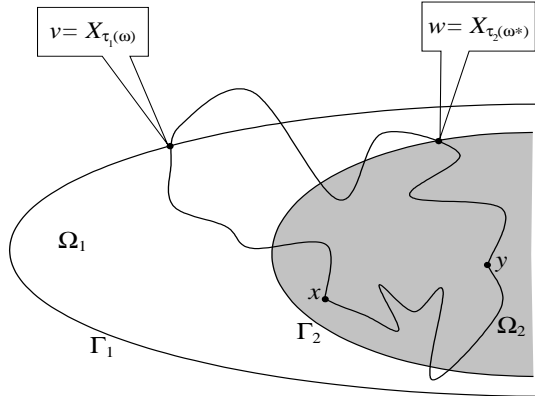
Figure 8: The case $\Omega_2 \subset \Omega_1$. Any path from x to y either stays in Ω_1 or crosses Γ_1 and Γ_2

Therefore, we obtain

$$1 \leq \mathbb{P}_{T,A,B}(\omega \subset \Omega_1) + \mathbb{P}_{T,A,B}(\tau_1(\omega) \leq T/2) + \mathbb{P}_{T,A,B}(\tau_2(\omega^*) \leq T/2). \qquad (3.19)$$

By (3.14) we have

$$\mathbb{P}_{T,A,B}(\tau_2(\omega^*) \leq T/2) = \mathbb{P}_{T,B,A}(\tau_2(\omega) \leq T/2).$$

Substituting into (3.19) and multiplying (3.19) by $\mathbb{P}_A(X_T \in B) = \mathbb{P}_B(X_T \in A)$ we obtain

$$\int_A \int_B p(T,x,y)\, d\mu(y)\, d\mu(x) \leq \mathbb{P}_A^{\Omega_1}(X_T \in B) + \mathbb{P}_A(\tau_1 \leq T/2, X_T \in B) + \mathbb{P}_B(\tau_2 \leq T/2, X_T \in A).$$

$$(3.20)$$

Clearly, we have

$$\mathbb{P}_A^{\Omega_1}(X_T \in B) = \int\limits_A \int\limits_B p_{\Omega_1}(T, x, y)\, d\mu(y)\, d\mu(x),$$

whereas by the strong Markov property and (3.8)

$$\mathbb{P}_A(\tau_1 \le T/2, X_T \in B) = \int\limits_A \int\limits_B \mathbb{E}_x\left(\mathbf{1}_{\{\tau_1 \le T/2\}} p(T - \tau_1, X_{\tau_1}, y)\right) d\mu(y)\, d\mu(x)$$

$$\le \int\limits_A \int\limits_B \psi_{\Gamma_1}(\tfrac{T}{2}, x) \sup_{\substack{T/2 \le s \le T \\ v \in \Gamma_1}} p(s, v, y)\, d\mu(y)\, d\mu(x).$$

Similarly, we obtain

$$\mathbb{P}_B(\tau_2 \le T/2, X_T \in A) \le \int\limits_B \int\limits_A \psi_{\Gamma_2}(\tfrac{T}{2}, x) \sup_{\substack{T/2 \le s \le T \\ w \in \Gamma_2}} p(s, w, x)\, d\mu(x)\, d\mu(y). \tag{3.21}$$

Substituting into (3.20), dividing by $\mu(A)\,\mu(B)$ and contracting the sets A and B to the points x and y, respectively, we finish the proof of (3.10).

Finally, let us prove (3.11). If Ω_1 and Ω_2 are disjoint then $p_{\Omega_1}(s, w, x) = 0$ because $x \in \Omega_1$ and $w \notin \Omega_1$. Therefore, by (3.12) $\widehat{p}_{\Omega_1}(s, w, x) = p(s, w, x)$ so that (3.11) is identical to (3.10). Assuming now that $\Omega_2 \subset \Omega_1$. The last term in (3.19) can be replaced by

$$\mathbb{P}_{T,A,B}(\tau_2(\omega^*) \le T/2 \text{ and } \omega^* \text{ crosses } \partial\Omega_1)$$
$$= \mathbb{P}_{T,B,A}(\tau_2(\omega) \le T/2 \text{ and } \omega \text{ crosses } \partial\Omega_1)$$
$$= \mathbb{P}_{T,B,A}(\tau_2(\omega) \le T/2) - \mathbb{P}_{T,B,A}(\tau_2(\omega) \le T/2 \text{ and } \omega \text{ does not cross } \partial\Omega_1).$$

Multiplying by $\mathbb{P}_B(X_T \in A)$ we obtain that the last term in (3.20) can be replaced by

$$\mathbb{P}_B(\tau_2 \le T/2, X_T \in A) - \mathbb{P}_B(\tau_2 \le T/2, X_t \notin \partial\Omega_1 \text{ for all } t \in [0, T], X_T \in A)$$
$$= \int\limits_B \int\limits_A \left[\mathbb{E}_y\left(\mathbf{1}_{\{\tau_2 \le T/2\}} p(T - \tau_2, X_{\tau_2}, x)\right) - \mathbb{E}_y\left(\mathbf{1}_{\{\tau_2 \le T/2\}} p_{\Omega_1}(T - \tau_2, X_{\tau_2}, x)\right)\right] d\mu(x)\, d\mu(y)$$
$$\le \int\limits_B \int\limits_A \psi_{\Gamma_2}(\tfrac{T}{2}, y) \sup_{\substack{T/2 \le s \le T \\ w \in \Gamma_2}} \widehat{p}_{\Omega_1}(s, w, x)\, d\mu(x)\, d\mu(y),$$

where $\widehat{p}_{\Omega_1}$ is defined by (3.12). Using this estimate instead of (3.21) we obtain (3.11). ∎

The next statement is the main result of this section.

Theorem 3.5 *Let Ω_1 and Ω_2 be two open sets in M with boundaries Γ_1 and Γ_2 respectively. Assume that Γ_2 separates Ω_2 from Γ_1. Write for simplicity $\psi_i(t, x) = \psi_{\Gamma_i}(t, x)$, $i = 1, 2$, and set*

$$\overline{G}(t) := \int_0^t \sup_{v \in \Gamma_1, w \in \Gamma_2} p(s, v, w)\, ds \quad and \quad \underline{G}(t) := \int_0^t \inf_{v \in \Gamma_1, w \in \Gamma_2} p(s, v, w)\, ds. \tag{3.22}$$

Then, for all $x \in \Omega_1$, $y \in \Omega_2$, and $t > 0$,

$$p(t, x, y) \le p_{\Omega_1}(t, x, y) + 2\left[\sup_{s \in [t/4, t]} \sup_{v \in \Gamma_1, w \in \Gamma_2} p(s, v, w)\right] \psi_1(t, x) \psi_2(t, y)$$

$$+ \overline{G}(t)\left[\sup_{s \in [t/4, t]} \psi_1'(s, x)\right] \psi_2(t, y) + \overline{G}(t)\left[\sup_{s \in [t/4, t]} \psi_2'(s, y)\right] \psi_1(t, x) \tag{3.23}$$

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and

$$2p(t,x,y) \geq p_{\Omega_1}(t,x,y) + 2 \left[\inf_{\substack{s \in [t/4,t]}} \inf_{v \in \Gamma_1, w \in \Gamma_2} p(s,v,w) \right] \psi_1(\tfrac{t}{4},x)\psi_2(\tfrac{t}{4},y) \tag{3.24}$$

$$+\underline{G}(\tfrac{t}{4}) \left[\inf_{s \in [t/4,t]} \psi_1'(s,x) \right] \psi_2(\tfrac{t}{4},y) + \underline{G}(\tfrac{t}{4}) \left[\inf_{s \in [t/4,t]} \psi_2'(s,y) \right] \psi_1(\tfrac{t}{4},x). \tag{3.25}$$

Proof. By (3.11) and the monotonicity of $\psi_i(t,x)$ in t we have

$$p(t,x,y) \leq p_{\Omega_1}(t,x,y) + \sup_{\substack{t/2 \leq s \leq t \\ v \in \Gamma_1}} p(s,v,y)\,\psi_1(t,x) + \sup_{\substack{t/2 \leq s \leq t \\ w \in \Gamma_2}} \widehat{p}_{\Omega_1}(s,w,x)\,\psi_2(t,y). \tag{3.26}$$

Applying (3.3) with $\Omega = \Omega_1$ and $\Gamma = \Gamma_1$ we obtain, for all $w \in \Gamma_2$ and $s > 0$,

$$\widehat{p}_{\Omega_1}(s,w,x) = p(s,x,w) - p_{\Omega_1}(s,x,w)$$

$$\leq \sup_{\substack{s/2 \leq \theta \leq s \\ z \in \Gamma_1}} p(\theta,z,w)\,\psi_1(s,x) + \sup_{s/2 \leq \theta \leq s} \psi_1'(\theta,x) \int_0^s \sup_{z \in \Gamma_1} p(\theta,z,w)d\theta\,.$$

Set

$$\overline{q}(\theta) := \sup_{z_1 \in \Gamma_1, z_2 \in \Gamma_2} p(\theta,z_1,z_2)\,.$$

As $\psi_i(t,x)$ is increasing in t, the above inequality gives, for $s \in [t/2,t]$,

$$\widehat{p}_{\Omega_1}(s,w,x) \leq \sup_{\theta \in [t/4,t]} \overline{q}(\theta)\,\psi_1(t,x) + \sup_{\theta \in [t/4,t]} \psi_1'(\theta,x) \int_0^t \overline{q}(\theta)\,d\theta.$$

Similarly, as $p_{\Omega_2}(s,y,v) = 0$, (3.3) with $\Omega = \Omega_2$ and $\Gamma = \Gamma_2$ implies that, for any $v \in \Gamma_1$ and $s \in [t/2,t]$,

$$p(s,v,y) \leq \sup_{\theta \in [t/4,t]} \overline{q}(\theta)\,\psi_2(t,y) + \sup_{\theta \in [t/4,t]} \psi_2'(\theta,y) \int_0^t \overline{q}(\theta)\,d\theta.$$

Using these two estimates in (3.26) yields (3.23).

The lower bound (3.24)-(3.25) is proved in a similar way. Indeed, by (3.9) we have

$$2p(t,x,y) \geq p_{\Omega_1}(t,x,y) + \inf_{\substack{t/2 \leq s \leq t \\ v \in \Gamma_1}} p(s,v,y)\,\psi_1(\tfrac{t}{4},x) + \inf_{\substack{t/2 \leq s \leq t \\ w \in \Gamma_2}} p(s,w,x)\,\psi_2(\tfrac{t}{4},y). \tag{3.27}$$

Setting

$$\underline{q}(t) := \inf_{z_1 \in \Gamma_1, z_2 \in \Gamma_2} p(\theta,z_1,z_2)$$

and using (3.4) we obtain for any $w \in \Gamma_2$ and $s \in [t/2,t]$

$$p(s,x,w) \geq \inf_{t/4 \leq \theta \leq t} \underline{q}(\theta)\,\psi_1(\tfrac{t}{4},x) + \inf_{t/4 \leq \theta \leq t} \psi_1'(\theta,x) \int_0^{t/4} \underline{q}(\theta)\,d\theta,$$

and a similar inequality for $p(s,y,v)$. Substituting into (3.27) finishes the proof. ∎

Remark 3.6 Since $\psi_i(t,x)$ is the $\mathbb{P}_x$-probability of X_t hitting Γ_i by time t, the function $\psi_i(t,x)$ is fully determined by the intrinsic geometry of the set Ω_i, and so is p_{Ω_i}. Thus, the estimates of $p(t,x,y)$ given by Theorem 3.5 are determined by the intrinsic geometries of Ω_i *and* by estimates of $p(t,v,w)$ where $v \in \Gamma_1$ and $w \in \Gamma_2$. To obtain the latter, we will use different techniques for upper and for lower bounds – see Sections 4.3 and 5.4.

4 Upper bound

4.1 Faber-Krahn inequalities and the heat kernel

Let (M, μ) be a non-compact complete weighted manifold, possibly with boundary. For any region $\Omega \subset M$, set

$$\lambda_1(\Omega) := \inf_{\phi \in C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla \phi|^2 \, d\mu}{\int_\Omega \phi^2 d\mu}.$$

In words, $\lambda_1(\Omega)$ is the smallest eigenvalue of L in Ω satisfying the Dirichlet condition on $\partial \Omega$ and the Neumann condition on $\delta \Omega$.

The classical Faber-Krahn theorem says that, for any open set $\Omega \subset \mathbb{R}^N$ and $L = \Delta$,

$$\lambda_1(\Omega) \geq c_N \mu(\Omega)^{-2/N}, \tag{4.1}$$

where μ is the Lebesgue measure in $\mathbb{R}^N$ (the constant c_N is such that equality is attained for balls; however, the exact value of c_N is of no importance for our purpose). For an arbitrary manifold, (4.1) may not be true. However, as balls in M are precompact, a compactness argument implies that for any ball $B(x, r)$ there exists $b(x, r) > 0$ such that for any open set $\Omega \subset B(x, r)$

$$\lambda_1(\Omega) \geq b(x, r) \mu(\Omega)^{-2/N}.$$

If we know the function $b(x, r)$ then we can control the heat kernel on M as follows.

Theorem 4.1 ([32, Theorem 5.2]) *Assume that (M, μ) is a complete weighted manifold such that, for any ball $B(x, r)$ and any open set $\Omega \subset B(x, r)$,*

$$\lambda_1(\Omega) \geq b(x, r) \mu(\Omega)^{-\alpha}, \tag{4.2}$$

where $b(x, r) > 0$ and $\alpha > 0$. Then, for all $x, y \in M$ and $t > 0$,

$$p(t, x, y) \leq \frac{C \exp\left(-c \frac{d^2(x,y)}{t}\right)}{\left(t^2 b(x, \sqrt{t}) b(y, \sqrt{t})\right)^{1/(2\alpha)}}.$$

One particular case of (4.2) will be frequently used so that we separate it out as the following condition:

(RFK) The *relative Faber-Krahn inequality*: there exist $\alpha > 0$ and $c > 0$ such that, for any ball $B(x, r) \subset M$ and for any precompact open set $\Omega \subset B(x, r)$,

$$\lambda_1(\Omega) \geq \frac{c}{r^2} \left(\frac{V(x, r)}{\mu(\Omega)}\right)^\alpha. \tag{4.3}$$

In other words, the condition (RFK) means that (4.2) holds with

$$b(x, r) = \frac{c}{r^2} V(x, r)^\alpha.$$

For example, (RFK) holds with $\alpha = 2/N$ if M is a complete Riemannian manifold with non-negative Ricci curvature (see [31, Theorem 1.4]).

Note that if (4.3) holds for some $\alpha = \alpha_0$ then it is satisfied also for any smaller value $\alpha < \alpha_0$ because $\mu(\Omega) \leq V(x, r)$.

Consider also the following properties which in general may be true or not.

(VD) The *volume doubling* property: for all $x \in M$ and $r > 0$,

$$V(x, 2r) \leq CV(x, r). \tag{4.4}$$

For a later reference, we also note that (VD) implies that for any $\varepsilon > 0$ and for all $x, y \in M$, $t > 0$,

$$\frac{V(y, \sqrt{t})}{V(x, \sqrt{t})} \leq \left(1 + \frac{d(x, y)}{\sqrt{t}}\right)^C \leq C_\varepsilon \exp\left(\varepsilon \frac{d^2(x, y)}{t}\right). \tag{4.5}$$

(UED) The *on-diagonal upper estimate* of the heat kernel: for all $x \in M$ and all $t > 0$,

$$p(t, x, x) \leq \frac{C}{V(x, \sqrt{t})}. \tag{4.6}$$

(UE) The *off-diagonal upper estimate* of the heat kernel: for all $x, y \in M$ and all $t > 0$,

$$p(t, x, y) \leq \frac{C}{V(x, \sqrt{t})} \exp\left(-c\frac{d^2(x, y)}{t}\right). \tag{4.7}$$

Theorem 4.2 ([32, Proposition 5.2]) *For any complete weighted manifold (M, μ), the following equivalences take place*

$$(RFK) \iff (VD) + (UED) \iff (VD) + (UE).$$

Proposition 4.3 ([35, Theorem 11.1]) *Let (M, μ) be a complete weighted manifold satisfying (RFK). Then (M, μ) is non-parabolic if and only if*

$$\int^\infty \frac{ds}{V(x, \sqrt{s})} < \infty. \tag{4.8}$$

4.2   Hitting probability

Given a complete weighted manifold (M, μ), fix a compact set K with non-empty interior and a reference interior point $o \in K$. Set

$$|x| := \sup_{y \in K} d(x, y), \; x \in M,$$

and

$$H_*(x, t) := \min\left\{1, \; \frac{|x|^2}{V(o, |x|)} + \left(\int_{|x|^2}^t \frac{ds}{V(o, \sqrt{s})}\right)_+\right\}, \tag{4.9}$$

where $(\cdot)_+$ is the positive part, that is, $\max(\cdot, 0)$. Note that $H_*(x, t)$ in increasing in t. The following result is a combination of Proposition 4.3 and Corollary 4.2 from [39].

Theorem 4.4 *Let (M, μ) be a complete non-compact manifold satisfying (RFK), $K \subset M$ be a compact set, $o \in K$ be an interior point of K, and $\delta > 0$. Then, for all $x \in M \setminus K_\delta$ and $t > 0$,*

$$\psi_K(t, x) \leq CH_*(x, t) \exp\left(-c\frac{|x|^2}{t}\right) \tag{4.10}$$

and

$$\partial_t \psi_K(t, x) \leq \frac{C}{V(o, \sqrt{t})} \exp\left(-c\frac{|x|^2}{t}\right). \tag{4.11}$$

Note that the function H used in [39, Corollary 4.2] is slightly different from the function H_* defined above, and this is why the present estimates require also Proposition 4.3 from [39].

4.3 Initial upper bound

In this and the next section, we assume that $M = M_1 \# ... \# M_k$ and will use the notation from Section 2.2. In particular, let us recall that M is the disjoint union of the central part K and the ends $E_1, ... E_k$ with respect to K. Each E_i is identified with the complement of a compact set in M_i.

Geodesic balls are denoted by $B(x, r)$ in M and by $B_i(x, r)$ in M_i. We also set $V(x, r) = \mu(B(x, r))$ and $V_i(x, r) = \mu_i(B_i(x, r))$. Observe that if $B_i(x, r) \subset E_i$ then $B_i(x, r) = B(x, r)$ and $V_i(x, r) = V(x, r)$. For each index $i \geq 1$, fix a reference point $o_i \in \partial E_i$, and set

$$V_i(r) = V_i(o_i, r), \qquad V_0(r) = \min_{1 \leq i \leq k} V_i(r). \tag{4.12}$$

It will also be useful to set

$$V_0(x, r) \equiv V_0(r)$$

for all $x \in M$. If all functions $V_i(r)$ satisfy the doubling property then so does $V_0(r)$.

For any $x \in M$, $r > 0$, set

$$F(x, r) := \begin{cases} V(x, r), & \text{if } B(x, r) \subset E_i, \ i \geq 1, \\ V_0(r), & \text{otherwise.} \end{cases} \tag{4.13}$$

Note that if r stays bounded and x varies in a compact neighbourhood of K then $V_i(x, r) \approx r^N$. For this range of x and r we have also

$$F(x, r) \approx V_0(r) \approx r^N \approx V(x, r). \tag{4.14}$$

Theorem 4.5 ([41, Proposition 3.6]) *Assume that for each $i = 1, ..., k$, the manifold (M_i, μ_i) satisfies (RFK). Then there exists $\alpha > 0$ and $c > 0$ such that for any ball $B = B(x, r) \subset M$ and for any open set $\Omega \subset B$*

$$\lambda_1(\Omega) \geq \frac{c}{r^2} \left(\frac{F(x, r)}{\mu(\Omega)} \right)^\alpha.$$

Combining with Theorem 4.1 we obtain the following result.

Corollary 4.6 *Assume that for each $i = 1, ..., k$, each manifold (M_i, μ_i) satisfies (RFK). Then the heat kernel on (M, μ) satisfies*

$$p(t, x, y) \leq \frac{C}{\sqrt{F(x, \sqrt{t}) F(y, \sqrt{t})}} \exp\left(-c \frac{d^2(x, y)}{t} \right), \tag{4.15}$$

for all $x, y \in M$ and $t > 0$, where F is defined at (4.13).

Corollary 4.7 *Let E_0 be a precompact open set with smooth boundary containing K. Referring to the setting of Corollary 4.6, we have:*

(i) For any positive finite t_0, for all $x, y \in M$ and $0 < t < t_0$,

$$p(t, x, y) \leq \frac{C}{\sqrt{V(x, \sqrt{t}) V(y, \sqrt{t})}} \exp\left(-c \frac{d^2(x, y)}{t} \right). \tag{4.16}$$

(ii) For all $x, y \in E_0$ and $t > 0$,

$$p(t, x, y) \leq \frac{C}{V_0(\sqrt{t})} \exp\left(-c \frac{d^2(x, y)}{t} \right). \tag{4.17}$$

Proof. (i) It suffices to show that, for all $x \in M$ and $0 < r < r_0 := t_0^2$,

$$F(x,r) \approx V(x,r).$$

If $B(x,r)$ is in some end E_i then $F(x,r) = V(x,r)$ by definition. Otherwise, the condition $r < r_0$ implies that x belongs to K_{r_0} and the claim follows from (4.14).

(ii) It suffices to show that for all $x \in E_0$ and $r > 0$,

$$F(x,r) \geq cV_0(r).$$

If $B(x,r)$ is in E_i then r has a bounded range and hence the claim follows from (4.14). Otherwise, we have $F(x,r) = V_0(r)$ by definition. ∎

Remark 4.8 The inequality (4.16) is equivalent to say that, for all $x, y \in M$ and $0 < t < t_0$,

$$p(t,x,y) \leq \frac{C}{V(x,\sqrt{t})} \exp\left(-c\frac{d^2(x,y)}{t}\right). \tag{4.18}$$

Indeed, (4.18) implies (4.16) by switching x, y in (4.18) and using the symmetry of the heat kernel. Conversely, (4.16) implies (4.18). To see this, observe that the function V satisfies $V(x,2r) \leq CV(x,r)$ for all $x \in M$ and all $r \in (0, r_0)$. It follows (see, e.g., [58, Lemma 5.2.7]) that

$$\frac{V(x,\sqrt{t})}{V(y,\sqrt{t})} \leq \exp\left(C\frac{d(x,y)}{\sqrt{t}}\right)$$

which easily shows that (4.16) implies (4.18).

4.4   Full upper bounds

For any $x \in M$ set

$$i_x = \begin{cases} i, & \text{if } x \in E_i, i \geq 1, \\ 0, & \text{if } x \in K. \end{cases}$$

Set also

$$|x| = \sup_{y \in K} d(x,y)$$

and notice that $|x|$ is bounded away from 0. Define the function $H(x,t)$ by

$$H(x,t) = \min\left\{1, \ \frac{|x|^2}{V_{i_x}(|x|)} + \left(\int_{|x|^2}^t \frac{ds}{V_{i_x}(\sqrt{s})}\right)_+\right\}. \tag{4.19}$$

Clearly, $H(x,t)$ is bounded away from 0 when $|x|$ is bounded from above. Let us spell out explicitly the simple relationship between H and the functions H_*^i obtained on each M_i by considering a compact set K_i such that $E_i = M_i \setminus K_i$ and applying Definition (4.9). Setting for convenience $H_*^0(x,t) = 1$, we have

$$H(x,t) \approx H_*^{i_x}(x,t). \tag{4.20}$$

Indeed, for bounded x, we have $H(x,t) \approx 1 \approx H_*^{i_x}(x,t)$ whereas, if $x \in E_i$ with $i \in \{1, \ldots, k\}$, then the volume functions used in (4.9) and (4.19) are comparable and thus $H(x,t) \approx H_*^i(x,t)$.

In the case the function $V_{i_x}(r)$ satisfies in addition the condition

$$\frac{V_{i_x}(R)}{V_{i_x}(r)} \geq c \left(\frac{R}{r}\right)^{2+\varepsilon} \quad \text{for all } R > r \geq 1,$$

with some $c > 0$ and $\varepsilon > 0$, one easily obtains from (4.19) that

$$H(x,t) \approx \frac{|x|^2}{V_{i_x}(|x|)} \tag{4.21}$$

(cf. the proof of Corollary 4.5 in [39]).

For $x, y \in M$, let us set

$$d_+(x,y) = \inf\{\, \text{length}(\gamma) : \gamma(0) = x, \ \gamma(1) = y, \ \gamma \cap K \neq \emptyset\,\}, \tag{4.22}$$

where the infimum is taken over all curves $\gamma : [0,1] \to M$ connecting x, y and *passing through* K. Let us define also

$$d_\emptyset(x,y) = \inf\{\, \text{length}(\gamma) : \gamma(0) = x, \ \gamma(1) = y, \ \gamma \cap K = \emptyset\,\}, \tag{4.23}$$

where the infimum is taken over all curves $\gamma[0,1] \to M$ connecting x, y, *without intersecting K*.

Note that always $d_+(x,y) \geq d(x,y)$ and $d_\emptyset(x,y) \geq d(x,y)$, and, moreover, one of these inequalities must in fact be an equality. For example, if $x \in E_i \cup K$, $y \in E_j \cup K$ and $i \neq j$, then $d_\emptyset(x,y) = \infty$ whence $d_+(x,y) = d(x,y)$. If $x, y \in E_i$ then the elementary argument with the triangle inequality shows that

$$|x| + |y| - 2\text{diam}K \leq d_+(x,y) \leq |x| + |y| \tag{4.24}$$

and

$$d(x,y) \leq d_\emptyset(x,y) \leq d(x,y) + C_K \tag{4.25}$$

where C_K is a constant depending on K.

The next theorem is one of the main results of this paper.

Theorem 4.9 *Assume that (M, μ) is a connected sum of complete non-compact weighted manifolds (M_i, μ_i), $i = 1, 2, ..., k$, each of which satisfies (RFK). Assume further that (M, μ) is non-parabolic. Then, for all $x, y \in M$ and $t > 0$, the heat kernel on M is bounded by*

$$p(t,x,y) \ \leq \ C \left(\frac{H(x,t)H(y,t)}{V_0(\sqrt{t})} + \frac{H(y,t)}{V_{i_x}(\sqrt{t})} + \frac{H(x,t)}{V_{i_y}(\sqrt{t})}\right) \exp\left(-c\frac{d_+^2(x,y)}{t}\right) \tag{4.26}$$

$$+ \frac{C}{\sqrt{V_{i_x}(x,\sqrt{t})V_{i_y}(y,\sqrt{t})}} \exp\left(-c\frac{d_\emptyset^2(x,y)}{t}\right). \tag{4.27}$$

Each term in (4.26)-(4.27) has a geometric meaning and corresponds to a certain way a Brownian particle may move from x to y. To start with, the term (4.27) estimates the probability of getting from x to y without touching K. This may happen only if x, y belong to the same end E_i, and the term (4.27) comes from estimating p_{E_i}. The third (and, similarly, the second) term in (4.26) estimates the probability that starting from x, the particle hits K before time t and then reaches y in time of order t. The first term in (4.26) estimates the probability that the particle hits K before time t, loops from K to K in time of order t and finally reaches y in time smaller than t. It is natural to use the distance d_+ in (4.26) since the corresponding events involve trajectories from x to y passing through K. Using the distance $d_\emptyset$ in (4.27) reflects the fact that the trajectories from x to y, corresponding to that term, avoid K.

Remark 4.10 If $x \in E_i \cup K$ and $y \in E_j \cup K$ with $i \neq j$ then $d_\emptyset(x, y) = \infty$ so that the term (4.27) vanishes, whereas $d_+(x, y)$ in (4.26) can be replaced by $d(x, y)$.

If x, y belong to the same end E_i and $t \geq t_0 > 0$ then, by (4.24), $d_+(x, y)$ in (4.26) can be replaced by $|x| + |y|$ and, by (4.25), $d_\emptyset(x, y)$ in (4.27) can be replaced by $d(x, y)$.

Remark 4.11 If $k = 2$ and $x \in E_1 \cup K$, $y \in E_2 \cup K$ then the term $\frac{H(x,t)H(y,t)}{V_0(\sqrt{t})}$ in (4.26) is dominated by the two other terms and, hence, can be neglected.

Remark 4.12 An equivalent heat kernel estimate is obtained by replacing the volume functions $V_{i_x}(\sqrt{t})$ and $V_{i_y}(\sqrt{t})$ in (4.26) by $V_{i_x}(x, \sqrt{t})$ and $V_{i_y}(y, \sqrt{t})$, respectively. Indeed, if $x \in K$ then $i_x = 0$ and

$$V_0\left(x, \sqrt{t}\right) = V_0\left(\sqrt{t}\right).$$

If $i_x = i \geq 1$ and $|x|$ is large enough then, by (4.5), for any $\varepsilon > 0$,

$$\frac{V_i\left(\sqrt{t}\right)}{V_i\left(x, \sqrt{t}\right)} = \frac{V_i\left(o_i, \sqrt{t}\right)}{V_i\left(x, \sqrt{t}\right)} \leq C_\varepsilon \exp\left(\varepsilon \frac{d_i^2(x, o_i)}{t}\right) \leq C_\varepsilon \exp\left(\varepsilon C \frac{d_+^2(x, y)}{t}\right). \tag{4.28}$$

If $|x|$ is bounded then (4.28) holds again because

$$V_i\left(\sqrt{t}\right) = V_i\left(o_i, \sqrt{t}\right) \approx V_i\left(x, \sqrt{t}\right).$$

Indeed, for small t all these functions are of the order $t^{N/2}$ and, for large t, (VD) applies. In the same way, we obtain

$$\frac{V_i\left(x, \sqrt{t}\right)}{V_i\left(\sqrt{t}\right)} \leq C_\varepsilon \exp\left(\varepsilon \frac{d_i^2(x, o_i)}{t}\right) \leq C_\varepsilon \exp\left(\varepsilon C \frac{d_+^2(x, y)}{t}\right).$$

Choosing ε small enough proves the claim.

Remark 4.13 Note that the term in (4.27) can be replaced by

$$\frac{C}{V_{i_x}(x, \sqrt{t})} \exp\left(-c \frac{d_\emptyset^2(x, y)}{t}\right) \quad \text{or by} \quad \frac{C}{V_{i_y}(y, \sqrt{t})} \exp\left(-c \frac{d_\emptyset^2(x, y)}{t}\right),$$

which can be seen by an argument similar to that of the previous remark.

Remark 4.14 Observe that the non-parabolicity of (M, μ) is equivalent to the fact that *one* of the manifolds (M_i, μ_i) is non-parabolic (see [35, Proposition 14.1]). However, the estimate (4.26)-(4.27) is sharp only if *all* (M_i, μ_i) are non-parabolic (see Sections 5.4 and 6).

Proof of Theorem 4.9. Set $\delta = \operatorname{diam} K$ and let K' and K'' be compact sets with smooth boundaries such that $K \subset K' \subset K''$ and

$$d\left(\partial K, \partial K'\right) \geq 2\delta \quad \text{and} \quad d\left(\partial K', \partial K''\right) \geq 2\delta.$$

Since the estimate (4.26)-(4.27) is symmetric in x, y, there are three essentially different cases:

1. $x, y \in K''$.

2. $x \in E_i \setminus K'$ and $y \in E_j \setminus K''$ where $i, j > 0$ may be the same or not.

3. $x \in K'$ and $y \in E_j \setminus K''$ for some $j > 0$.

Case 1. Let $x, y \in K''$. By Corollary 4.7, we have

$$p(t, x, y) \leq \frac{C}{V_0(\sqrt{t})} \exp\left(-c\frac{d^2(x, y)}{t}\right). \tag{4.29}$$

If $d_+(x, y) = d(x, y)$ then (4.29) implies

$$p(t, x, y) \leq \frac{C}{V_0(\sqrt{t})} \exp\left(-c\frac{d_+^2(x, y)}{t}\right) \leq C\frac{H(x, t)H(y, t)}{V_0(\sqrt{t})} \exp\left(-c\frac{d_+^2(x, y)}{t}\right), \tag{4.30}$$

(because $H(x, t)$ and $H(y, t)$ are separated from 0 for $x, y \in K''$), which in turn yields (4.26)-(4.27). Moreover, if $d_+(x, y) \leq \sqrt{t}$ then the same argument works because the Gaussian factor in (4.30) is comparable to 1.

Assume now that $d_+(x, y) > d(x, y)$ and $d_+(x, y) > \sqrt{t}$. Then x and y belong to the same end E_i, $i \geq 1$, and $d_\emptyset(x, y) = d(x, y)$. Also, t is bounded by $4\mathrm{diam}^2(K'')$ whence

$$V_0(\sqrt{t}) \approx t^{N/2} \approx \sqrt{V_i(x, \sqrt{t})V_i(y, \sqrt{t})}.$$

Therefore, the right hand side of (4.29) is majorized by the term (4.27).

Before we consider the other two cases, let us set

$$J := \int_0^\infty \sup_{\substack{v, w \in K'' \\ d(v, w) \geq \delta}} p(s, v, w)\, ds. \tag{4.31}$$

It follows from (4.29) and from the condition $d(v, w) \geq \delta$ that the integral (4.31) converges at 0. The fact that M is non-parabolic, ensures the convergence of the integral (4.31) at ∞. Hence, $J < \infty$. The number J will enter the heat kernel upper bounds as a constant.

Case 2. ("the main case") Let $x \in E_i \setminus K'$ and $y \in E_j \setminus K''$ (see Fig. 9 when $i \neq j$).
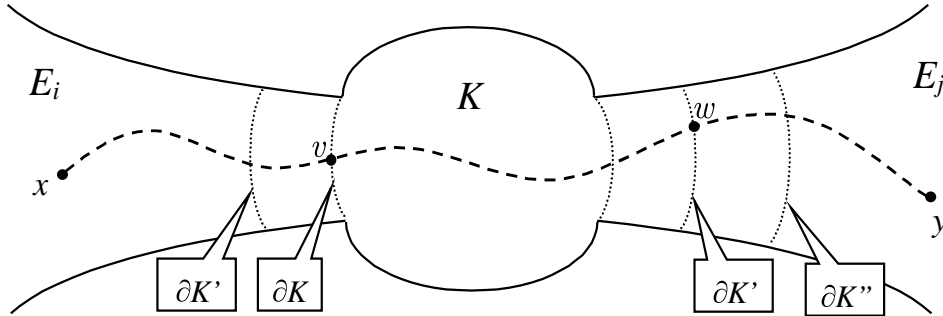

Figure 9: Case $x \in E_i \setminus K'$ and $y \in E_j \setminus K''$, $i \neq j$

By Theorem 3.5 with $\Omega_1 = E_i$ and $\Omega_2 = E_j \setminus K'$ we obtain

$$p(t,x,y) \;\leq\; p_{E_i}(t,x,y) + \left(\sup_{s\in[t/4,t]} \sup_{\substack{v\in\partial K \\ w\in\partial K'}} p(s,v,w) \right) \psi_K(t,x)\psi_{K'}(t,y) \tag{4.32}$$

$$+ \int_0^t \sup_{\substack{v\in\partial K \\ w\in\partial K'}} p(s,v,w)ds \left(\sup_{s\in[t/4,t]} \psi'_K(s,x) \right) \psi_{K'}(t,y) \tag{4.33}$$

$$+ \int_0^t \sup_{\substack{v\in\partial K \\ w\in\partial K'}} p(s,v,w)ds \left(\sup_{s\in[t/4,t]} \psi'_{K'}(s,y) \right) \psi_K(t,x). \tag{4.34}$$

If $i \neq j$ then $p_{E_i}(t,x,y) = 0$ whereas for $i = j$ Theorem 4.2 yields

$$p_{E_i}(t,x,y) \leq p_{M_i}(t,x,y) \leq \frac{C}{\sqrt{V_i(x,\sqrt{t})\,V_i(y,\sqrt{t})}} \exp\left(-c\frac{d_\emptyset^2(x,y)}{t} \right),$$

where we have used the fact that $d_{M_i}(x,y) \approx d_\emptyset(x,y)$ for all $x,y \in E_i \setminus K'$.

As $x \in E_i \setminus K'$, the hitting probability $\psi_K(t,x)$ depends only on the intrinsic properties of the manifold (M_i, μ_i). Since (M_i, μ_i) satisfies (RFK), Theorem 4.4 and (4.20) yield

$$\psi_K(t,x) \leq C\,H(x,t) \exp\left(-c\frac{|x|^2}{t} \right) \tag{4.35}$$

and

$$\psi'_K(t,x) \leq \frac{C}{V_i(x,\sqrt{t})} \exp\left(-c\frac{|x|^2}{t} \right). \tag{4.36}$$

Since $x \in E_i \setminus K'$, we have $d_i(x_i, o_i) \approx |x|$ and, by Remark 4.12,

$$\psi'_K(t,x) \leq \frac{C}{V_i(\sqrt{t})} \exp\left(-c\frac{|x|^2}{t} \right). \tag{4.37}$$

Similar estimates take place for $\psi_{K'}(t,y)$ and its time derivative for $y \in E_j \setminus K''$. By (4.29), we have for all $v,w \in K'$

$$\sup_{s\in[t/4,t]} p(s,v,w) \leq \frac{C}{V_0(\sqrt{t})}. \tag{4.38}$$

Finally, each integral in (4.33) and (4.34) is bounded from above by the constant J defined by (4.31) because $d(v,w) \geq \delta$. Substituting the above estimates into (4.32)-(4.34) and observing that

$$|x|^2 + |y|^2 \approx d_+^2(x,y),$$

we obtain (4.26)-(4.27).

Case 3. Let $x \in K'$ and $y \in E_j \setminus K''$ for some $j > 0$. Ω be an open subset of E_j containing $\overline{E_j \setminus K''}$ but such that $d(\partial\Omega, K') \geq \delta$ (see Fig. 10). By inequality (3.3) of Lemma 3.1 for this Ω and for $\Gamma = \partial\Omega$ we have

$$p(t,x,y) \leq \sup_{\substack{t/2\leq s\leq t \\ z\in\Gamma}} p(s,z,x)\,\psi_\Gamma(t,y) + \sup_{t/2\leq s\leq t} \psi'_\Gamma(s,y) \int_0^t \sup_{z\in\Gamma} p(\theta,z,x)d\theta. \tag{4.39}$$
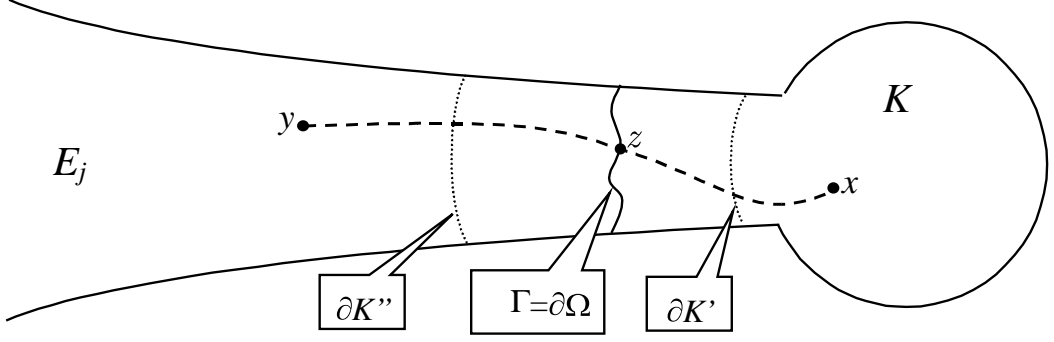
Figure 10: Case $x \in K'$ and $y \in E_j \setminus K''$

By (4.29) we obtain, for all $x \in K'$, $z \in \Gamma$, and $s \in [t/2, t]$,

$$p(s, z, x) \leq \frac{C}{V_0(\sqrt{t})}.$$

The integral in (4.39) is bounded from above by the constant J because $d(x, z) \geq \delta$. The functions $\psi_\Gamma(t, y)$ and $\psi'_\Gamma(s, y)$ are estimated as in (4.35) and (4.37), respectively. From (4.39) we obtain

$$p(t, x, y) \leq C \left(\frac{H(y, t)}{V_0(\sqrt{t})} + \frac{1}{V_j(\sqrt{t})} \right) \exp \left(-c \frac{|y|^2}{t} \right), \tag{4.40}$$

which implies (4.26)-(4.27) because $H(x, t) \approx 1$ and $|y| \approx d_+(x, y)$. ∎

Remark 4.15 Alternatively, Case 3 can be obtained directly from Case 2 by extending the range of x using a local Harnack inequality argument (see Section 5.1 below).

Theorem 4.9 provides a heat kernel upper bound for all $x, y \in M$ and $t > 0$. Still, it may be useful and convenient to write more explicit estimates for certain ranges of x, y, t.

Corollary 4.16 *Let E_0 be a precompact open set with smooth boundary containing K. Referring to the setting of Theorem 4.9, we have the following estimates:*

0. For any fixed $t_0 > 0$, if $t \leq t_0$ and $x, y \in M$ then

$$p(t, x, y) \leq \frac{C}{V(x, \sqrt{t})} \exp \left(-c \frac{d^2(x, y)}{t} \right). \tag{4.41}$$

1. If $x, y \in E_0$ then, for all $t > 0$,

$$p(t, x, y) \leq \frac{C}{V_0(\sqrt{t})} \exp \left(-c \frac{d^2(x, y)}{t} \right). \tag{4.42}$$

2. If $x \in E_i$, $i \geq 1$, and $y \in E_0$ then, for all and $t > 0$,

$$p(t, x, y) \leq C \left(\frac{H(x, t)}{V_0(\sqrt{t})} + \frac{1}{V_i(\sqrt{t})} \right) \exp \left(-c \frac{d^2(x, y)}{t} \right). \tag{4.43}$$

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3. If $x \in E_i$, $y \in E_j$, $i \neq j$, $i, j \geq 1$, then, for all $t > 0$,

$$p(t, x, y) \leq C \left(\frac{H(x,t)H(y,t)}{V_0(\sqrt{t})} + \frac{H(y,t)}{V_i(\sqrt{t})} + \frac{H(x,t)}{V_j(\sqrt{t})} \right) \exp \left(-c \frac{d^2(x,y)}{t} \right). \tag{4.44}$$

4. If $x, y \in E_i$, $i \geq 1$, then, for all $t > 0$,

$$\begin{aligned} p(t, x, y) \quad \leq \quad & C \frac{H(x,t)H(y,t)}{V_0(\sqrt{t})} \exp \left(-c \frac{|x|^2 + |y|^2}{t} \right) \\ & + \frac{C}{V_i(x, \sqrt{t})} \exp \left(-c \frac{d^2(x,y)}{t} \right). \end{aligned} \tag{4.45}$$

Proof. Parts 0,1 follow from Corollary 4.7 and Remark 4.8.

Part 2. If $|x|$ is bounded then the result follows as in Part 1, so we can assume in the sequel that $|x|$ is large enough. If $y \in K$ then using $i_y = 0$, $H(y, t) \approx 1$ and $d_+(x, y) \geq d(x, y)$ in (4.26)-(4.27), we obtain (4.43). Assume now $y \in E_0 \setminus K$. Then, for $j = i_y$, we have $y \in E_j \cap E_0$, and (4.26)-(4.27) yields

$$\begin{aligned} p(t, x, y) \quad \leq \quad & C \left(\frac{H(x,t)}{V_0(\sqrt{t})} + \frac{1}{V_i(\sqrt{t})} + \frac{H(x,t)}{V_j(\sqrt{t})} \right) \exp \left(-c \frac{d^2(x,y)}{t} \right) \tag{4.46} \\ & + \frac{C}{\sqrt{V_i(x, \sqrt{t}) V_j(y, \sqrt{t})}} \exp \left(-c \frac{d_\emptyset^2(x,y)}{t} \right). \tag{4.47} \end{aligned}$$

Since $V_0(\sqrt{t}) \leq V_j(\sqrt{t})$, the third term in (4.46) can be absorbed into the first one. If $j \neq i$ then the term in (4.47) vanishes and (4.43) follows. Assuming now $i = j$, we have

$$d_\emptyset(x, y) \geq d(x, y) \approx |x|$$

and, by Remark 4.12, the term (4.47) is dominated by the middle term in (4.46).

Part 3. Inequality (4.44) coincides with (4.26)-(4.27) since in this case $d_\emptyset(x, y) = \infty$ and $d_+(x, y) = d(x, y)$.

Part 4. Assume first that $|x|$ and $|y|$ are bounded. If also t is bounded then the last term in (4.45) is comparable with the right hand side of (4.42) and the claim follows from Part 1. If t is large enough then the first term in (4.45) is comparable with the right hand side of (4.42) and the claim again follows from Part 1.

If one of $|x|$, $|y|$ is bounded and the other is large enough then the claim follows from Part 2. Now let us assume that both $|x|$ and $|y|$ are large enough. We always have $d_\emptyset(x, y) \geq d(x, y)$ and in the present case $d_+(x, y) \approx |x| + |y|$. Thus, the first term in (4.26) and the term (4.27) are dominated by the right hand side of (4.45).

To finish the proof, it suffices to show that the second term (and similarly, the third term) in (4.26) is dominated by the last term of (4.45), that is

$$\frac{H(y,t)}{V_i(\sqrt{t})} \exp \left(-c \frac{d_+^2(x,y)}{t} \right) \leq \frac{C}{V_i(x, \sqrt{t})} \exp \left(-c' \frac{d^2(x,y)}{t} \right).$$

As $H(y, t) \leq 1$, this follows from Remark 4.12. ∎

4.5 Examples

Let us assume, under the hypotheses of Theorem 4.9, that

$$V_i(r) \approx \begin{cases} r^{N_i}, & \text{if } r > r_0 \\ r^N, & \text{if } r \le r_0 \end{cases},$$

where all $N_i > 2$. By definition (4.12), we have

$$V_0(r) \approx \begin{cases} r^n, & \text{if } r > r_0 \\ r^N, & \text{if } r \le r_0 \end{cases},$$

where

$$n := \min_i N_i > 2.$$

By definition (4.19) we have, for any $x \in E_i$,

$$H(x,t) \approx \frac{1}{|x|^{N_i-2}}.$$

Thus, the estimates (4.44) and (4.45) yield, for $t > t_0 := r_0^2$, $x \in E_i$, $y \in E_j$,

$$p(t,x,y) \le C \left(\frac{1}{t^{n/2} |x|^{N_i-2} |y|^{N_j-2}} + \frac{1}{t^{N_i/2} |y|^{N_j-2}} + \frac{1}{t^{N_j/2} |x|^{N_i-2}} \right) \exp\left(-c \frac{d^2(x,y)}{t} \right) \quad (4.48)$$

when $i \ne j$, $i,j \ge 1$, and

$$p(t,x,y) \le \frac{C}{t^{n/2} |x|^{N_i-2} |y|^{N_i-2}} \exp\left(-c \frac{|x|^2 + |y|^2}{t} \right) + \frac{C}{t^{N_i/2}} \exp\left(-c \frac{d^2(x,y)}{t} \right) \quad (4.49)$$

when $i = j \ge 1$. These upper bounds yield the estimates (1.14) and (1.19) of the Introduction.

Note that if N_i or N_j is equal to n (which is the case when M has only two ends), then (4.48) simplifies to

$$p(t,x,y) \le C \left(\frac{1}{t^{N_i/2} |y|^{N_j-2}} + \frac{1}{t^{N_j/2} |x|^{N_i-2}} \right) \exp\left(-c \frac{d^2(x,y)}{t} \right).$$

This gives (1.6).

5 Lower bounds

5.1 Parabolic Harnack inequality

Fix $R_0 \in (0, +\infty]$ and consider the following property of a weighted manifold (M, μ), which in general may be true or not:

(PH_{R_0}) The *parabolic Harnack inequality* (up to scale R_0): there exists $C > 0$ such that any positive solution $u(t,x)$ of the heat equation $\partial_t u = Lu$ in a cylinder
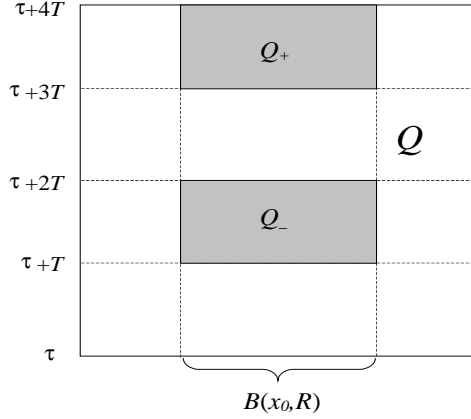
$$Q = (\tau, \tau + 4T) \times B(x_0, 2R),$$

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Figure 11: Cylinders Q_+ and Q_-

where $x_0 \in M$, $0 < R < R_0$, $T = R^2$, and $\tau \in (-\infty, +\infty)$, satisfies the inequality

$$\sup_{Q_-} u(t, x) \leq C \inf_{Q_+} u(t, x), \tag{5.1}$$

where

$$Q_- = (\tau + T, \tau + 2T) \times B(x_0, R), \quad Q_+ = (\tau + 3T, \tau + 4T) \times B(x_0, R).$$

For simplicity, we will write (PH) for (PH_∞). For example, (PH) holds for Riemannian manifolds with non-negative Ricci curvature (see [48]). Moreover, (PH) still holds if the weighted manifold M is quasi-isometric to a manifold of non-negative Ricci curvature and σ, σ^{-1} are uniformly bounded on M (see [31, 55]). Other examples are described in [55] and [57].

Consider also the following properties of M which, in general, may be true or not.

(PI) The *Poincaré inequality*: for any $x \in M$, $r > 0$ and for any function $f \in C^1(B(x, 2r))$

$$\int_{B(x,2r)} |\nabla f|^2 \, d\mu \geq \frac{c}{r^2} \inf_{\xi \in \mathbb{R}} \int_{B(x,r)} (f - \xi)^2 \, d\mu. \tag{5.2}$$

(ULE) The *upper and lower estimate* of the heat kernel: for all $x, y \in M$, $t > 0$,

$$\frac{c_2}{V(x, \sqrt{t})} \exp(-C_2 \frac{d^2}{t}) \leq p(t, x, y) \leq \frac{C_1}{V(x, \sqrt{t})} \exp(-c_1 \frac{d^2}{t}), \tag{5.3}$$

where $d = d(x, y)$.

The following theorem combines results of [31] and [55]. For this statement, recall that (VD) and (RFK) are defined Section 4.1.

Theorem 5.1 *Let (M, μ) be a complete weighted manifold. Then the following is true:*

1. $(VD) + (PI) \Longleftrightarrow (PH) \Longleftrightarrow (ULE)$.

2. $(VD) + (PI) \Longrightarrow (RFK)$.

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Remark 5.2 Clearly, assertion 2 follows from assertion 1 and Theorem 4.2.

Theorem 5.1 admits an extension treating (PH_{R_0}) with $R_0 < \infty$. In this case, (VD) and (PI) are also restricted to balls of radii $< R_0$, and (ULE) holds for all $x, y \in M$ and $t < R_0^2$ (see [2, 53, 31, 33, 55, 57] or [58, Section 5.5.1]).

The following standard consequence of (PH_{R_0}) will be useful (see [58, Corollary 5.4.4]).

Lemma 5.3 *Assume that M satisfies (PH_{R_0}) for some $R_0 > 0$ and let $u(t, x)$ be a positive solution to the heat equation $\partial_t u = Lu$ in $(0, \infty) \times M$. Then, for all positive ρ, c, C there exists a constant $a = a(\rho, c, C) > 0$ such that*

$$u(t, x) \geq au(s, y) \quad \text{if } t > s \geq c\rho^2, \quad c\rho^2 \leq t - s \leq C\rho^2, \quad d(x, y) \leq C\rho. \tag{5.4}$$

5.2 Dirichlet heat kernel

For any open set Ω of a complete weighted manifold (M, μ), the Dirichlet heat kernel p_Ω in Ω satisfies

$$p_\Omega(t, x, y) \leq p(t, x, y). \tag{5.5}$$

The next theorem provides for non-parabolic manifolds a lower bound for p_Ω, which matches the upper bound (5.5).

Theorem 5.4 ([38, Theorem 3.3])*Let (M, μ) be a non-parabolic, complete weighted manifold. Assume that the parabolic Harnack inequality (PH) holds on (M, μ). Let $K \subset M$ be a compact set and $\Omega := M \setminus K$. Then there exists $\delta > 0$ and, for each $t_0 > 0$, there exist positive constants C and c such that, for all $t \geq t_0$ and all $x, y \notin K_\delta$,*

$$p_\Omega(t, x, y) \geq \frac{c}{V(x, \sqrt{t})} \exp\left(-C\frac{d^2(x, y)}{t}\right). \tag{5.6}$$

Remark 5.5 Recall that (PH) implies the upper bound (ULE). Thus, under the hypotheses of Theorem 5.4, inequality (5.5) implies that, for all $x, y \notin K$ and $t > 0$,

$$p_\Omega(t, x, y) \leq \frac{C}{V(x, \sqrt{t})} \exp\left(-c\frac{d^2(x, y)}{t}\right). \tag{5.7}$$

Hence the lower bound (5.6) is, in a sense, optimal. Furthermore (5.6) means that the Dirichlet heat kernel $p_\Omega(t, x, y)$ is essentially of the same order of magnitude than the global heat kernel $p(t, x, y)$. This hangs on the transience of the process X_t which escapes to infinity without touching K, with a positive probability. Therefore, the influence of the killing condition on the boundary ∂K becomes negligible in the long term. If, instead, the process X_t is recurrent then p_Ω may be substantially smaller than p (see [43]).

Remark 5.6 Since the heat kernel $p_\Omega(t, x, y)$ is symmetric in x, y, (5.6) implies also the symmetric inequality

$$p_\Omega(t, x, y) \geq \frac{c}{V(y, \sqrt{t})} \exp\left(-C\frac{d^2(x, y)}{t}\right). \tag{5.8}$$

5.3 Hitting probability

Theorem 4.4 gives an upper bound for the hitting probability $\psi_K(t, x)$. Here we will also need a lower bound.

Theorem 5.7 ([39, Theorem 4.4]) *Let (M, μ) be a complete non-compact non-parabolic weighted manifold satisfying (PH), and let K be a compact subset of M with non-empty interior. Then, for any $\delta > 0$ and for all $x \notin K_\delta$, $t > 0$,*

$$cH_*(x,t) \exp\left(-C\frac{|x|^2}{t}\right) \leq \psi_K(t, x) \leq CH_*(x,t) \exp\left(-c\frac{|x|^2}{t}\right), \tag{5.9}$$

where $|x|$ and $H_*(x,t)$ are as in Section 4.2.

For the application of this theorem, we will need the following elementary lemma.

Lemma 5.8 *On an arbitrary manifold M, we have, for all $x \in M$ and $t > 0$,*

$$H_*(x, 2t) \leq 2H_*(x, t).$$

Proof. We have (see definition (4.9) of H_*)

$$H_*(x,t) := \min\left\{1, \frac{r^2}{V(r)} + \left(\int_{r^2}^{t} \frac{ds}{V(\sqrt{s})}\right)_+\right\},$$

where $V(r) := V(o,r)$, $o \in K$ is a fixed point, and $r = |x| > 0$. It suffices to show

$$\frac{r^2}{V(r)} + \left(\int_{r^2}^{2t} \frac{ds}{V(\sqrt{s})}\right)_+ \leq \frac{2r^2}{V(r)} + 2\left(\int_{r^2}^{t} \frac{ds}{V(\sqrt{s})}\right)_+. \tag{5.10}$$

We will use only the fact that the function $V(r)$ is increasing. If $t \leq r^2$ then (5.10) follows from

$$\int_{r^2}^{2t} \frac{ds}{V(\sqrt{s})} \leq \frac{r^2}{V(r)}.$$

If $t > r^2$ then, by change of variable $s = \xi/2$, we obtain

$$2\int_{r^2}^{t} \frac{ds}{V(\sqrt{s})} \geq \int_{2r^2}^{2t} \frac{d\xi}{V(\sqrt{\xi})}$$

and

$$\begin{aligned}
\int_{r^2}^{2t} \frac{ds}{V(\sqrt{s})} - 2\int_{r^2}^{t} \frac{ds}{V(\sqrt{s})} &\leq \int_{r^2}^{2t} \frac{ds}{V(\sqrt{s})} - \int_{2r^2}^{2t} \frac{ds}{V(\sqrt{s})} \\
&= \int_{r^2}^{2r^2} \frac{ds}{V(\sqrt{s})} \leq \frac{r^2}{V(r)},
\end{aligned}$$

whence (5.10) follows. ∎

5.4 Full lower bounds

This section applies the results of Section 3, 5.2, and 5.3 to obtain global lower bounds for the connected sum $M = M_1 \# M_2 \# ... \# M_k$. We start with a simple lemma dealing with the small time behavior of the heat kernel.

Lemma 5.9 *Assume that (M, μ) is a connected sum of complete non-compact weighted manifolds (M_i, μ_i), $i = 1, 2, ..., k$, each of which satisfies (PH). Then, for all $x, y \in M$ and $t > 0$, For any finite R_0, the manifold M satisfies (PH_{R_0}). Moreover, for any finite t_0 there exist positive constants c, C such that for $0 < t \leq t_0$ and all $x, y \in M$,*

$$\frac{c}{V(x, \sqrt{t})} \exp\left(-C\frac{d^2(x,y)}{t}\right) \leq p(t, x, y) \leq \frac{C}{V(x, \sqrt{t})} \exp\left(-c\frac{d^2(x,y)}{t}\right). \qquad (5.11)$$

Proof. Let first R_0 be so small that any ball of radius $2R_0$ on M lies either in one of the ends E_i or in E_0. Then one can apply either the Harnack inequality from M_i or the one from E_0 which holds just due to the compactness of $\overline{E_0}$. By a standard chaining argument, (PH_{R_0}) holds to any finite R_0. The estimate (5.11) follows then from Theorem 5.1 and Remark 5.2. Note that no hypotheses concerning the parabolicity/non-parabolicity of the ends are required for Lemma 5.9. ∎

In the rest of this section we use the same notation as in Sections 2.2, 4.3, and 4.4. In particular, the function H on M is defined at (4.19).

Theorem 5.10 *Assume that (M, μ) is a connected sum of complete non-compact weighted manifolds (M_i, μ_i), $i = 1, 2, ..., k$, each of which satisfies (PH). Assume further that each (M_i, μ_i) is non-parabolic. Then, for all $x, y \in M$ and $t > 0$,*

$$p(t, x, y) \geq c\left(\frac{H(x, t)H(y, t)}{V_0(\sqrt{t})} + \frac{H(y, t)}{V_{i_x}(\sqrt{t})} + \frac{H(x, t)}{V_{i_y}(\sqrt{t})}\right)\exp\left(-C\frac{d_+^2(x,y)}{t}\right) \qquad (5.12)$$

$$+ \frac{c}{\sqrt{V_{i_x}(x, \sqrt{t})V_{i_y}(y, \sqrt{t})}}\exp\left(-C\frac{d_\emptyset^2(x,y)}{t}\right). \qquad (5.13)$$

Remark 5.11 Since (PH) implies (RFK) and the non-parabolicity of one end (M_i, μ_i) implies the non-parabolicity of (M, μ) (see [35, Proposition 14.1]), the heat kernel upper bound (4.26)-(4.27) of Theorem 4.9 applies in the present setting. The upper bound (4.26)-(4.27) matches the lower bound (5.12)-(5.13).

Remark 5.12 In (5.12) one can replace $V_{i_x}(\sqrt{t})$ and $V_{i_y}(\sqrt{t})$ respectively by $V_{i_x}(x, \sqrt{t})$ and $V_{i_y}(y, \sqrt{t})$ (see Remark 4.12).

Remark 5.13 The term $\sqrt{V_{i_x}(x, \sqrt{t})V_{i_y}(y, \sqrt{t})}$ in (5.13) can be replaced by either $V_{i_x}(x, \sqrt{t})$ or $V_{i_y}(y, \sqrt{t})$ – see Remark 4.13.

We precede the proof by a series of lemmas in which the hypotheses of Theorem 5.10 are always implicitly assumed. It will be useful to choose the neighborhood E_0 of K as follows. By Theorem 5.4 applied to M_i, there exists $\delta_i > 0$ such that, for all $x, y \in E_i \setminus K_{\delta_i}$ and $t \geq t_0$

$$p_{E_i}(t, x, y) \geq \frac{c}{V_i(x, \sqrt{t})}\exp\left(-C\frac{d_i^2(x,y)}{t}\right), \qquad (5.14)$$

where $t_0 > 0$ is arbitrary. Set

$$\delta = \max_{1 \leq i \leq k} \delta_i$$

and choose E_0 so that it contains K_δ.

Lemma 5.14 *Fix $t_0 > 0$. For all $x, y \in E_i$, $i \geq 1$, and $t \geq t_0$,*

$$p(t, x, y) \geq \frac{c}{V_i(x, \sqrt{t})} \exp\left(-C\frac{d^2(x, y)}{t}\right). \tag{5.15}$$

Proof. Observe that, for all $x, y \in E_i$ and $t > 0$,

$$p(t, x, y) \geq p_{E_i}(t, x, y), \tag{5.16}$$

where p_{E_i} is the Dirichlet heat kernel of E_i. For $x, y \in E_i' := E_i \setminus K_\delta$, we have $d(x, y) \approx d_i(x, y)$. Hence, for such x, y, (5.15) follows from (5.14) and (5.16).

In general, for $x, y \in E_i$ and $t \geq t_0$, find the points $x' \in E_i'$ and $y' \in E_i'$ such that $d(x, x') \leq 2\delta$ and $d(y, y') \leq 2\delta$.
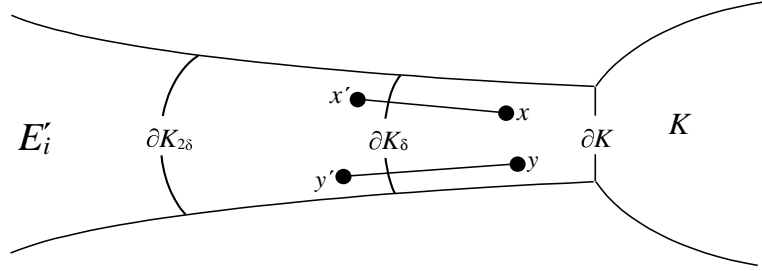
Figure 12: Points $x, y \in E_i$ and $x', y' \in E_i'$

By Lemma 5.9, M satisfies (PH_{R_0}) for any finite R_0. Setting $t' = t - t_0/4$ and $t'' = t - t_0/2$ and applying (5.4) twice, we obtain

$$p(t, x, y) \geq cp(t', x', y) \geq c'p(t'', x', y') \geq \frac{c}{V_i(x', \sqrt{t''})} \exp\left(-C\frac{d^2(x', y')}{t''}\right),$$

where we have used (5.15) for $p(t'', x', y')$. Inequality (5.15) for $p(t, x, y)$ follows from $t'' \approx t$,

$$d^2(x', y') \leq C(d^2(x, y) + \delta^2),$$

and

$$V_i(x', \sqrt{t''}) \leq CV_i(x, \sqrt{t}), \tag{5.17}$$

which is a consequence of (VD). ∎

Lemma 5.15 *Fix $t_0 > 0$. For all $x, y \in E_0$ and all $t \geq t_0$,*

$$p(t, x, y) \geq \frac{c}{V_0(\sqrt{t})}. \tag{5.18}$$

Proof. Fix $t \geq t_0$ and choose $i \geq 1$ so that (see the definition of V_0 at (4.12))

$$V_0(\sqrt{t}) = V_i(\sqrt{t}).$$

Fix a point $z \in E_i \cap E_0$. Using (5.4) observe that, for all $x, y \in E_0$ and $t \geq t_0$,

$$p(t, x, y) \geq cp(t', z, z),$$

where $t' = t - t_0/2$. By Lemma 5.14 and the doubling property (VD) in M_i, we have

$$p(t', z, z) \geq \frac{c}{V_i(z, \sqrt{t'})} \geq \frac{c'}{V_i(z, \sqrt{t})} .$$

Applying again the doubling property (VD) in M_i, we get

$$V_i(z, \sqrt{t}) \leq CV_i(o_i, \sqrt{t}) = CV_i\left(\sqrt{t}\right) = CV_0\left(\sqrt{t}\right).$$

Combining the above three estimates, we obtain (5.18). ∎

Lemma 5.16 *Fix $t_0 > 0$. For all $x, y \in M$ and $t \geq t_0$, we have*

$$p(t, x, y) \geq \frac{cH(x, t)H(y, t)}{V_0(\sqrt{t})} \exp\left(-C\frac{|x|^2 + |y|^2}{t}\right). \tag{5.19}$$

Proof. Case 1. Assume $x, y \in K$. Then (5.19) follows from Lemma 5.15 and $H \leq 1$. By symmetry, we can now assume $y \notin K$.

Case 2. Assume that $x \in E_i$ and $y \in E_j$ (where $i, j \geq 1$ may be equal or not) and apply Theorem 3.5 with $\Omega_1 = E_i$ and $\Omega_2 = E_j$ (see Fig. 13). Using also Theorem 5.7, Lemma 5.8, (4.20) and Lemma 5.15, we obtain, for $t \geq t_0/2$,

$$
\begin{aligned}
p(t, x, y) &\geq \left(\inf_{s \in [t/4, t]} \inf_{v, w \in K} p(s, v, w)\right) \psi_K(t/4, x)\psi_K(t/4, y) \\
&\geq \frac{c}{V_0(\sqrt{t})} H(x, t)H(y, t) \exp\left(-C\frac{|x|^2 + |y|^2}{t}\right).
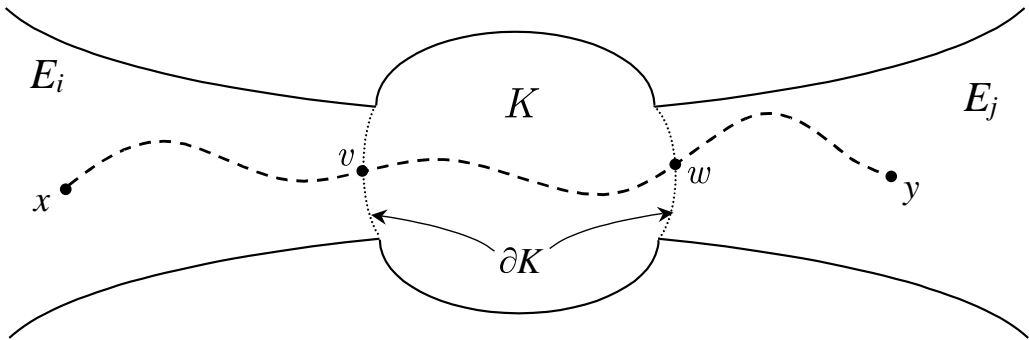\end{aligned}
$$

Figure 13: Points $x \in E_i$ and $y \in E_j$

Case 3. Finally, assume $x \in K$ and $y \in E_j$, $j \geq 1$. Let x' be a fixed point in $E_j \cap E_0$ (hence at bounded distance from x). By Case 2, for $t \geq t_0$ and $t' := t - t_0/2$, we have

$$p\left(t', x', y\right) \geq \frac{c}{V_0(\sqrt{t})} H(x,t)H(y,t) \exp\left(-C\frac{|x|^2 + |y|^2}{t}\right),$$

where we have used $H\left(x', t\right) \approx H\left(x, t\right) \approx 1$ and $|x'| \approx |x| \approx 1$. By the local Harnack inequality (5.4), we have

$$p\left(t, x, y\right) \geq cp\left(t', x', y\right),$$

whence (5.19) follows. ∎

Lemma 5.17 *Fix $t_0 > 0$. For all $x, y \in M$ and $t \geq t_0$, we have*

$$p(t, x, y) \geq \frac{cH(x,t)}{V_{i_y}(\sqrt{t})} \exp\left(-C\frac{|x|^2 + |y|^2}{t}\right). \tag{5.20}$$

Proof. Case 1. Assume $y \in E_0$. Then (5.20) follows from (5.19) because $H\left(y, t\right) \approx 1$ and $V_{i_y}\left(\sqrt{t}\right) \geq V_0\left(\sqrt{t}\right)$.

Case 2. Assume that $x, y \notin E_0$. Then, for some $i, j \geq 1$, we have $x \in E_i' := E_i \setminus K_\delta$ and $y \in E_j'$. Lemma 3.1 with $\Gamma = K$ and $\Omega = M \setminus K$ gives

$$p(t, x, y) \geq \inf_{\substack{t/2 \leq s \leq t \\ z \in K}} p(s, z, y)\psi_K(t/2, x).$$
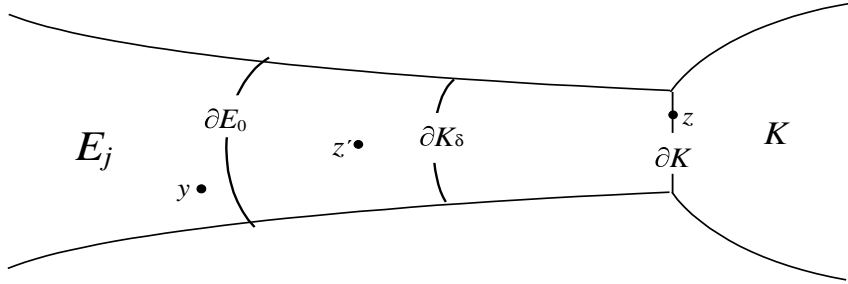
Figure 14: Points $y \in E_j'$, $z \in K$ and $z' \in E_0 \cap E_j'$

Fix $z' \in E_0 \cap E_j'$. As $z' \in E_0$ and $z \in K$ (see Fig. 14), the local Harnack inequality (5.4) yields

$$p(s, z, y) \geq cp(s', z', y),$$

for all $s \geq t_0/2$ and $s' = s - t_0/4$. Since

$$p(s', z', y) \geq p_{E_j}(s', z', y),$$

we obtain, for all $t \geq t_0$,

$$p(t, x, y) \geq c \inf_{t/4 \leq s' \leq t} p_{E_j}(s', z', y)\psi_K(t/2, x). \tag{5.21}$$

As $x \in E_i'$, Theorem 5.7 gives the lower bound for $\psi_K(t/2, x)$. As y and z' are in E_j', Theorem 5.4 gives the lower bound for $p_{E_j}(s', z', y)$. Using also $s' \approx t$ and $d_j(y, z') \le C |y|$, (5.21) gives

$$p(t, x, y) \ge \frac{cH(x, t/2)}{V_j(y, \sqrt{t})} \exp\left(-C \frac{|x|^2 + |y|^2}{t}\right).$$

Finally, by Remark 4.12, Lemma 5.8 and (4.20), we obtain (5.20).

Case 3. Assume $x \in E_0$ and $y \notin E_0$. Fix a point x' in $M \setminus E_0$. By the local Harnack inequality (5.4), we have

$$p(t, x, y) \ge cp(t', x', y)$$

where $t' = t - t_0/2$. Using the previous case to estimate $p(t', x', y)$ and noticing that $H(x, t) \approx H(x', t) \approx 1$, we finish the proof. ∎

Proof of Theorem 5.10. Fix $t_0 > 0$ and let E_0 be large enough so that is contains $K_{\sqrt{t_0}}$. Let us assume $t \le t_0$ and deduce the estimate (5.12)-(5.13) from the lower bound in (5.11) of Lemma 5.9 and its symmetric version in x, y. Since $H \le 1$, $d(x, y) \le d_+(x, y)$, it suffices to prove

$$V(x, \sqrt{t}) \le CV_{i_x}(x, \sqrt{t}) \quad \text{and} \quad V(x, \sqrt{t}) \le C_\varepsilon V_0(\sqrt{t}) \exp\left(\varepsilon \frac{d_+(x, y)^2}{\sqrt{t}}\right), \quad \forall \varepsilon > 0. \qquad (5.22)$$

If $x \in E_0$ then all the functions $V(x, \sqrt{t})$, $V_0(\sqrt{t})$, $V_{i_x}(x, \sqrt{t})$ are of the order $t^{N/2}$. Assume that $x \in E_i \setminus E_0$ for some $i \ge 1$. Then $B(x, \sqrt{t}) \subset E_i$ and

$$V(x, \sqrt{t}) = V_i(x, \sqrt{t}),$$

which proves the first inequality in (5.22). Next, by (4.5) we have

$$\frac{V(x, \sqrt{t})}{V_i(o_i, \sqrt{t})} = \frac{V_i(x, \sqrt{t})}{V_i(o_i, \sqrt{t})} \le C_\varepsilon \exp\left(\varepsilon \frac{d_i^2(x, o_i)}{\sqrt{t}}\right) \le C_\varepsilon \exp\left(C\varepsilon \frac{d_+^2(x, y)}{\sqrt{t}}\right).$$

Finally, since $V_i(o_i, \sqrt{t}) \approx V_0(\sqrt{t}) \approx t^{N/2}$, (5.22) follows.

Let now $t \ge t_0$. Note that $d_\emptyset(x, y)$ is finite only when x, y are in the same end, say E_i. This is the only case when we need to prove (5.13), and it follows from Lemma 5.14, its symmetric version in x, y, and $d(x, y) \le d_\emptyset(x, y)$. We are left to prove (5.12). If both $x, y \in E_0$ then this follows from Lemma 5.15. If one of the points x, y is outside E_0 then $d_+(x, y) \approx |x| + |y|$, and (5.12) follows by adding up the inequalities (5.19) of Lemma 5.19, (5.20) of Lemma 5.17 and its symmetric version in x, y. ∎

The next corollary gives lower bounds for x, y in different regions of M. Taken together, they are equivalent to the lower bound of Theorem 5.10 but are more explicit. These lower bounds match case by case the upper bounds of Corollary 4.16.

Corollary 5.18 *Referring to the setting of* Theorem 5.10, *the following estimates of* $p(t, x, y)$ *hold.*

0. *For any fixed* $t_0 > 0$, *if* $t \le t_0$ *and* $x, y \in M$ *then*

$$p(t, x, y) \ge \frac{c}{V(x, \sqrt{t})} \exp\left(-C \frac{d^2(x, y)}{t}\right).$$

1. If $x, y \in E_0$, then, for all $t > 0$,

$$p(t, x, y) \geq \frac{c}{V_0(\sqrt{t})} \exp\left(-C \frac{d^2(x, y)}{t}\right).$$

2. If $x \in E_i, i \geq 1, y \in E_0$ then, for all $t > 0$,

$$p(t, x, y) \geq c \left(\frac{H(x, t)}{V_0(\sqrt{t})} + \frac{1}{V_i(\sqrt{t})}\right) \exp\left(-C \frac{d^2(x, y)}{t}\right).$$

3. If $x \in E_i$, $y \in E_j$, $i \neq j$, $i, j \geq 1$ then, for all $t > 0$,

$$p(t, x, y) \geq c \left(\frac{H(x, t) H(y, t)}{V_0(\sqrt{t})} + \frac{H(x, t)}{V_j(\sqrt{t})} + \frac{H(y, t)}{V_i(\sqrt{t})}\right) \exp\left(-C \frac{d^2(x, y)}{t}\right).$$

4. If $x, y \in E_i$, $i \geq 1$, then, for all $t > 0$,

$$p(t, x, y) \geq \frac{c H(x, t) H(y, t)}{V_0(\sqrt{t})} \exp\left(-C \frac{|x|^2 + |y|^2}{t}\right) + \frac{c}{V_i(x, \sqrt{t})} \exp\left(-C \frac{d(x, y)^2}{t}\right). \quad (5.23)$$

Proof. Part 0 coincides with the lower bound of Lemma 5.9. This lemma implies also Part 1 for $t \leq t_0$ because $V_0\left(\sqrt{t}\right) \approx V\left(x, \sqrt{t}\right)$ for bounded x and t. For $t > t_0$, Part 1 follows from Lemma 5.15.

In Part 2, if $|x|$ is bounded then the estimate follows from Part 1. If $|x|$ is large enough then $d_+(x, y) \approx d(x, y)$, and the estimate follows from Theorem 5.10 and $H(y, t) \approx 1$.

Part 3 also follows from Theorem 5.10 because in this case $d_+(x, y) = d(x, y)$.

Part 4 for $t \geq t_0$ follows by adding up the estimates of Lemmas 5.14 and 5.16. If $t < t_0$ then the second term in (5.23) dominates because by (4.5)

$$\frac{V_i\left(x, \sqrt{t}\right)}{V_0\left(\sqrt{t}\right)} \approx \frac{V_i\left(x, \sqrt{t}\right)}{V_i\left(\sqrt{t}\right)} \leq C_\varepsilon \exp\left(\varepsilon \frac{d_i^2(x, o_i)}{t}\right) \leq C_\varepsilon \exp\left(\varepsilon \frac{|x|^2}{t}\right)$$

and $|x| + |y| \geq d(x, y)$. Then (5.23) follows from Part 0 and the first inequality in (5.22). ∎

Remark 5.19 In the proofs in this section, we have used some lower bounds for the heat kernel obtained in Section 3, namely, the middle term in (3.4) (Lemma 3.1) and the last term in (3.24) (Theorem 3.5). Alternatively, we could have used the full estimate (3.24)-(3.25) of Theorem 3.5. However, this would require using the lower estimates of the time derivative $\psi_K'(t, x)$ obtained in [39]. These estimates are more involved than the estimates of $\psi_K(t, x)$ given by Theorem 5.7.

5.5 Examples

Assume that each M_i is a complete non-compact Riemannian manifold of non-negative Ricci curvature equipped with its Riemannian measure μ_i. Assume in addition that

$$\int^\infty \frac{dt}{V_i(x, \sqrt{t})} < \infty$$

so that each M_i is non-parabolic. Then all hypotheses of Theorem 5.10 are satisfied, and we have the heat kernel lower bounds on $M = M_1 \# M_2 \# ... \# M_k$ implied by this theorem. For the case when

$$V_i(r) \approx \begin{cases} r^N, & r \leq 1, \\ r^{N_i}, & r > 1 \end{cases}$$

with $N_i > 2$, the upper bounds for p were considered in Example 4.5. Now Theorem 5.10 yields the matching lower bounds, which proves the estimates (1.15), (1.20) of the Introduction.

Another source of examples are domains in Euclidean space with ends isometric to convex domains of revolution. For instance, in $\mathbb{R}^3$ with coordinates $x = (x_1, x_2, x_3)$, set $x^i = (x_j, x_k)$, $i, j, k \in \{1, 2, 3\}$, $i \notin \{j, k\}$, $j < k$, and $\|x^i\| = \sqrt{x_j^2 + x_k^2}$. For $i = 1, 2, 3$, consider the (closed) domains of revolution

$$D_i = \{x = (x_1, x_2, x_3) : x_i \geq 0, \|x^i\| \leq f_i(x_i)\}$$

where the functions f_i are smooth, concave with $f_i(0) = 0$ and all derivatives equal to $+\infty$ at 0. Let M be the closure of a domain in $\mathbb{R}^3$ such that there exists a compact $K \subset M$ for which $M \setminus K$ has 3 connected components E_1, E_2, E_3 with E_i isometric to the complement of a compact set in D_i. Thus, $M = D_1 \# D_2 \# D_3$. Convex domains in $\mathbb{R}^n$ satisfy (VD), (PI) and (PH). Hence Theorems 4.9 and 5.10 apply and yield matching upper and lower bounds for the heat kernel on M. Assume for instance that for each i and $r \geq 1$, $f_i(r) \leq r$. Then, in the notation of Theorems 4.9 and 5.10, for $i = 1, 2, 3$ and $r \geq 1$, we have

$$V_i(r) \approx \int_0^r f_i(s)^2 ds \approx r f_i(r)^2.$$

In particular, if $f_i(r) = \sqrt{r \log_i^\alpha(2+r)}$, $\alpha_i > 0$, then for r large enough

$$V_i(r) \approx r^2 \log^{\alpha_i} r.$$

Of course, examples with more than 3 ends are easily constructed in a similar fashion.

More generally, consider a weighted manifold $M = M_1 \# ... \# M_k$ satisfying the hypothesis of Theorem 5.10 and such that for all $i = 1, 2, ...k$,

$$V_i(r) \approx r^2 \log^{\alpha_i} r , \quad \forall r > 2, \tag{5.24}$$

where $\alpha_i > 1$. Set $\alpha = \min_i \alpha_i$. Clearly, by (4.12) we have,

$$V_0(r) \approx r^2 \log^\alpha r , \quad \forall r > 2.$$

Therefore, by Theorems 4.9 and 5.10, the long term behavior of the heat kernel is given by

$$p(t, x, y) \approx \frac{1}{V_0(\sqrt{t})} \approx \frac{1}{t \log^\alpha t}, \quad t \to \infty.$$

Let us compute $H(x, t)$ assuming $x \in E_i$ and $|x| > 2$. Definition (4.19) of H gives

$$H(x, t) \approx \log^{-\alpha_i} |x| + \left(\log^{1-\alpha_i} |x|^2 - \log^{1-\alpha_i} t \right)_+ . \tag{5.25}$$

In particular, one has

$$H(x, t) \approx \log^{-\alpha_i} |x| \quad \text{if} \quad |x| \geq c\sqrt{t} \tag{5.26}$$

and

$$H(x,t) \approx \log^{1-\alpha_i} |x| \quad \text{if} \quad |x| \le Ct^\varepsilon, \quad \varepsilon < 1/2. \tag{5.27}$$

Suppose now that $x \in E_i$, $y \in E_j$, $i \ne j$, $i,j \ge 1$, and

$$|x| \le C\sqrt{t} \quad \text{and} \quad |y| \le C\sqrt{t}. \tag{5.28}$$

Hence, Corollary 4.16(3) and Corollary 5.18(3) yield

$$p(t,x,y) \approx \frac{H(x,t)}{t \log^{\alpha_j} t} + \frac{H(y,t)}{t \log^{\alpha_i} t} + \frac{H(x,t)H(y,t)}{t \log^\alpha t}. \tag{5.29}$$

Let t be large enough. For $|x| \approx |y| \approx \sqrt{t}$, (5.26) and (5.29) give

$$p(t,x,y) \approx \frac{1}{t \log^{\alpha_i+\alpha_j} t}.$$

For $|x| \le Ct^\varepsilon$ and $|x| \le Ct^\varepsilon$ with $\varepsilon < 1/2$, (5.29) and (5.27) give

$$p(t,x,y) \approx \frac{1}{t \log^{\alpha_j} t \log^{\alpha_i-1} |x|} + \frac{1}{t \log^{\alpha_i} t \log^{\alpha_j-1} |y|} + \frac{1}{t \log^\alpha t \log^{\alpha_i-1} |x| \log^{\alpha_j-1} |y|}.$$

In particular, if $|x| \approx |y| \approx t^\varepsilon$ with $\varepsilon < 1/2$ then

$$p(t,x,y) \approx \frac{1}{t \log^{\alpha_i+\alpha_j-1} t}.$$

Remark 5.20 In the examples above, the ends can easily be ordered according to their volume growth and, in particular, one can identify the "smallest " end (or ends). However, in the settings of Theorems 4.9 and 5.10, it is well possible that no such ordering exists. Indeed, one can construct two pointed manifolds (M_1, o_1), (M_2, o_2), both satisfying (PH) and such that the volume functions $V_i(r) = V_i(o_i, r)$ satisfy

$$\limsup_{r \to \infty} \frac{V_1(r)}{V_2(r)} = \limsup_{r \to \infty} \frac{V_2(r)}{V_1(r)} = \infty.$$

Nevertheless, Theorems 4.9 and 5.10 yield matching upper and lower bounds for the heat kernel of $M_1 \# M_2$.

6 The mixed case

This section is devoted to heat kernel estimates on connected sums $M = M_1 \# \ldots \# M_k$ when manifold M is nonparabolic but some of the ends M_i's are parabolic (recall that M is non-parabolic if and only if at least one of the M_i's is non-parabolic [35, Proposition 14.1]) The case when M is parabolic (i.e., all M_i's are parabolic) will be treated in a forthcoming paper [43] (a special case is considered in Section 7 below).

6.1 Harmonic functions

Let (M, μ) be a weighted manifold. We say that a function u defined in a region $\Omega \subset M$ is *harmonic* if $\Delta_\mu u = 0$ in Ω. If the boundary δM is non-empty, then we require in addition that u satisfies the Neumann boundary condition on $\delta\Omega := \Omega \cap \delta M$, that is

$$\left. \frac{\partial u}{\partial \mathbf{n}} \right|_{\delta\Omega} = 0. \tag{6.1}$$

For the purposes of this section, we need to be able to construct a harmonic function in an exterior domain with a controlled rate of growth at infinity. For that, we introduce a new geometric assumption. Fix a reference point $o \in M$ and consider the following condition that in general may be true or not:

(RCA) *Relative connectedness of the annuli*: there exists $A > 1$ such that, for all R large enough and for any two points $x, y \in M$ both at distance R from o, there is a continuous path γ connecting x to y and staying in the annulus $B(o, AR) \setminus B(o, R/A)$ (see Fig. 15).
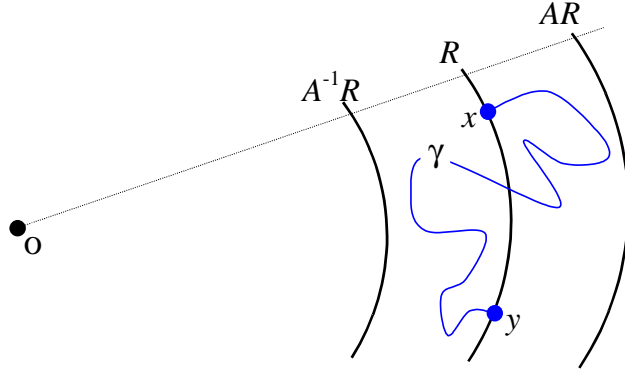

Figure 15: A path γ connecting x and y in the annulus

In particular, M has property (RCA) if the annuli $B(o, AR) \setminus B(o, R/A)$ are connected. Note that, although property (RCA) is defined for a pointed manifold (M, o), the role of the pole o is very limited. Indeed, (M, o) has property (RCA) if and only if (M, o') has it for some other point o' (the value of the constant A may change, as well as how large R has to be before the relevant connectedness property holds true).

Note that $\mathbb{R}^n$ satisfies (RCA) if and only if $n \geq 2$. The manifolds $\mathcal{R}^n$ introduced above satisfy (RCA) for all $n \geq 1$. It is easy to see that any two-dimensional convex surface in $\mathbb{R}^3$ satisfies (RCA) provided it is complete and unbounded. It was shown in [51] that any complete weighted manifold M satisfying the Poincaré inequality (PI), the doubling volume property (VD), and the condition

$$\frac{V(o, r)}{V(o, s)} \geq c \left(\frac{r}{s} \right)^N,$$

for some point $o \in M$, and all $r > s > 1$, where $c > 0$ and $N > 2$, satisfies (RCA) (see also a previous result of [44, Proposition 4.5]).

The following statement is a consequence of [38, Lemma 4.5].

Lemma 6.1 *If a complete weighted manifold M satisfies (PH) and (RCA) then, for any non-empty compact set $K \subset M$ with smooth boundary, there exists a non-negative continuous function u on M such that $u = 0$ on K, u is positive and harmonic in $M \setminus K$ and satisfies the following estimate*

$$u(x) \approx \int_1^{r^2} \frac{ds}{V(o, \sqrt{s})},$$

for all large enough $r = d(x, o)$.

Note that in the case when M is non-parabolic, the function u is bounded while in the parabolic case $u(x) \to \infty$ as $x \to \infty$.

From now on, let $M = M_1 \# ... \# M_k$ be a connected sum as in Section 2.2. As before, let K be the central part of M, that is, a compact set with smooth boundary such that $M \setminus K$ is the disjoint union of k connected components $E_1, ... E_k$ (the ends with respect to K), and each end E_i is identified with the complement of a compact set in M_i. Define a subset $I \subset \{1, \ldots, k\}$ by

$$i \in I \iff M_i \text{ is parabolic.} \tag{6.2}$$

The structure of various spaces of harmonic functions on complete Riemannian manifolds with finitely many ends has been studied in [59], and these results are easily extended to weighted manifolds. The following statement is a consequence of [59, Lemma 3.1 and Proposition 2.7].

Proposition 6.2 *Let $M = M_1 \# \ldots \# M_k$ be a connected sum of complete weighted manifolds (M_i, μ_i), and assume that M is non-parabolic. For each $i \in I$, let u_i be a non-negative continuous function on M which vanishes on $M \setminus E_i$ and is harmonic on E_i. Then there exists a positive harmonic function h defined on all of M and such that $|h - \sum_{i \in I} u_i|$ is uniformly bounded on M.*

Note that the assumption that M is non-parabolic cannot be omitted. It is known (see [59] and the references therein) that each parabolic end E_i, $i \in I$, admits a continuous non-negative harmonic function u_i vanishing on the boundary ∂E_i and such that $\sup_{E_i} u_i = \infty$. Thus, Proposition 6.2 produces an unbounded positive harmonic function on M, whereas on any parabolic manifold any positive harmonic function is constant.

In what follows, we will use property (RCA) as one of our basic assumption on the components M_i of M. From this viewpoint, (RCA) is a very natural assumption. It implies that the ends E_i are connected at infinity, i.e., that M has exactly k "true ends". Furthermore, it prohibits the situation when M_i consists of two "nearly" disjoint unbounded parts connected only by a rare sequence of small tubes.

Proposition 6.3 *Let $M = M_1 \# \ldots \# M_k$ be a connected sum of complete weighted manifolds (M_i, μ_i). Assume that M is non-parabolic and that, for each $i = 1, \ldots, k$, M_i satisfies (PH). Assume further that, for each $i \in I$, M_i satisfies (RCA). Then there exists a positive harmonic function h on M such that, for all $x \in M$,*

$$h(x) \approx 1 + \left(\int_1^{|x|^2} \frac{ds}{V_{i_x}(\sqrt{s})} \right)_+.$$

Remark 6.4 It follows from the above estimate of the function h that $h(x) \approx 1$ if x stays in any non-parabolic end whereas $h(x) \to \infty$ if $x \to \infty$ within a parabolic end.

Proof. For each $i \in I$, manifold M_i satisfies the hypotheses of Lemma 6.1. Hence, there is a continuous function u_i on M that vanishes on $M \setminus E_i$, is positive and harmonic on E_i, and satisfies the estimate

$$u_i(x) \approx \int_1^{|x|^2} \frac{ds}{V_i(\sqrt{s})}.$$

for all $x \in E_i$ and $|x|$ large enough. Applying Proposition 6.2, we obtain a desired harmonic function h. ∎

6.2 Doob's transform

Let $(M = M_1 \# \ldots \# M_k, \mu)$ be a connected sum of complete non-compact weighted manifolds (M_i, μ_i). Let h be an arbitrary positive harmonic function on M. We can then consider the new weighted manifold $\widetilde{M} = (M, \widetilde{\mu})$ where

$$d\widetilde{\mu} = h^2 d\mu.$$

Moreover, by restricting h to $E_i = M_i \setminus K_i$ and extending the resulting function smoothly to a function h_i defined on M_i, $i \in \{1, \ldots, k\}$, we obtain new weighted manifolds $\widetilde{M}_i = (M_i, \widetilde{\mu}_i)$ where $d\widetilde{\mu}_i = h_i^2 d\mu_i$ and such that

$$\widetilde{M} = \widetilde{M}_1 \# \ldots \# \widetilde{M}_k.$$

As h is harmonic on M, the weighted Laplacian

$$\widetilde{L} = L_{\widetilde{\mu}} = h^{-2} \mathrm{div}_\mu(h^2 \nabla)$$

of $\widetilde{M}$ is related to the weighted Laplacian L of M by the formula

$$\widetilde{L} = h^{-1} \circ L \circ h.$$

This implies that the heat kernels on M and $\widetilde{M}$ are related exactly by

$$\widetilde{p}(t, x, y) = \frac{p(t, x, y)}{h(x)h(y)}. \tag{6.3}$$

Thus, in this situation, any estimate of $\widetilde{p}(t, x, y)$ translates easily into an estimate of $p(t, x, y)$.

Proposition 6.5 *Let $M = M_1 \# \ldots \# M_k$ be a connected sum of complete non-compact weighted manifolds (M_i, μ_i). Assume that M is non-parabolic and that, for each $i = 1, \ldots, k$, M_i satisfies (PH). Assume further that, for each $i \in I$, M_I satisfies (RCA). Let h be the harmonic function from Proposition 6.3 and let $\widetilde{M}_i = (M_i, \widetilde{\mu}_i)$, $i = 1, \ldots, k$, be the corresponding weighted manifolds constructed as above. Then each $\widetilde{M}_i$, $i = 1, \ldots, k$, is non-parabolic and satisfies (PH).*

Proof. This is essentially [38, Lemma 4.8]. More precisely, if i is such that M_i is non-parabolic then $h_i \approx 1$. It then follows that $\widetilde{M}_i$ is still non-parabolic and satisfies (PH) (see Theorem 5.1). If instead i is such that M_i is parabolic, that is, $i \in I$, then

$$h_i(x) \approx 1 + \left(\int_1^{|x|^2} \frac{ds}{V_i(\sqrt{s})} \right)_+ .$$

In this case, the hypothesis that M_i satisfies (RCA) and (PH) (hence (VD)) together with [40, Theorem 5.7] shows that $\widetilde{M}_i$ satisfies (PH). A classical argument (see, e.g., the proof of [38, Lemma 4.8]) shows that $\widetilde{M}_i$ is non-parabolic. ∎

6.3 Full two-sided bounds

For a connected sum $M = M_1 \# \ldots \# M_k$ satisfying the hypotheses of Proposition 6.5, let $\widetilde{M}$, $\widetilde{M}_i, i \in \{1, \ldots, k\}$ be the weighted manifolds constructed in Section 6.2 using the function h of Proposition 6.3. We will use a tilde $\widetilde{}$ to denote objects relative to the manifold $\widetilde{M}$. In particular, $\widetilde{H}$ denotes the function defined at (4.19) relative to $\widetilde{M}$.

Proposition 6.5 allows us to apply Theorems 4.9 and 5.10 to $\widetilde{M}$. This yields two-sided heat kernel estimates for $\widetilde{p}(t, x, y)$ which we can transfer to the heat kernel $p(t, x, y)$ of M using (6.3). The resulting estimates are recorded in the following Theorem that gathers in one statement all the main results of this paper, that is, the upper bounds of Theorems 4.9 and the lower bounds of 5.10. Recall that I defined by (6.2), is the set of indices i such that M_i is parabolic.

Theorem 6.6 *Let $M = M_1 \# \ldots \# M_k$ be a connected sum of complete non-compact weighted manifolds M_i. Assume that M is non-parabolic and that each M_i, $i = 1, \ldots, k$, satisfies (PH). Assume further that, for each $i \in I$, M_i satisfies (RCA). Referring to the weighted manifolds $\widetilde{M}_i$ introduced above, the heat kernel $p(t, x, y)$, $t > 0$, $x, y \in M$, of the weighted manifold M satisfies*

$$p(t,x,y) \leq Ch(x)h(y)\left(\frac{\widetilde{H}(x,t)\widetilde{H}(y,t)}{\widetilde{V}_0(\sqrt{t})} + \frac{\widetilde{H}(x,t)}{\widetilde{V}_{i_y}(\sqrt{t})} + \frac{\widetilde{H}(y,t)}{\widetilde{V}_{i_x}(\sqrt{t})} \right) \exp\left(-c\frac{d_+^2(x,y)}{t} \right)$$
$$+ \frac{Ch(x)h(y)}{\sqrt{\widetilde{V}_{i_x}(x,\sqrt{t})\widetilde{V}_{i_y}(y,\sqrt{t})}} \exp\left(-c\frac{d_\emptyset^2(x,y)}{t} \right)$$

and

$$p(t,x,y) \geq ch(x)h(y)\left(\frac{\widetilde{H}(x,t)\widetilde{H}(y,t)}{\widetilde{V}_0(\sqrt{t})} + \frac{\widetilde{H}(x,t)}{\widetilde{V}_{i_y}(\sqrt{t})} + \frac{\widetilde{H}(y,t)}{\widetilde{V}_{i_x}(\sqrt{t})} \right) \exp\left(-C\frac{d_+^2(x,y)}{t} \right)$$
$$+ \frac{ch(x)h(y)}{\sqrt{\widetilde{V}_{i_x}(x,\sqrt{t})\widetilde{V}_{i_y}(y,\sqrt{t})}} \exp\left(-C\frac{d_\emptyset^2(x,y)}{t} \right)$$

Remark 6.7 In the estimates above one can replace $\widetilde{V}_{i_x}(\sqrt{t})$ and $\widetilde{V}_{i_y}(\sqrt{t})$ respectively by $\widetilde{V}_{i_x}(x, \sqrt{t})$ and $\widetilde{V}_{i_y}(y, \sqrt{t})$ (see Remarks 4.12 and 5.12). Similarly, $\sqrt{\widetilde{V}_{i_x}(x, \sqrt{t})\widetilde{V}_{i_y}(y, \sqrt{t})}$ can be replaced by either $\widetilde{V}_{i_x}(x, \sqrt{t})$ or $\widetilde{V}_{i_y}(y, \sqrt{t})$ (see Remarks 4.13 and 5.13).

Remark 6.8 Any geodesically complete non-compact manifold M with asymptotically non-negative sectional curvature can be written as $M = M_1 \# \ldots \# M_k$ where each end M_i satisfies (PH) and (RCA) (cf. Example 2.1). Thus, Theorem 6.6 yields heat kernel bounds on any such manifold as long as it is non-parabolic. The same applies to manifolds with non-negative Ricci curvature outside a compact set, provided each end satisfies (RCA) (cf. Example 2.2).

Let us give some general formulas for computing the various terms in Theorem 6.6. For $i = 0, 1, \ldots, k$, set

$$\eta_i(r) := 1 + \left(\int_1^{r^2} \frac{ds}{V_i(\sqrt{s})} \right)_+ \tag{6.4}$$

and note that

$$h(x) \approx \eta_{i_x}(|x|) \geq c\frac{|x|^2}{V_{i_x}(|x|)}. \tag{6.5}$$

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By [38, Lemma 4.8], we have, for $x \in E_i$, $i = 0, 1, \ldots, k$,

$$\widetilde{V}_i(x, r) \approx (\eta_i^2(|x|) + \eta_i^2(r))V_i(x, r). \tag{6.6}$$

Hence

$$\widetilde{V}_i(r) \approx \eta_i^2(r)V_i(r). \tag{6.7}$$

and

$$\widetilde{H}(x, t) \approx \frac{|x|^2}{\eta_{i_x}^2(|x|)V_{i_x}(|x|)} + \frac{1}{\eta_{i_x}(|x|)\,\eta_{i_x}(\sqrt{t})}\left(\int_{|x|^2}^{t} \frac{ds}{V_{i_x}(\sqrt{s})}\right)_+. \tag{6.8}$$

When comparing (6.8) with Definition 4.19 note that, by (6.5), the right-hand side of (6.8) is always bounded above so that there is no need to take the minimum with 1.

The following statement follows by inspection from the estimates of Theorem 6.6.

Corollary 6.9 *Under the hypotheses and notation of* Theorem 6.6*, we have, for any fixed $x, y \in M$ and all large enough t,*

$$\begin{aligned}
\sup_{x', y'} p\left(t, x', y'\right) &\approx \max_i \frac{1}{V_i\left(\sqrt{t}\right)} \\
\sup_{y'} p\left(t, x, y'\right) &\approx \max_i \frac{1}{V_i\left(\sqrt{t}\right) \eta_i\left(\sqrt{t}\right)} \\
p\left(t, x, y\right) &\approx \max_i \frac{1}{V_i\left(\sqrt{t}\right) \eta_i^2\left(\sqrt{t}\right)}.
\end{aligned}$$

In particular, if $I = \emptyset$ then, for any fixed $x, y \in M$ and all large t,

$$p\left(t, x, y\right) \approx \sup_{y'} p\left(t, x, y'\right) \approx \sup_{x', y'} p\left(t, x', y'\right) \approx \max_i \frac{1}{V_i\left(\sqrt{t}\right)}.$$

Using the parabolicity test (4.8) and its consequence that $\eta_i(r) \to \infty$ as $r \to \infty$ on any parabolic end, one can prove that if $I \neq \emptyset$ then

$$\liminf_{t \to \infty} \frac{p\left(t, x, y\right)}{\sup_{y'} p\left(t, x, y'\right)} = 0 \text{ and } \liminf_{t \to \infty} \frac{\sup_{y'} p\left(t, x, y'\right)}{\sup_{x', y'} p\left(t, x', y'\right)} = 0.$$

Using the remarks from Section 2.2, we obtain the following.

Corollary 6.10 *Let M be a complete non-parabolic Riemannian manifold without boundary. Assume that either M has asymptotically non-negative sectional curvature, or M has non-negative Ricci curvature outside a compact set and each end satisfies (RCA). Then M has a parabolic end if and only if for some/any $x, y \in M$,*

$$\liminf_{t \to \infty} \frac{p(t, x, y)}{\sup_{x', y'} p(t, x', y')} = 0.$$

6.4 Examples

Example 6.11 Let $M_1 = \mathcal{R}^1 := \mathbb{R}_+ \times \mathbb{S}^{N-1}$, $M_2 = \mathcal{R}^3$ and consider the connected sum $M = \mathcal{R}^1 \# \mathcal{R}^3$. We have, for $r > 1$,

$$V_1(r) \approx r \quad \text{and} \quad V_2(r) \approx r^3.$$

By (6.4), we obtain, for $r > 1$,

$$\eta_1(r) \approx r \quad \text{and} \quad \eta_2(r) \approx 1.$$

Then, by (6.7), $\widetilde{V}_i$ satisfies

$$\widetilde{V}_1(r) \approx r^3 \quad \text{and} \quad \widetilde{V}_2(r) \approx r^3,$$

whence

$$\widetilde{V}_0(r) = \min(\widetilde{V}_1(r), \widetilde{V}_2(r)) \approx r^3.$$

Using (6.8) to compute $\widetilde{H}(x,t)$, we find that $\widetilde{H}(x,t) \approx \eta_1(|x|)^{-1}$ if $x \in E_1$ and $\widetilde{H}(x,t) \approx |x|^2/V_2(|x|)$ if $x \in E_2$. It follows that, for all $x \in M$ and all $t > 0$,

$$\widetilde{H}(x,t) \approx |x|^{-1}.$$

Hence, Theorem 6.6 yields the following estimates:

1. For $x \in E_0 \cup E_1$, $y \in E_0 \cup E_2$ and $t \geq 1$,

$$\frac{c}{t^{3/2}} \left(1 + \frac{|x|}{|y|} \right) e^{-Cd^2(x,y)/t} \leq p(t,x,y) \leq \frac{C}{t^{3/2}} \left(1 + \frac{|x|}{|y|} \right) e^{-cd^2(x,y)/t}. \tag{6.9}$$

2. For $x, y \in E_0 \cup E_1$ and $t \geq 1$,

$$p(t,x,y) \geq \frac{c\,|x|\,|y|}{\sqrt{t(t+|y|^2)(t+|x|^2)}} e^{-Cd^2(x,y)/t}$$

and

$$p(t,x,y) \leq \frac{C\,|x|\,|y|}{\sqrt{t(t+|y|^2)(t+|x|^2)}} e^{-cd^2(x,y)/t}. \tag{6.10}$$

Note that for $|x| \geq |y| \geq \sqrt{t}$, the above two estimates reduce to

$$\frac{c}{t^{1/2}} e^{-Cd(x,y)^2/t} \leq p(t,x,y) \leq \frac{C}{t^{1/2}} e^{-cd(x,y)^2/t}$$

as it should whereas for $|x| \approx \sqrt{t} \geq 1$ and $|y| \approx t^\epsilon$, $\epsilon \in [0, 1/2]$, we get $p(t,x,y) \approx t^{-(1-\epsilon)}$.

3. For $x, y \in E_0 \cup E_2$, and $t \geq 1$,

$$\frac{c}{t^{3/2}} e^{-C\frac{d(x,y)^2}{t}} \leq p(t,x,y) \leq \frac{c}{t^{3/2}} e^{-C\frac{d(x,y)^2}{t}}. \tag{6.11}$$

In (6.9), the contributions of both ends $\mathcal{R}^1$ and $\mathcal{R}^3$ to the long time behavior of the heat kernel on M are of the same order $t^{-3/2}$. This may seem surprising in view of the heat kernel estimates (1.6)-(1.7) for the manifold $\mathcal{R}^n \# \mathcal{R}^m$ with $n, m > 2$, which contains both terms $t^{-n/2}$ and $t^{-m/2}$. The explanation is that what counts for the manifold $\mathcal{R}^1 \# \mathcal{R}^3$ is the heat kernel long time behavior on $\widetilde{\mathcal{R}}^1$ rather than that on $\mathcal{R}^1$. On $\widetilde{\mathcal{R}}^1$, we have $\widetilde{V}_1(r) \approx r^3$ and therefore a heat kernel behavior of order $t^{-3/2}$. This effect was first observed by E.B.Davies [25] in a model situation of a one-dimensional complex.

Example 6.12 Let us now generalize the previous example and describe a situation where the formulas (6.4), (6.7) and (6.8) can be simplified. Assume that, for r large enough and $i \in I$ (i.e., E_i is a parabolic end),

$$\int_1^{r^2} \frac{ds}{V_i(\sqrt{s})} \approx \frac{r^2}{V_i(r)}. \tag{6.12}$$

Then, for r large enough, we have

$$\eta_i(r) \approx \frac{r^2}{V_i(r)}, \quad \widetilde{V}_i(r) \approx \frac{r^4}{V_i(r)}$$

$$\widetilde{V}_{i_x}(x,r) \approx \left(\frac{|x|^4}{V_i(|x|)^2} + \frac{r^4}{V_i(r)^2} \right) V_{i_x}(x,r)$$

$$\widetilde{H}(x,t) \approx |x|^{-2} V_{i_x}(|x|).$$

For example, (6.12) holds if $V_i(r) \approx r^{\alpha_i}$ with $0 < \alpha_i < 2$ in which case we obtain

$$\eta_i(r) \approx r^{2-\alpha_i}, \quad \widetilde{V}_i(r) \approx r^{4-\alpha_i} = r^{\alpha_i^*}$$

$$\widetilde{V}_{i_x}(x,r) \approx \left(|x|^{4-2\alpha_{i_x}} + r^{4-2\alpha_{i_x}} \right) V_{i_x}(x,r)$$

$$\approx \left(|x|^{2\alpha_{i_x}^* - 4} + r^{2\alpha_{i_x}^* - 4} \right) V_{i_x}(x,r)$$

$$\widetilde{H}(x,t) \approx |x|^{\alpha_i - 2} = |x|^{2-\alpha_i^*}$$

where

$$\alpha_i^* := 4 - \alpha_i.$$

We see that $\widetilde{V}_i(r)$ and $\widetilde{H}(x,t)$ behave like the corresponding functions on a *non-parabolic* manifold with volume growth $r^{\alpha_i^*}$. Hence, to some extent, the parabolic manifold with volume growth r^{α_i} can be regarded as dual to the non-parabolic manifold with volume growth $r^{\alpha_i^*}$. This leads to the following statement.

Corollary 6.13 *Referring to the setting of* Theorem 6.6, *assume further that for each manifold M_i there is a positive real $n_i \neq 2$ such that $V_i(r) \approx r^{n_i}$ for $r \geq 1$. Set*

$$n_i^* := \begin{cases} 4 - n_i, & n_i < 2 \\ n_i, & n_i > 2 \end{cases}$$

and

$$n := \min_{1 \leq i \leq k} n_i^*.$$

1. *If $x \in E_0 \cup E_i$, $y \in E_0 \cup E_j$, $1 \leq i \neq j \leq k$, and $t \geq 1$ then*

$$p(t,x,y) \leq C \left(\frac{1}{t^{n/2} |x|^{n_i^* - 2} |y|^{n_j^* - 2}} + \frac{1}{t^{n_j^*/2} |x|^{n_i^* - 2}} + \frac{1}{t^{n_i^*/2} |y|^{n_j^* - 2}} \right)$$
$$\times |x|^{(2-n_i)_+} |y|^{(2-n_j)_+} \exp\left(-c \frac{d^2(x,y)}{t} \right) \tag{6.13}$$

and

$$p(t,x,y) \geq c \left(\frac{1}{t^{n/2} |x|^{n_i^* - 2} |y|^{n_j^* - 2}} + \frac{1}{t^{n_j^*/2} |x|^{n_i^* - 2}} + \frac{1}{t^{n_i^*/2} |y|^{n_j^* - 2}} \right)$$
$$\times |x|^{(2-n_i)_+} |y|^{(2-n_j)_+} \exp\left(-C \frac{d^2(x,y)}{t} \right). \tag{6.14}$$

2. *If $x, y \in E_i$, $1 \le i \le k$, and $t \ge 1$ then*

$$p(t, x, y) \quad \le \quad \frac{C(|x||y|)^{(2-n_i)-}}{t^{n/2}} \exp\left(-c\frac{|x|^2 + |y|^2}{t}\right) \tag{6.15}$$

$$+ \left(\frac{C|x||y|}{(|x| + \sqrt{t})(|y| + \sqrt{t})}\right)^{(2-n_i)+} \frac{1}{\sqrt{V_i(x, \sqrt{t})V_i(y, \sqrt{t})}} \exp\left(-c\frac{d^2(x, y)}{t}\right)$$

and

$$p(t, x, y) \quad \ge \quad \frac{c(|x||y|)^{(2-n_i)-}}{t^{n/2}} \exp\left(-C\frac{|x|^2 + |y|^2}{t}\right) \tag{6.16}$$

$$+ \left(\frac{c|x||y|}{(|x| + \sqrt{t})(|y| + \sqrt{t})}\right)^{(2-n_i)+} \frac{1}{\sqrt{V_i(x, \sqrt{t})V_i(y, \sqrt{t})}} \exp\left(-C\frac{d^2(x, y)}{t}\right).$$

In particular, (6.13)-(6.14) gives (1.6)-(1.7) and (1.14)-(1.15) for the manifold

$$\mathcal{R}^{N_1} \# \mathcal{R}^{N_2} \# ... \# \mathcal{R}^{N_k}$$

when all N_i are larger than 2. The estimate (6.9) for the manifold $\mathcal{R}^1 \# \mathcal{R}^3$ is also a straightforward consequence of (6.13)-(6.14). Similarly, (6.15)-(6.16) gives (1.19)-(1.20). It also gives (6.10)-(6.11) for $\mathcal{R}^1 \# \mathcal{R}^3$ although in that case there are additional simplifications due to the similarity of the behavior of both ends.

The long time asymptotic in Corollary 6.13 is determined by the term $t^{-n/2}$ where $n = \min_i n_i^*$. This was noticed by Davies [25] for an one-dimensional complex, modelling manifolds with ends. If $n_i > 2$ for all $i = 1, \ldots k$ then $n/2$ can be interpreted as the exponent of the largest heat kernel of the M_i's. However, in general this is not true. It turns out that $t^{-n_i^*/2}$ is the long time decay rate of the *Dirichlet* heat kernel of E_i, that is, the heat kernel on E_i with the vanishing boundary condition on ∂E_i. Therefore, the term $t^{-n/2}$ is determined in general by the largest Dirichlet heat kernel on the E_i's. In fact, we have used precise estimates of the Dirichlet heat kernel on each E_i as crucial tools for the proof of the results described above.

Assume that $x \in E_i$, $y \in E_j$, $1 \le i \ne j \le k$. Consider the long time asymptotic regime $|x| \le \eta(t)$, $|y| \le \eta(t)$ where η is a positive function going to infinity slower than any positive power of t (see (1.9)). In this case, Corollary 6.13 gives

$$p(t, x, y) \approx \frac{q(x, y)}{t^{n/2}}$$

with

$$q(x, y) = |x|^{(2-n_i)+}|y|^{(2-n_j)+} \times \begin{cases} |y|^{2-n_j^*} & \text{if} & n = n_j^* < n_i^*, \\ |x|^{2-n_i^*} & \text{if} & n = n_i^* < n_j^* \\ |x|^{2-n_i^*} + |y|^{2-n_j^*} & \text{if} & n = n_i^* = n_j^*, \\ |x|^{2-n_i^*}|y|^{2-n_j^*} & \text{if} & n < \min\{n_i^*, n_j^*\}. \end{cases}$$

This generalizes (1.17) which treats the case where all n_i are greater than 2.

Next consider the medium time asymptotic regime when $|x|, |y| \approx \sqrt{t}$ and $t \to \infty$. In this case Corollary 6.13 gives

$$p(t, x, y) \approx t^{[(2-n_i)+ + (2-n_j)+ - n_i^* - n_j^* + 2]/2}. \tag{6.17}$$

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If both n_i, n_j are greater than 2 then (6.17) gives $p(t, x, y) \approx t^{-(n_i+n_j)/2+1}$ as in (1.13). Similarly, if both n_i, n_j are less than 2 (in this case there must be another end that is non-parabolic) then (6.17) becomes $p(t, x, y) \approx t^{-(n_i^*+n_j^*)/2+1}$. However, if $n_i < 2$ and $n_j > 2$ then (6.17) gives $p(t, x, y) \approx t^{-n_j/2}$. Thus, in this third case, the medium time asymptotic is determined only by the larger end, in contrast to the previous two cases where both ends contribute.

However, the most interesting paradoxical effect in (6.13)-(6.14) occurs if $n_i < 2$, $n_j > 2$, $|x| \approx \sqrt{t}$, and $|y| \approx 1$. In this case, the middle term in (6.13)-(6.14) dominates and gives

$$p(t, x, y) \approx t^{-1}, \tag{6.18}$$

regardless of the exponents n_i, n_j! Therefore, if x moves away at the rate $\sqrt{t}$ in a parabolic end and y stays in E_0, then $p(t, x, y) \approx t^{-1}$ is *larger* than $p(t, y, y) \approx p(t, u, v)$, $u, v \in E_0$, since the latter satisfies $p(t, u, v) \approx t^{-n/2}$. Note that $p(t, x, x) \approx t^{-n_i/2}$ in this situation. The explanation is that if x and y are close to the central part and t is large then the process X_t started at x tends to escape to infinity within the larger end so that its chances to loop back to y are relatively small. On the contrary, if X_t starts at the point x located at the smaller end at the distance $\sqrt{t}$ from the central part, then it cannot escape to infinity within this end because of its *parabolicity*. Hence, it moves towards the central part and hits y in time t with a higher probability. Note that, in this type of heuristic explanation, it is easy to forget that $p(t, x, y)$ is symmetric!

To describe what the above estimates say concerning the approximate hot spot for fixed x and large t, consider the function

$$\mathcal{H}(y) = \frac{p(t, x, y)}{\sup_{y'} p(t, x, y')}.$$

- If all manifolds M_i are non-parabolic then $\mathcal{H}(y) \approx 1$ on the set

$$\{|y| \approx 1\} \cup \bigcup_{\{i:n_i=m\}} \left\{ y \in E_i : |y| \leq C\sqrt{t} \right\}$$

where $m = \min n_i$ (see Fig. 16). Moreover, in this region

$$p(t, x, y) \approx \frac{1}{V_0(\sqrt{t})} \approx \frac{1}{t^{m/2}} \approx p(t, x, x).$$

- If some M_i are parabolic and some non-parabolic then $\mathcal{H}(y) \approx 1$ on the set

$$\bigcup_{\{i \in I:n_i=m\}} \left\{ y \in E_i : |y| \approx \sqrt{t} \right\},$$

where $m = \min n_i$ (see Fig. 17). Moreover, in this region

$$p(t, x, y) \approx \frac{1}{t} \gg \frac{1}{t^{n/2}} \approx p(t, x, x).$$

Example 6.14 Let us take $M_1 = \mathcal{R}^2$ and $M_2 = \mathcal{R}^3$. For $r > 2$ and $|x| > 2$, we have

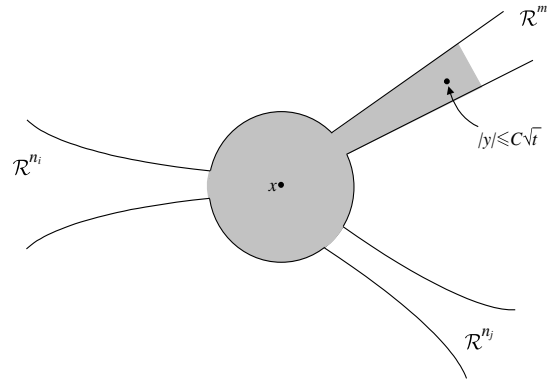$$V_1(r) \approx r^2, \quad V_2(r) \approx r^3,$$

Figure 16: Non-parabolic case: the highest temperature (up to a constant factor) is attained in the shaded area.
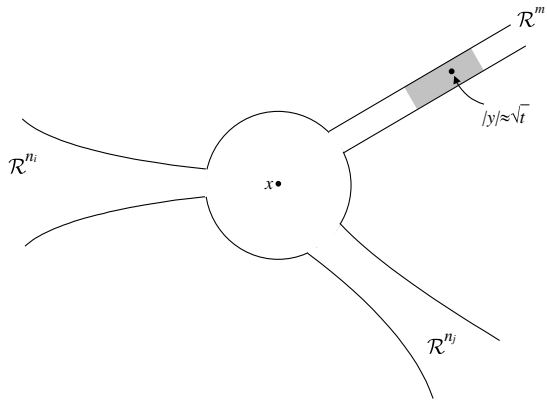

Figure 17: Mixed case: the highest temperature (up to a constant factor) is attained in the shaded area.

$$h_1(x) \approx \log(1 + |x|), \quad h_2(x) \approx 1,$$
$$\widetilde{V_1}(r) \approx r^2 \log^2(2 + r), \quad \widetilde{V_2}(r) \approx r^3,$$
$$\widetilde{V_0}(r) = \min(\widetilde{V_1}(r), \widetilde{V_2}(r)) \approx r^2 \log^2(2 + r),$$

and

$$\widetilde{V_1}(x, r) \approx [\log^2(2 + |x|) + \log^2(2 + r)]r^2, \quad \widetilde{V_2}(x, r) \approx r^3.$$

We first discuss the case where $x \in E_1$, $y \in E_2$. Then, for $t > 1$, we have $\widetilde{H}(y, t) \approx |y|^{-1}$ whereas

$$\widetilde{H}(x, t) \approx \frac{1}{\log^2(1 + |x|)} + \left(\frac{1}{2\log(1 + |x|)} - \frac{1}{\log(1 + t)} \right)_+.$$

Hence, for such x, y, t, we obtain

$$p(t, x, y) \leq C \left(\frac{\log(1 + |x|)}{|y|\, t \log^2(1 + t)} + \frac{1}{t^{3/2}} \left[\frac{1}{\log(1 + |x|)} + \left(\frac{1}{2} - \frac{\log(1 + |x|)}{\log(1 + t)} \right)_+ \right] \right) e^{-c\frac{d^2(x,y)}{t}},$$

$$p(t, x, y) \geq c \left(\frac{\log(1 + |x|)}{|y|\, t \log^2(1 + t)} + \frac{1}{t^{3/2}} \left[\frac{1}{\log(1 + |x|)} + \left(\frac{1}{2} - \frac{\log(1 + |x|)}{\log(1 + t)} \right)_+ \right] \right) e^{-C\frac{d^2(x,y)}{t}}.$$

In particular, for fixed x, y, the long time asymptotic is given by

$$p(t, x, y) \approx \frac{1}{t \log^2 t}.$$

The medium time asymptotic when $|x| \approx |y| \approx \sqrt{t}$ is given by

$$p(t, x, y) \approx \frac{1}{t^{3/2} \log t}.$$

If instead $|x| \approx \sqrt{t}$ and $|y| \approx 1$ then (compare with (6.18))

$$p(t, x, y) \approx \frac{1}{t \log t}.$$

Next, assume that $x, y \in E_1$ with $|x|, |y| \leq C\sqrt{t}$. Then we have

$$p(t, x, y) \approx \frac{\log(1 + |x|) \log(1 + |y|)}{t \log^2 t}.$$

Finally, if $x, y \in E_2$ and $|y| \leq |x| \leq C\sqrt{t}$, then

$$p(t, x, y) \approx \frac{1}{|y| t \log^2 t} + \frac{1}{t^{3/2}}.$$

Example 6.15 Our last example is $M = \mathcal{R}^1 \# \mathcal{R}^2 \# \mathcal{R}^3$ (this is essentially the same as the manifold with boundary on Fig. 2). For this example, we will only write down the long time and medium time estimates. The various functions entering the inequalities of Theorem 6.6 have been already computed in the previous examples. The long time asymptotic for any fixed $x, y \in M$ is given by

$$p(t, x, y) \approx \frac{1}{t \log^2 t}.$$

Setting $M_i = \mathcal{R}^i$, we obtain for the medium time regime $|x| \approx |y| \approx \sqrt{t}$, that

$$p(t,x,y) \approx \begin{cases} t^{-1/2} & \text{if} \quad x,y \in E_1 \\ t^{-1} & \text{if} \quad x,y \in E_2 \\ (t\log t)^{-1} & \text{if} \quad x \in E_1 \cup E_3, \ y \in E_2 \\ t^{-3/2} & \text{if} \quad x \in E_1 \cup E_3, \ y \in E_3 \end{cases}.$$

If $|x| \approx 1$ and $|y| \le C\sqrt{t}$, we get

$$p(t,x,y) \approx \begin{cases} \frac{1}{t}\left(\frac{1}{\log^2 t} + \frac{|y|}{\sqrt{t}}\right) & \text{if} \quad y \in E_1 \\ \frac{1+\log|y|}{t\log^2 t} & \text{if} \quad y \in E_2 \\ \frac{1}{t}\left(\frac{1}{\sqrt{t}} + \frac{1}{|y|\log^2 t}\right) & \text{if} \quad y \in E_3. \end{cases}$$

This proves the estimates (1.21) from the Introduction and allows to allocate the approximate hot spots as follows. For fixed x, t, set again

$$\mathcal{H}(y) = \frac{p(t,x,y)}{\sup_{y'} p(t,x,y')}.$$

Then, for large enough t, we have the following (see Fig. 18):

- $\mathcal{H}(y) \approx 1$ occurs only in the annulus $E_1 \cap \{|y| \approx \sqrt{t}\}$ so that the approximate hot spot contains such an annulus and is contained in one.

- $\mathcal{H}(y) \approx \frac{1}{\log t}$ on $E_1 \cap \{|y| \approx \frac{\sqrt{t}}{\log t}\}$ and on $E_2 \cap \{\log y \approx \log t\}$ (for $y \in E_2$ this is the approximate maximal value of $\mathcal{H}$).

- $\mathcal{H}(y) \approx \frac{1}{\log^2 t}$ on $\{|y| \approx 1\}$ (for $y \in E_3$ this is the approximate maximal value of $\mathcal{H}$).
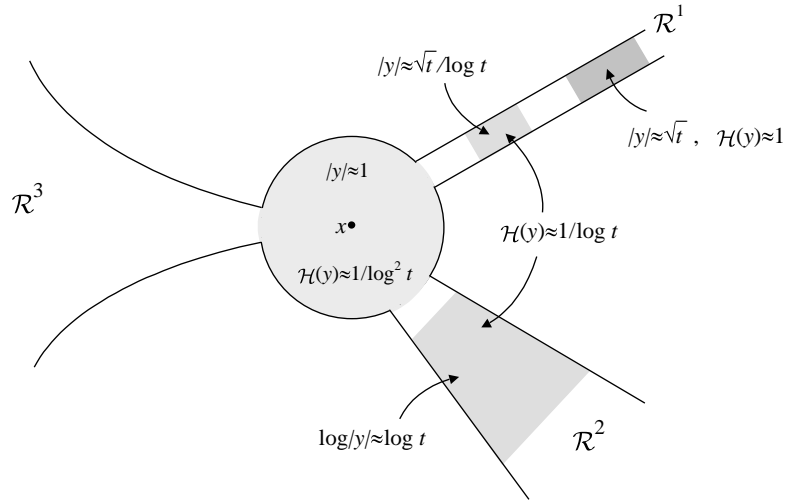

Figure 18: The approximate hot spot (darkest shade) and other relatively hot regions on the manifold $\mathcal{R}^1 \# \mathcal{R}^2 \# \mathcal{R}^3$.

7 The homogeneous parabolic case

In this section we consider a very restricted class of parabolic manifolds with ends for which the results from [38], [39], and Theorem 3.5 suffice to obtain sharp two-sided bounds (an example of such a manifold is the catenoid). Let $M = M_1 \# \ldots \# M_k$ be a connected sum of complete non-compact weighted manifolds. We assume that each M_i satisfies (PH), (RCA) and is parabolic. We assume further that M is homogeneous in the sense that, for any $i, j \in \{1, \ldots, k\}$ and all $r > 0$, we have

$$V_i(r) \approx V_j(r) \approx V_0(r). \tag{7.1}$$

Thus all the ends M_i of M have essentially the same volume growth. In this case, set

$$\eta(r) = 1 + \left(\int_1^{r^2} \frac{ds}{V_0(\sqrt{s})} \right)_+,$$

$$Q(x, t) = \frac{|x|^2}{\eta(|x|)V_0(|x|)} + \frac{1}{\eta(\sqrt{t})} \left(\int_{|x|^2}^t \frac{ds}{V_0(\sqrt{s})} \right)_+,$$

and

$$D(x, t) = \frac{\eta(|x|)}{\eta(|x|) + \eta(\sqrt{t})}.$$

With this notation, we have the following result.

Theorem 7.1 *Let $M = M_1 \# \ldots \# M_k$ be a connected sum of complete non-compact weighted manifolds. Assume that M is parabolic and that each M_i satisfies (PH), (RCA). Assume further that M satisfies (7.1). Referring to the notation introduced above, the heat kernel on M satisfies, for all $x, y \in M$ and $t > 0$,*

$$
\begin{aligned}
p(t, x, y) \;\leq\; & \frac{C}{V_0(\sqrt{t})} \left(Q(x,t)Q(y,t) + Q(x,t)D(y,t) + D(x,t)Q(y,t) \right) \exp\left(-c\frac{d_+(x,y)^2}{t} \right) \\
& + \frac{CD(x,t)D(y,t)}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}} \exp\left(-c\frac{d_\emptyset(x,y)^2}{t} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
p(t, x, y) \;\geq\; & \frac{c}{V_0(\sqrt{t})} \left(Q(x,t)Q(y,t) + Q(x,t)D(y,t) + D(x,t)Q(y,t) \right) \exp\left(-C\frac{d_+(x,y)^2}{t} \right) \\
& + \frac{cD(x,t)D(y,t)}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}} \exp\left(-C\frac{d_\emptyset(x,y)^2}{t} \right).
\end{aligned}
$$

    Proof. For any fixed t_0 and $t \in (0, t_0)$ these bounds reduce to the two-sided estimate given by Lemma 5.9. For $t > t_0$, using the local Harnack inequality provided by Lemma 5.9, it suffices to consider the case where $|x|, |y|$ are large enough. In this case, we either have $d(x,y) \approx d_+(x,y) \approx |x| + |y|$ or $d(x,y) \approx d_\emptyset(x,y)$ depending on whether or not x, y are in different ends.

    In order to use Theorem 3.5, we need two-sided estimates for the following quantities:

1. $p(t, u, v)$ when $|u|, |v|$ are bounded;

2. $\psi(t,x)$ when $|x|$ is large enough;

3. $\psi'(t,x)$ when $|x|$ is large enough.

4. $p_{E_i}(t,x,y)$ when $|x|,|y|$ are large enough, $x,y \in E_i$.

Here ψ is the hitting probability for the central part K of M and p_{E_i} is the Dirichlet heat kernel in the end E_i.

We start with point 1. Because, by hypothesis, the volume functions V_i, $i \in \{1,\ldots,k\}$ are all comparable, Theorems 5.1(2) and 4.5 show that the manifold M has the doubling volume property (VD) and satisfies the relative Faber-Krahn inequality (4.3) for some $\alpha > 0$. In particular, it follows from Theorem 4.1 that, for all $t > t_0$ and $|u|,|v|$ bounded,

$$p(t,u,v) \leq \frac{C}{V_0(\sqrt{t})}. \tag{7.2}$$

By [18, Theorem 7.2], we also have the matching lower bound

$$p(t,u,v) \geq \frac{c}{V_0(\sqrt{t})}. \tag{7.3}$$

Note that the above argument strongly uses the homogeneity hypothesis, i.e., the fact that all ends have comparable volume growth. Without this hypothesis, Theorem 4.5 does not provide a sharp central upper bound when M is parabolic. Under the present hypotheses, (7.2)-(7.3) takes care of point 1 above.

For points 2 and 3, i.e., two-sided bounds on $\psi(t,x)$ and $\psi'(t,x)$, observe that the problem is localized to each of the different ends, separately. The desired two-sided bounds are given in [39, Theorem 4.6].

Finally, a two-sided bound on the Dirichlet heat kernel of each end is provided by [38, Theorem 4.27], taking care of point 4.

Given those results, the rest of the proof of Theorem 7.1 reduces to bookkeeping and we omit the details. ∎

We will illustrate Theorem 7.1 with two different examples.

Example 7.2 Consider the connected sum $\mathcal{R}^2 \# \mathcal{R}^2$ of two Euclidean planes (the same estimates apply to the catenoid). The hypotheses of Theorem 7.1 are satisfied and $\eta(r) \approx \log(2+r)$. Hence,

$$Q(x,t) \approx \frac{1}{\log(2+|x|)} + \left(\frac{1}{2} - \frac{\log(2+|x|)}{\log(2+t)}\right)_+$$

and

$$D(x,t) \approx \frac{\log(2+|x|)}{\log(2+|x|) + \log(2+t)}.$$

For all $t > 0$, and x,y in the same end $E_i \cup E_0$, $i = 1$ or 2, we obtain

$$\frac{c}{t}\exp\left(-C\frac{d(x,y)^2}{t}\right) \leq p(t,x,y) \leq \frac{C}{t}\exp\left(-c\frac{d(x,y)^2}{t}\right).$$

Indeed, for $t \in (0,1)$, this follows from Lemma 5.9. Fix $\epsilon \in (0,1/2)$. If $t \geq 1$ and $|x|,|y| \geq t^\epsilon$, then the term involving $d_\emptyset$ dominates (essentially) and gives the desired two-sided bound. If $|x| \leq t^\epsilon$ and $|y| \geq t^\epsilon$ then $Q(x,t) \approx 1 \approx D(y,t)$ whereas if $|x|,|y| \leq t^\epsilon$ then $D(x,t) \approx 1 \approx D(y,t)$. In these two cases, the term involving d_+ dominates (essentially) and gives the desired result.

For $t \geq 1$, $x \in E_1$, $y \in E_2$ and $|x|, |y| \leq C\sqrt{t}$, we have

$$p(t, x, y) \approx \frac{C}{t} \left(Q(x, t) D(y, t) + D(x, t) Q(y, t) + Q(x, t) Q(y, t) \right).$$

In particular, for $t \geq 1$, $x \in E_1$, $y \in E_2$ and $|x|, |y| \approx \sqrt{t}$, we have

$$p(t, x, y) \approx \frac{1}{t \log t}$$

because

$$D(x, t) \approx D(y, t) \approx 1, \quad Q(x, t) \approx Q(x, t) \approx \frac{1}{\log(2 + t)}.$$

For $t \geq 1$, $x \in E_1$, $|y| \in E_2$ and $|x| \approx \sqrt{t}$, $|y| \approx t^\epsilon$, $\epsilon \in [0, 1/2)$, we have

$$p(t, x, y) \approx \frac{1}{t}$$

because $Q(y, t) \approx 1$ and $D(x, t) \approx 1$.

Example 7.3 In our second example, we assume that the function V_0 satisfies the following additional condition

$$\int_1^{r^2} \frac{ds}{V_0(\sqrt{s})} \leq C \frac{r^2}{V_0(r)}. \tag{7.4}$$

In particular, (7.4) is satisfied when $V_0(r) \approx r^\alpha$, $r > 1$, for a real $\alpha \in (0, 2)$.

Corollary 7.4 *Under the hypotheses of Theorem 7.1, assume further that V_0 satisfies (7.4). Then, for all $x, y \in M$ and $t > 0$, the heat kernel satisfies (ULE), that is,*

$$\frac{c}{V(x, \sqrt{t})} \exp\left(-C \frac{d(x, y)^2}{t} \right) \leq p(t, x, y) \leq \frac{C}{V(x, \sqrt{t})} \exp\left(-c \frac{d(x, y)^2}{t} \right). \tag{7.5}$$

and M satisfies the parabolic Harnack inequality (PH).

    Proof. Although (7.5) can be proved by inspection of the upper and lower bound in Theorem 7.1, it is simpler to first observe that the upper bound immediately follows from Corollary 4.6. Indeed, under the hypothesis that each M_i satisfy the volume doubling property and that (7.1) holds, the function F defined at (4.13) satisfy

$$F(x, r) \approx V(x, r).$$

Indeed, if the ball $B(x, r)$ is contained in one of the ends E_i then $F(x, r) = V(x, r)$ by definition. If not, then $B(x, r) \cap K \neq \emptyset$ and it follows that $F(x, r) = V_0(r) \approx V(x, r)$ because of the doubling property. Now the upper bound in (7.5) follows immediately from (4.5) and Corollary 4.6. Note that we have not used the additional hypothesis (7.4) to prove this upper bound.

    For $t \leq t_0$, the matching lower bound follows from Lemma 5.9.

    To prove the matching lower bound for $t \geq t_0$ we use Theorem 7.1. Observe that (7.4) implies

$$\eta(r) \approx 1 + \frac{r^2}{V(r)}, \quad Q(x, t) \approx 1.$$

By Theorem 7.1, this implies that for $|x| \leq \sqrt{t}$

$$p(t,x,y) \geq \frac{c}{V_0(\sqrt{t})} \exp\left(-c\frac{|x|^2 + |y|^2}{t}\right).$$

By the volume inequality (4.5), this gives

$$p(t,x,y) \geq \frac{c}{V(x,\sqrt{t})} \exp\left(-C\frac{|x|^2 + |y|^2}{t}\right).$$

As $d(x,y) \geq |y|^2 - |x|^2 - \mathrm{diam}(K)$, the last inequality implies (7.5) if $|x| \leq \sqrt{t}$. By symmetry, we can now assume that $|x|$ and $|y|$ are larger than $\sqrt{t}$ and thus $D(x,t) \approx D(y,t) \approx 1$. If x,y are in the same end, then $d_\emptyset(x,y) \approx d(x,y)$ whereas if they are in different ends, $d_+(x,y) \approx d(x,y) \approx |x| + |y|$. In the first case, the lower bound in (7.5) follows directly from the bounds of Theorem 7.1 using the term involving $d_\emptyset$. In the second case, it follows from (4.5) and Theorem 7.1, using the term involving d_+. This finishes the proof of (7.5).

The fact that M satisfies (PH) follows from (7.5) and Theorem 5.1. ■

The statement of Corollary 7.4 was proved by a different method in [40, Theorem 7.1].

8 One-dimensional Schrödinger operator

In this section we apply our main result to estimate the heat kernel $q(t,x,y)$ of the operator $\frac{d^2}{dx^2} - \Phi$ in $\mathbb{R}$ where Φ is a smooth non-negative function on $\mathbb{R}$. Assume that there is a smooth positive function h in $\mathbb{R}$ satisfying the equation $h'' - \Phi h = 0$. Let λ be the Lebesgue measure in $\mathbb{R}$ and μ be a measure in $\mathbb{R}$ defined by

$$d\mu = h^2 d\lambda. \tag{8.1}$$

It is easy to verify the following identity

$$\frac{d^2}{dx^2} - \Phi = h \circ L \circ h^{-1} \tag{8.2}$$

where

$$L = \frac{1}{h^2}\frac{d}{dx}\left(h^2 \frac{d}{dx}\right)$$

is the Laplace operator for the weighted manifold $(\mathbb{R}, \mu)$ (cf. the discussion in Section 6.2). This implies that the operator $\frac{d^2}{dx^2} - \Phi$ in $L^2(\mathbb{R}, \lambda)$ is unitary equivalent to the operator L in $L^2(\mathbb{R}, \mu)$. Consequently, if $p(t,x,y)$ is the heat kernel for L then we have the identity

$$q(t,x,y) = p(t,x,y)h(x)h(y). \tag{8.3}$$

The manifold $(\mathbb{R}, \mu)$ can be considered as a connected sum of $(\mathbb{R}_+, \mu)$ and $(\mathbb{R}_-, \mu)$. If $(\mathbb{R}_+, \mu)$ and $(\mathbb{R}_-, \mu)$ satisfy (PH) and are non-parabolic, then the heat kernel $p(t,x,y)$ can be estimated by Theorems 4.9 and 5.10. This and (8.3) lead to the desired estimates of $q(t,x,y)$.

A particularly interesting case, which received attention in literature is when $\Phi(x) = c|x|^{-2}$ for large x (see, e.g., [27], [49], [50], [65]). In this case, as we will see below, the exponent of the long time decay of $q(t,x,y)$ depends on c. In $\mathbb{R}^n$, $n \geq 2$, this problem is actually easier and the result is simpler than in $\mathbb{R}^1$ because $\mathbb{R}^n$ satisfies (RCA) and the gluing techniques are not necessary (see [36, Section 10.4]).

We start with the following lemma.

Lemma 8.1 *Let $\Phi(x)$ be a smooth function on $\mathbb{R}$ such that*

$$0 \le \Phi(x) \le C_0 |x|^{-2},$$

for some $C_0 > 0$ and for all $x \in \mathbb{R}$. Then the solution h of the initial value problem

$$\begin{aligned} h'' - \Phi h &= 0 \\ h(0) &= 1 \\ h'(0) &= 0 \end{aligned}$$

is a smooth positive function on $\mathbb{R}$, and there exists a constant $C = C(C_0) > 1$ such that

$$C^{-1} \le \frac{h(x)}{h(y)} \le C \tag{8.4}$$

for all $x, y \in \mathbb{R}$ of the same sign such that

$$\frac{1}{2}|y| \le |x| \le 2|y|. \tag{8.5}$$

If in addition $\Phi(0) > 0$ then there there exists a constant $\delta > 0$ such that

$$\frac{h(x)}{h(y)} \ge \delta \frac{|x|}{|y|} \tag{8.6}$$

for all x, y of the same sign such that $|x| \ge |y| \ge 1$.

Proof. Note that if h is positive on some interval then h is convex on this interval, due to the equation $h'' = \Phi h$. Since h is positive in a neighborhood of 0, there is a maximal open interval I containing 0 where h is positive. It follows that h is convex in I and since $h'(0) = 0$, h increases in the positive part of I and decreases in the negative part of I. Hence, if I has a finite end, then h will have a non-zero limit at that end, which contradicts the maximality of I. Thus, $I = \mathbb{R}$ which finishes the proof of the positivity of h.

To prove (8.4), let us consider the function $g = \frac{h'}{h}$. It suffices to show that, for some constant A,

$$|g(x)| \le \frac{A}{|x|} \tag{8.7}$$

because then we have, for positive x, y satisfying (8.5),

$$\ln \frac{h(x)}{h(y)} = \int_y^x g(t)\, dt \le A \ln \frac{x}{y} \le A \ln 2$$

whence (8.4) follows. Negative x, y are handled similarly.

We will prove (8.7) with A being the unique positive root of the equation

$$A^2 - A = C_0.$$

Assume that (8.7) is not true for some $x > 0$. Since (8.7) holds for $x = 0$, there is the minimal $a > 0$ such that $g(a) = \frac{A}{a}$ and $g(x) > \frac{A}{x}$ in a right neighborhood of a, say in (a, b). It is easy to see that g satisfies the equation

$$g' + g^2 = \Phi. \tag{8.8}$$

It follows that in the interval (a, b)

$$g'(x) = \Phi(x) - g^2(x) \le \frac{C_0 - A^2}{x^2} = -\frac{A}{x^2}.$$

Integrating this differential inequality from a to $x \in (a, b)$, we obtain

$$g(x) - g(a) \le -A\left(\frac{1}{a} - \frac{1}{x}\right),$$

whence

$$g(x) \le \frac{A}{x} + g(a) - \frac{A}{a} = \frac{A}{x},$$

which contradicts the choice of the interval (a, b).

If $\Phi(0) > 0$ then $h'(0) = 0$ implies that $h'(x) > 0$ for $x > 0$. Hence, also $g(x) > 0$ for $x > 0$. It follows from (8.8) that in $(0, +\infty)$

$$\frac{g'}{g^2} + 1 \ge 0.$$

Integrating this differential inequality, we obtain, for $x \ge 1$,

$$\frac{1}{g(x)} - \frac{1}{g(1)} \le x - 1$$

whence

$$g(x) \ge \frac{1}{x + \alpha}$$

where $\alpha = \left(\frac{1}{g(1)} - 1\right)_+$. Using $g = (\ln h)'$ and integrating again, we obtain, for all $x \ge y \ge 1$,

$$\frac{h(x)}{h(y)} \ge \frac{x + \alpha}{y + \alpha} \ge \delta \frac{x}{y}$$

where $\delta = \frac{1}{1+\alpha}$. The case $x \le y \le -1$ is handled similarly. $\blacksquare$

To state the next result, we will use the notation $f \asymp g_{c,C}$, which means that both inequalities $f \le g_{c,C}$ and $f \ge g_{c,C}$ hold but with different values of the positive constants c, C.

Theorem 8.2 *Let $\Phi(x)$ be a smooth function on $\mathbb{R}$ such that*

$$0 \le \Phi(x) \le C_0 |x|^{-2},$$

for some $C_0 > 0$ and for all $x \in \mathbb{R}$ and $\Phi(0) > 0$, and let $h(x)$ be defined as in Lemma 8.1. *Then the heat kernel $q(t, x, y)$ of the operator $\frac{d^2}{dx^2} - \Phi$ satisfies the estimates:*

0. For all $x, y \in \mathbb{R}$ and $0 < t \le 1$,

$$q(t, x, y) \asymp \frac{C}{\sqrt{t}} \exp\left(-c\frac{|x - y|^2}{t}\right).$$

1. For all $x \le 1$, $y \ge -1$, $t \ge 1$,

$$q(t, x, y) \asymp \frac{C}{\sqrt{t}}\left(\frac{h(y)(1 + |x|)}{h(x) h^2(\sqrt{t})} + \frac{h(x)(1 + |y|)}{h(y) h^2(-\sqrt{t})}\right) \exp\left(-c\frac{|x - y|^2}{t}\right).$$

2. For all x, y of the same sign σ such that $|x|, |y| \geq 1$ and for all $t \geq 1$,

$$q(t, x, y) \asymp \frac{C}{\sqrt{t}} \frac{|x| |y|}{h(x) h(y)} \left(\frac{1}{h^2(\sqrt{t})} + \frac{1}{h^2(-\sqrt{t})} \right) \exp\left(-c \frac{|x|^2 + |y|^2}{t} \right)$$
$$+ \frac{Ch(x) h(y)}{\sqrt{t} h(x + \sigma\sqrt{t}) h(y + \sigma\sqrt{t})} \exp\left(-c \frac{|x - y|^2}{t} \right).$$

Proof. Define measure μ on $\mathbb{R}$ by (8.1). The main point of this proof is to estimate the heat kernel $p(t, x, y)$ of the weighted manifold $(\mathbb{R}, \mu)$ and use it to estimate $q(t, x, y)$ by (8.3).

Set $M_1 = M_2 = [0, +\infty)$ so that $\mathbb{R} = M_1 \# M_2$ where we follows the agreement that M_1 maps to the positive half-line of $\mathbb{R}$ and M_2 maps to the negative half-line. Let the central part be $K = [-1, 1]$.

Define on M_i the function h_i by

$$h_1(x) = h(x), \quad h_2(x) = h(-x).$$

Then $(\mathbb{R}, \mu)$ is a connected sum of (M_1, μ_1) and (M_2, μ_2) where $d\mu_i = h_i^2 d\lambda$.

Since (M_i, λ), $i = 1, 2$, satisfies (PH) and (RCA), the weighted manifold (M_i, μ_i) also satisfies (PH) because the function h is increasing in $|x|$ and satisfies (8.4) (see [40, Theorem 5.7] and [38, Theorem 2.11]). The volume function $V_i(x, r)$ on (M_i, μ_i) is estimated by

$$V_i(x, r) \approx rh_i^2(x + r), \tag{8.9}$$

where the factor r comes from the volume of balls in (M_i, λ). It follows from (8.6) that $h_i(r) \geq cr$ for $r \geq 1$ whence $V_i(x, r) \geq cr^3$ for $r \geq 1$. By Proposition 4.3, this implies that (M_i, μ_i) is non-parabolic.

Hence, all the hypotheses of Theorems 4.9, 5.10 are satisfied and we can apply these theorems to estimate the heat kernel $p(t, x, y)$ of $(\mathbb{R}, \mu)$. Using the notation of Section 4.3, we have

$$V_i(r) \approx rh_i^2(r), \ i = 1, 2$$

and

$$V_0(r) \approx r \min(h_1(r), h_2(r))^2.$$

Due to (8.6), we have

$$\frac{V_i(R)}{V_i(r)} \geq c \left(\frac{R}{r} \right)^3$$

for $R \geq r \geq 1$. Therefore, the function $H(x, t)$ defined in Section 4.4, can be estimated by (4.21) as follows:

$$H(x, t) \approx \frac{x^2}{V_i(|x|)} \approx \frac{|x|}{h^2(x)} \ \text{ if } |x| > 1 \text{ and } x \in M_i,$$

and

$$H(x, t) \approx 1 \text{ if } |x| \leq 1.$$

Applying Corollaries 4.16, 5.18 for the cases 0 and 2 and Theorems 4.9, 5.10 for the case 1 (cf. Remark 4.10), we obtain estimates for $p(t, x, y)$, which by (8.3) imply the desired estimates for $q(t, x, y)$. ∎

Corollary 8.3 *Let $\Phi(x) \geq 0$ for all x, $\Phi(0) > 0$, and*

$$\Phi(x) = \begin{cases} \alpha_+ |x|^{-2}, & x > x_0, \\ \alpha_- |x|^{-2}, & x < -x_0, \end{cases}$$

where $x_0 > 0$ and α_+, α_- are non-negative constants. Set

$$\beta_\pm = \frac{1}{2} + \sqrt{\frac{1}{4} + \alpha_\pm} \ . \tag{8.10}$$

Then the heat kernel $q(t, x, y)$ of the operator $\frac{d^2}{dx^2} - \Phi(x)$ satisfies the estimates:

0. For all $x, y \in \mathbb{R}$ and $0 < t \leq 1$,

$$q(t, x, y) \asymp \frac{C}{\sqrt{t}} \exp\left(-c\frac{|x - y|^2}{t}\right).$$

1. For all $x \leq 1$, $y \geq -1$, $t \geq 1$,

$$q(t, x, y) \asymp C\left(\frac{\langle x \rangle^{1-\beta_-} \langle y \rangle^{\beta_+}}{t^{\beta_+ + 1/2}} + \frac{\langle x \rangle^{\beta_-} \langle y \rangle^{1-\beta_+}}{t^{\beta_- + 1/2}}\right) \exp\left(-c\frac{|x - y|^2}{t}\right).$$

where $\langle \cdot \rangle = 1 + |\cdot|$.

2. For all $x, y \geq 1$ and all $t \geq 1$,

$$\begin{aligned}
q(t, x, y) \quad \asymp \quad & C\frac{|x|^{1-\beta_+} |y|^{1-\beta_+}}{t^{\beta + 1/2}} \exp\left(-c\frac{|x|^2 + |y|^2}{t}\right) \\
& + \frac{C}{t^{1/2}}\left(1 + \frac{\sqrt{t}}{|x|}\right)^{-\beta_+}\left(1 + \frac{\sqrt{t}}{|y|}\right)^{-\beta_+} \exp\left(-c\frac{|x - y|^2}{t}\right),
\end{aligned}$$

where $\beta = \min(\beta_+, \beta_-)$. A similar estimate holds for $x, y \leq -1$ with β_- instead of β_+.

Proof. Consider function $f(x) = x^\gamma$ for $x > 0$. It is easy to see that $f''/f = \gamma(\gamma - 1)x^{-2}$. Therefore, if $\gamma^2 - \gamma = \alpha$ then f satisfies $f'' - \alpha x^{-2}f = 0$. For $\alpha > 0$ this quadratic equation has two roots

$$\gamma_1 = \frac{1}{2} + \sqrt{\frac{1}{4} + \alpha} \quad \text{and} \quad \gamma_2 = \frac{1}{2} - \sqrt{\frac{1}{4} + \alpha}$$

and $\gamma_1 > 0 > \gamma_2$. It follows that any solution to the equation $u'' - \alpha x^{-2}u = 0$ on an interval $(a, +\infty)$ is a linear combination of the functions x^{γ_1} and x^{γ_2}. This implies that either $u(x) \sim cx^{\gamma_1}$ or $u(x) \sim cx^{\gamma_2}$ as $x \to +\infty$.

As a consequence, we obtain that the function $h(x)$ from Lemma 8.1 satisfies

$$h(x) \approx x^{\beta_+} \text{ on } [1, +\infty) \quad \text{and} \quad h(x) \approx |x|^{\beta_-} \text{ on } (-\infty, 1],$$

where β_+, β_- are defined by (8.10). Substituting into Theorem 8.2, we finish the proof. ∎

9 Appendix - the list of conditions

(RFK) - relative Faber-Krahn inequality, Section 4.1,

(VD) - volume doubling, Introduction and Section 4.1,

(PI) - Poincaré inequality, Introduction and Section 5.1,

(PH) - parabolic Harnack inequality, Section 5.1,

(ULE) - upper and lower estimates of the heat kernel, Section 5.1,

(RCA) - relative connectedness of annuli, Section 6.1.

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