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BOUNDED SOLUTIONS OF THE SCHRÖDINGER EQUATION ON NONCOMPACT RIEMANNIAN MANIFOLDS

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Necessary and sufficient geometric conditions are proved for the equation $\Delta u - Q(x)u = 0$, $Q(x) \geq 0$, to have a bounded nontrivial solution on a noncompact Riemannian manifold. The results imply as corollaries conditions for parabolicity and stochastic completeness of a manifold, previously established by other methods.

INTRODUCTION

Let M be a smooth connected noncompact Riemannian manifold, Δ the Laplace–Beltrami operator on M, $Q(x)$ a smooth nonnegative function on M, not identically zero. Considering the Schrödinger equation on M,

$$\Delta u - Q(x)u = 0 \tag{0.1}$$

we ask the following question: For what manifolds M and potentials Q does equation (0.1) have a unique bounded solution $u = 0$? If this is the case, we say that Liouville's Theorem is true for equation (0.1).

It is well known that when $Q = 1$ Liouville's Theorem for equation (0.1) is equivalent to stochastic completeness of M [4]. A manifold is said to be stochastically complete if the total probability of a minimal Wiener process is preserved and equals 1; this is equivalent to uniqueness of the solution to the Cauchy problem for the heat equation $u_t - \Delta u = 0$ in L^∞.

It can be proved (see below, Sec. 1), that if Q has compact support Liouville's theorem is true for equation (0.1) if and only if the manifold is parabolic, i.e., the operator $-\Delta$ has no positive fundamental solution.

It is well known that M is parabolic if and only if the Wiener process is recurrent. The theories of stochastically complete and parabolic manifolds, developed independently in [1-3, 6, 9-12], have brought to light various beautiful analogies. For example, Cheng and Yau [1] proved that if M is geodesically complete and the volume $V(R)$ of a geodesic ball of radius R with fixed center O is such that as $R \to \infty$

$$V(R) \leqslant CR^2, \tag{0.2}$$

then M is parabolic. In my paper [2] I proved that if

$$V(R) \leqslant \exp CR^2 \tag{0.3}$$

then M is stochastically complete. Several authors [12] refer to a preprint of Karp and Li in which condition (0.3) is in fact established.

The reason underlying the analogy between (0.2) and (0.3) is that these conditions are special cases of one more general theorem, stating a sufficient condition for Liouville's

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Theorem to be true for an arbitrary Schrödinger equation (see Sec. 3).

Quite similarly, several necessary conditions for parabolicity and stochastic completeness, pharased in terms of curvature growth [6, 9, 10], follow in a unified fashion from suitable theorems for the Schrödinger equation (see Sec. 2).

1. SOME PRELIMINARY INFORMATION

Throughout this paper M will denote a smooth, connected, noncompact Riemannian manifold, possibly with boundary.

If the boundary is not empty, equation (0.1) must be considered together with a Neumann condition on the boundary:

$$\frac{\partial u}{\partial N}=0, \tag{1.1}$$

where N is the normal to ∂M.

Proposition 1.1. Let G be a precompact open subset of M with smooth boundary transversal to ∂M. The following conditions are equivalent:

a) Equation (0.1) with condition (1.1) has a nontrivial bounded solution on M.

b) Equation (0.1) with condition (1.1) has a positive bounded solution on M.

c) There exists a function $v \in C^2(M\setminus\overline{G})$ such that $0 \le v \le 1$, $\Delta v - Qv \ge 0$ in $M\setminus\overline{G}$, $v|_{\partial G} = 0$ (and moreover v satisfies condition (1.1) on $\partial M\setminus G$), $v \not\equiv 0$.

Proof. a) $\Rightarrow$ c). Let $\{B_k\}$ be an exhaustion of M by precompact open sets.

We may assume that all the boundaries ∂B_k are smooth and transversal to ∂M, and also that $B_k \supset \overline{G}$ for all k.

Let u be a solution as described in part a). We may assume without loss of generality that $\sup_M u = 1$. By the strong maximum principle, $m = \sup_G u_+ < 1$, where $u_+ = \max(u, 0)$. Let us solve the following sequence of boundary-value problems in $B_k\setminus\overline{G}$:

$$\Delta v_k - Qv_k = 0,\ v_k|_{\partial G} = 0,\ v_k|_{\partial B_k} = (u-m)_+$$

(we are assuming that the v_k also satisfy condition (1.1)). Since $\Delta(u - m) - Q(u - m) = Qm \ge 0$, $v_k \ge u - m$ on the boundary of $B_k\setminus\overline{G}$, it follows from the maximum principle that $v_k \ge u - m$ throughout the domain. It is also obvious that $v_k \ge 0$, whence $v_k \ge (u - m)_+$. By the maximum principle one easily infers that $v_k \le v_{k+1} \le 1$, so that the sequence $\{v_k\}$ is monotone increasing and converges to the required function v. That v is nontrivial follows from $v \ge (u - m)_+$ and the strong maximum principle.

c) $\Rightarrow$ b) The required function u will be the limit of solutions u_k of the following boundary-value problems in B_k:

$$\Delta u_k - Qu_k = 0,\ \ u_k|_{\partial B_k} = v.$$

b) $\Rightarrow$ a) Obvious.

COROLLARY 1.1. If M is parabolic, then for any Q Liouville's Theorem is true for equation (0.1).

Indeed, if u is a solution as in part b) of Proposition 1.1, then $\Delta u \ge 0$. But any bounded subharmonic function on a parabolic manifold is a constant [5], and this, together with (0.1), implies u = 0.

For compact-supported Q the converse is also valid (see Sec. 2).

For the sequel we need the Green's function of the Laplacian.

Definition 1.1. The Green's function g(x, y) of the Laplace operator on M is the smallest positive fundamental solution of the operator $-\Delta$ (with condition (1.1)).

The Green's function g(x, y) is constructed as the limit of Green's functions of the boundary-value problem for the Laplace operator in the exhausting domains B_k (with a Dirichlet condition on ∂B_k and a Neumann condition on $\partial M \cap B_k$).

In the case of a parabolic manifold this limit is ∞, so that we put $g = \infty$. For nonparabolic manifolds the limit satisfies Definition 1.1 (for more details see [5] or [3]).

The procedure just described to construct the Green's function readily implies

Proposition 1.2. Let f be a nonnegative function on M and let

$$u(x)=\int_M g(x,\ y)f(y)\,dy. \tag{1.2}$$

If $u(x_0) < \infty$ at some point x_0, then $u(x)$ is defined for all x and is the smallest nonnegative solution of the equation $\Delta u = -f$ on M (with condition (1.1)).

COROLLARY 1.2. The function (1.2) satisfies $\inf\limits_M u = 0$ (provided that $u < \infty$).

Indeed, if $m = \inf\limits_M u > 0$, then $u - m$ is also a nonnegative solution of the equation $\Delta u = -f$, contrary to Proposition 1.2.

Another result of the construction is

Proposition 1.3. If B is some precompact open set containing a point y, then

$$\sup_{x\in M\setminus B} g(x,\ y)=\sup_{x\in\partial B} g(x,\ y),$$

in particular, $g(x, y)$ is bounded for $x\in M\setminus B$.

2. EXISTENCE OF NONTRIVIAL BOUNDED SOLUTIONS OF THE SCHRÖDINGER EQUATION

THEOREM 2.1. Let Q be a potential such that

$$\int_M g(x,\ y)Q(y)\,dy<\infty,$$

where $g(x, y)$ is the Green's function of the Laplace operator (see Sec. 1). Then equation (0.1) with condition (1.1) has a nontrivial bounded solution on M.

Remark. The converse is unfortunately false.

Proof. As in Sec. 1, let $\{B_k\}$ be an exhausting sequence. In B_k, solve the boundary-value problem $\Delta u_k - Qu_k = 0$, $u_k|\partial B_k = 1$ (condition (1.1) is assumed to hold on $\partial M\cap B_k$). Obviously, $0 < u_k \le 1$. It follows easily from the maximum principle that the sequence $\{u_k\}$ is monotone decreasing and its limit is the desired function u. We have to prove that $u\not\equiv 0$. Denote the integral on the left of (2.1) by $v(x)$. Let us compare $v(x)$ with the function $\bar{u}_k = 1 - u_k$. Since $\bar{u}_k = Q\bar{u}_k - Q \ge -Q$, $\bar{u}_k|_{\partial B_k} = 0$, while ν is nonnegative and satisfies the equation $\Delta\nu = -Q$, it follows from the maximum principle that $v \ge \bar{u}_k$. Letting $k \to \infty$ we get $v \ge 1 - u$. But by Corollary 1.2 $\inf v = 0$, and so $\sup u \ge 1$, $u\not\equiv 0$.

COROLLARY 2.1. If Q is summable on M and M is not parabolic, then equation (0.1) (with condition (1.1)) has a nontrivial bounded solution.

The proof follows in an obvious way from the local summability of the Green's function and Proposition 1.3. The conditions of Corollary 2.1 hold, in particular, for potentials Q with compact support.

COROLALRY 2.2. Assume that Liouville's Theorem is true for equation (0.1) (with condition (1.1)) on M. Then if u is a nonnegative harmonic (or superharmonic) function which is summable on M with weight Q, i.e.,

$$\int_M u(x)Q(x)\,dx<\infty, \tag{2.2}$$

then u = const (it is assumed that u satisfies condition (1.1)).

Proof. If M is parabolic then u = const without the need to assume condition (2.2) [5]. Assume, therefore, that the Green's function $g(x, y)$ exists. We claim that (2.2) implies (2.1), so that the conclusion will follow from Theorem 2.1.

Let $\{B_k\}$ be an exhaustion of M, $g_k(x, y)$ the Green's function in B_k. If $\min\limits_{\partial B_1} u = 0$, then $u \equiv 0$ by the strong maximum principle. Let $\min\limits_{\partial B_1} u > 0$. Find a constant C such that $Cu(x) \geq g(x, y)$ for all $x \in \partial B_1$. Then for any k

$$Cu(x) \geq g_k(x, y) \tag{2.3}$$

for all $x \in \partial B_1$. Since $g_k(x, y) = 0$ for $x \in \partial B_k$, condition (2.3) holds on ∂B_k. By the maximum principle it is true for all $x \in B_k \setminus B_1$. Letting $k \to \infty$, we obtain $Cu(x) \geq g(x, y)$ for $x \in M \setminus B_1$. Hence, using (2.2), we readily obtain (2.1).

We now apply Theorem 2.1 to manifolds with a pole. a point $O \in M$ is called a pole if the exponential map $\exp_O$ is a diffeomorphism of $T_O M$ and M. In particular, a manifold with a pole is geodesically complete and has no boundary. Hence we have a globally defined polar coordinate system (r, β), where r is the geodesic distance to O and β a point on the unit sphere in $T_O M$. The Laplace operator in this coordinate system is

$$\Delta u = u_{rr} + H(r, \beta) u_r + \Delta_{S_r} u, \tag{2.4}$$

where S_r is the geodesic sphere of radius r centered at O, ΔS_r the inner Laplacian on S_r, $H(r, \beta)$ the mean curvature of S_r at the point (r, β) in the direction of the outer normal (see [8]). For example, in $\mathbb{R}^n$ $H = \frac{n-1}{r}$, while in Lobachevskii space $\mathbb{H}_k^n$ $H = (n - 1) \times k \operatorname{cth} kr$.

It is also known that $H(r, \beta)$ characterizes the relative expansion of S_r with increasing r. This means, in particular, that

$$\frac{d}{dr} s(r) = \int\limits_{S_r} H,$$

where $s(r) = \operatorname{mes} S_r$.

A manifold with pole O is called a model if the geodesic rotations about O are isometries. In that case $H(r, \beta) = H(r)$.

It is immediately verifiable that the Green's function $g(r) = g(x, O)$ on a model is given by

$$g(r) = \int\limits_r^\infty \frac{dR}{s(R)}.$$

Theorem 2.1 yields

COROLLARY 2.3. If M is a model manifold, $Q(x)|_{S_r} = \text{const} = Q(r)$ and

$$\int\limits^\infty \frac{1}{s(R)} \int\limits_0^R s(r) Q(r)\, dr\, dR < \infty, \tag{2.5}$$

then equation (0.1) has a nontrivial bounded solution.

Remark. It can be shown that if the integral in (2.5) is divergent, then, on the contrary, Liouville's Theorem is true.

Proposition 2.1. Let M be a manifold with pole, $\hat{M}$ a model. Suppose that for $r \geq R_0$ one has $H(r, \beta) \geq \hat{H}(r)$, $Q(r, \beta) \leq \hat{Q}(r)$. Then if the equation $\Delta u - \hat{Q} u = 0$ has a nontrivial bounded solution on $\hat{M}$, the same is true of equation (0.2) on M.

Proof. Let v be a positive bounded solution of the equation $\Delta v - \hat{Q} v = 0$ on $\hat{M}$. Averaging v over spheres, we may assume that v is independent of β. It follows from the maximum principle that $v(r)$ as a function on M. By (2.4),

$$\Delta v - Qv = v_{rr} + H(r, \beta) v_r - Qv \geq v_{rr} + \hat{H} v_r - \hat{Q} v = 0.$$

Define $u(r) = v(r) - v(R_0)$. Then $u > 0$, $\Delta u - Qu \geq 0$ in $M \setminus \overline{B}_{R_0}$, $u|_{\partial B_{R_0}} = 0$ (where B_{R_0} is the ball of radius R_0 centered at O), and it remains to apply Proposition 1.1.

COROLLARY 2.4. Let M be a manifold with a pole and let $q(r)$, $h(r)$ be nonnegative smooth functions such that $q(r) \geq Q(r, \beta)$ for $r \geq R_0$ and

$$\int^{\infty} \frac{dr}{h(r)} < \infty. \tag{2.6}$$

Suppose moreover that for $r \geq R_0$ we have $H(r, \beta) \geq \frac{h'}{h} + gh$, then equation (0.1) has a nontrivial bounded solution.

Proof. We construct a model $\hat{M}$ such that $\hat{H} = \frac{h'}{h} + gh$ for $r \geq R_0$. For example, working in $\mathbf{R}^n$, one can conformally modify the metric on every sphere centered at the origin in such a way that its volume becomes $\hat{s}(r)$, where $\hat{s}(r)$ is a function such that for $r \geq R_0$ one has $\hat{s}'/\hat{s} = \frac{h'}{h} + gh$, i.e., $\hat{s}(r) = Ch(r) \exp \int_{R_0}^{r} qh$.

Then condition (2.5) for the model $\hat{M}$ with potential q is precisely (2.6), and the rest follows immediately.

Examples. 1. If Q has compact support and $h = r^{1+c}$, $c > 0$, we obtain a sufficient condition for nonparabolicity: $H \geq \frac{1+c}{r}$. The corresponding manifolds expand rather more quickly than $\mathbf{R}^2$: for them $s(r) \geq$ const r^{1+c}.

2. If $Q \equiv 1$ we obtain a sufficient condition for stochastic incompleteness: $H \geq Cr^{1+c}$. The corresponding manifolds expand very rapidly: $s(r) \geq \exp C_1 r^{2+c}$.

3. Let M be the domain of revolution in $\mathbf{R}^n$ obtained by rotating the subgraph $y = x_1{}^{a}$ about the x_1 axis, where $\frac{1}{n-1} < a \leq 1$.

Let

$$Lu = \sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i})_{x_j}$$

be a uniformly elliptic operator in M with smooth coefficients. Defining a metric in M whose Laplace–Beltrami operator is proportional to L and using an estimate for the Green's function established in [3], we see that if

$$\int_M x_1^{1-(n-1)a} Q(x)\, dx < \infty$$

there exists in M a nontrivial bounded solution of the equation $Lu - Qu = 0$ with a Neumann condition along the conormal on the boundary of M.

3. LIOUVILLE'S THEOREM FOR THE SCHRÖDINGER EQUATION

Let $r(x)$ be a fixed locally Lipschitzian exhaustion function on M, i.e., a function whose sublevel sets $B_R = \{x : r(x) < R\}$ are precompact for all R.

Denote $S_R = \{x : r(x) = R\}$. If $r(x)$ is smooth, then for almost all r one can define the flow

$$p(R) = \int_{S_R} |\nabla r|.$$

Let $q(R)$ be a nonnegative continuous function such that $Q(x) \geq |\nabla r|^2 q(r)$ for all $x \in M$. Define

$$F(R) = \int_0^R \sqrt{q(t)}\, dt. \tag{3.1}$$

THEOREM 3.1. If $r(x)$ is a smooth exhaustion function and for some $C > 0$

$$\int^{\infty} \frac{\exp(CF(R)^2)}{p(R)}\, dR = \infty, \tag{3.2}$$

then equation (0.1) (with condition (1.1)) has a unique bounded solution $u = 0$.

COROLLARY 3.1. Let $r(x)$ be a Lipschitzian exhaustion function (e.g., in particular, a distance function). Let $V(R)$ denote the volume of the set B_R. Let $q(R) \le \min_{S_R} Q$. If there exists a sequence $R_k \to \infty$ such that

$$V(2R_k) \leqslant CR_k^2 \exp(CF(R_k)^2), \tag{3.3}$$

then Liouville's Theorem is true for equation (0.1) (with condition (1.1)) on M, where F is defined in terms of q as in (3.1).

In particular, if $V(R_k) \le CR_k^2$ the manifold is parabolic (this was first proved in [1]), while if $V(R_k) \le \exp CR_k^2$ it is stochastically complete.

Remark. Obviously, a Lipschitzian exhaustion function exists if and only if the manifold is metrically complete.

Proof of Theorem 3.1. Suppose that Liouville's Theorem is false on M, i.e., by Proposition 1.1, there exists a solution u of equation (0.1) (with condition (1.1)) such that $0 < u < 1$.

We claim that condition (3.2) necessarily implies that $u \le 0$ on $\operatorname{supp} Q$; this contradiction will complete the proof.

Considering the direct product $M \times [0, +\infty)$, we define $v(x, t) = e^t u(x) - 1$. Obviously, $v(x, t)$ satisfies the parabolic equation

$$Qv_t - \Delta v = 0 \tag{3.4}$$

with initial condition

$$v(x, 0) \leqslant 0 \tag{3.5}$$

(as well as a Nuemann boundary condition on ∂M).

From this fact, using (3.2), we shall infer that $v(x, t) \le 0$ for all $x \in \operatorname{supp} Q$, $t > 0$, whence it will follow that $u(x) \le e^{-t}$, and so as $t \to \infty$ $u(x) \le 0$ for all $x \in \operatorname{supp} Q$.

Let $\varphi(r)$, $g(r, t)$ be locally Lipschitzian functions of one and two variables, respectively, and assume in addition that φ has compact support. Considering r as a function on M, multiply (3.4) by $v_+\varphi^2 e^g$ and integrate over $M \times [T_1, T]$ (where $0 \le T_1 < T$):

$$\int_M \int_{T_1}^T Qv_t v_+ \varphi^2 e^g \, dt dx = \int_{T_1}^T \int_M \Delta v v_+ \varphi^2 e^g \, dx dt.$$

Integrating by parts, we obtain (the integral vanishes along ∂M by the Neumann condition)

$$\frac{1}{2}\int_M Qv_+^2 \varphi^2 e^g \Big|_{T_1}^T - \frac{1}{2}\int_M \int_{T_1}^T Qv_+^2 \varphi^2 g_t e^g = -\int_{T_1}^T \int_M |\nabla v_+|^2 \varphi^2 e^g -$$

$$-2\int_{T_1}^T \int_M v_+ (\nabla v, \nabla \varphi) \varphi e^g - \int_{T_1}^T \int_M v_+ (\nabla v, \nabla g) \varphi^2 e^g.$$

Applying suitable inequalities of the type $2ab \le ca^2 + c^{-1}b^2$, on the right, we can easily ensure that the integrals containing $|\nabla v_+|^2 \varphi^2 e^g$, cancel out. We finally obtain

$$\int_M Qv_+^2 \varphi^2 e^g \Big|_{T_1}^T \leqslant \int_{T_1}^T \int_M (Qg_t + |\nabla g|^2) v_+^2 \varphi^2 e^g + 4\int_{T_1}^T \int_M v_+^2 |\nabla \varphi|^2 e^g. \tag{3.6}$$

Now put $g(r, t) = \frac{f(r)^2}{t-T}$, where $f(r) = 0$ for $r \le R$, and $f(r) = \frac{1}{2}\int_R^r \sqrt{q(s)}\, ds$ for $r \ge R$ (the number $R > 0$ is fixed for the present). With g thus defined it is readily seen that $Qg_t + |\nabla g|^2 \le 0$. We now demand that the function $\varphi(r)$ satisfy the condition $\varphi(r) = 1$ for $r \le R$ and $\varphi(r) = 0$ for $r \ge R_1$, where $R_1 > R$. Noting that $g(r, t) \le 0$ and that for $r \le R$ $g(r, t) = 0$, and also that $v(x, t) \le e^t$, we infer from (3.6) that

$$\int_{B_R} Qv_+^2(x,T) - \int_{B_{R_1}} Qv_+^2(x,T_1) \leqslant 2e^{2T} \int_{B_{R_1}\setminus B_R} |\nabla\varphi|^2 \exp\left(-\frac{f(r)^2}{T-T_1}\right). \tag{3.7}$$

Denoting $E(R, T) = \int_{B_R} Qv_+^2(x, T)dx$ and using Federer's formula [7] for integration over level sets for the integral on the right of (3.7), we obtain

$$E(R,T) - E(R_1,T_1) \leqslant 2e^{2T} \int_R^{R_1} p(r)\varphi'(r)^2 \exp\left(-\frac{f(r)^2}{T-T_1}\right) \tag{3.8}$$

(we have used the fact that $|\nabla\varphi| = \varphi'|\nabla r|$, and the definition of $p(r)$). Minimizing the integral on the right of (3.8) with respect to all functions φ such that $\varphi(R) = 1, \varphi(R_1) = 0$, we obtain

$$E(R,T) - E(R_1,T_1) \leqslant 2e^{2T} \left\{ \int_R^{R_1} \frac{\exp\left(\frac{f(r)^2}{T-T_1}\right)}{p(r)} dr \right\}^{-1}.$$

It is easy to see that for sufficiently small $c > 0$

$$\frac{1}{c} f(r)^2 = \frac{1}{4c} \left(\int_R^r \sqrt{q} \right)^2 \geqslant CF(r)^2 - C_R,$$

where $C > 0$ is the constant in the assumptions of the theorem and C_R depends on C and R. Therefore, if $T - T_1 \leq c$, then

$$E(R,T) - E(R_1,T_1) \leqslant 2e^{2T+C_R} \left\{ \int_R^{R_1} \frac{\exp(CF(r)^2)}{p(r)} dr \right\}^{-1}.$$

Letting $R_1 \to \infty$, we see that the right-hand side of this inequality tends to zero because of (3.2), so that $E(R, T) \leq E(\infty, T_1)$. For $T_1 = 0$, the initial condition (3.5) yields $E(\infty, T_1 = 0$, so that also $E(R, T) = 0$ for all $T \leq c$, $R > 0$. Now, putting $T_1 = c$, we obtain $E(R, T) = 0$ for $T \leq 2c$, and so on. Thus, $Qv_+^2 \equiv 0$, $v(x, t) \leq 0$ for $x \in \operatorname{supp} Q$ and all $t > 0$.

Proof of Corollary 3.1. Obviously, it will be enough to consider the case in which $r(x)$ is a bounded Lipschitzian function (this follows from approximation arguments). For simplicity's sake, let us assume that the Lipschitz constant is 1, i.e., $|\nabla r| \leq 1$. Then q satisfies the assumptions of Theorem 3.1 and

$$\int_R^{2R} p(r)\,dr = \int_{B_{2R}\setminus B_R} |\nabla r|^2\,dx \leqslant V(2R).$$

Therefore

$$\int_{R_k}^{2R_k} p(r)\,dr \leqslant CR_k^2 \exp(CF(R_k)^2). \tag{3.9}$$

By the Cauchy–Bunyakovskii inequality,

$$\int_{R_k}^{2R_k} \frac{\exp(CF(r)^2)}{p(r)} dr \geqslant R_k^2 \left\{ \int_{R_k}^{2R_k} \frac{p(r)\,dr}{\exp(CF(r)^2)} \right\}^{-1} \geqslant C^{-1}$$

by virtue of (3.9). Hence it follows that (3.2) holds, and the rest follows from Theorem 3.1.

The next theorem somewhat weakens condition (3.2).

THEOREM 3.2. Let $r(x)$ be a Lipschitzian exhaustion function on M, $q(R) \leq \min_{S_R} Q$. Assume that $\int^\infty \sqrt{q}\, dR = \infty$, $F(R)$ is defined as in (3.1), and $h(R)$ is a positive continuous function such that

$$\int^\infty \frac{dR}{\left(R + \frac{F(R)}{F'(R)}\right) h(R)} = \infty. \tag{3.10}$$

Suppose that for sufficiently large R

$$V(2R)\leqslant C\exp(h(R)F(R)^2). \tag{3.11}$$

Then Liouville's Theorem is true for equation (0.1) with condition (1.1)).

Proof. We confine attention to the case in which $r(x)$ is smooth and $|\nabla r| \leq 1$. We shall use inequality (3.9), established in the proof of Theorem 3.1.

Fixing R and T, we inductively define sequences $\{R_k\}$, $\{P_k\}$, $\{T_k\}$, $R_{k-1} < P_k < R_k$, $T_k < T_{k-1}$, such that $P_0 = R$, $T_0 = T$, $R_k = aP_k$,

$$\int_{R_{k-1}}^{P_k} \sqrt{q} \geqslant (a-1)\int_0^{R_{k-1}} \sqrt{q}, \tag{3.12}$$

$$0 < T_{k-1} - T_k \leqslant \frac{b}{h(P_k)}, \tag{3.13}$$

where $a\in(1, 2)$ is some positive constant, $b = \frac{1}{4}\left(\frac{a-1}{a}\right)^2$ (obviously, conditions (3.12) and (3.13) leave us some leeway for choice – we shall use this later). Renaming the limits of integration in inequality (3.8) and taking $\varphi(r)$ equal to 1 in $[r_{k-1}, P_k]$, zero for $r > R_k$ and linear in $[P_k, R_k]$, we obtain

$$E(R_{k-1}, T_{k-1}) - E(R_k, T_k) \leqslant \frac{2e^{2T}}{(R_k - P_k)^2}\int_{P_k}^{R_k} p(r)\times$$

$$\times\exp\left\{-\frac{h(P_k)}{4b}\left(\int_{R_{k-1}}^{r}\sqrt{q}\right)^2\right\}dr \leqslant$$

$$\leqslant \frac{2e^{2T}}{(a-1)^2P_k^2}\exp\left\{-\frac{h(P_k)}{4b}\left(\int_{R_{k-1}}^{P_k}\sqrt{q}\right)^2\right\}\int_{P_k}^{R_k} p(r)\,dr \leqslant$$

$$\leqslant \frac{2e^{2T}}{(a-1)^2P_k^2}\exp(-h(P_k)F(P_k)^2)V(2P_k) \leqslant \frac{C_1}{P_k^2}.$$

Here we have used (3.11), (3.12) and (3.13). Summing all these inequalities over k and noting that $P_{k+1} \geq aP_k$, we obtain $E(aR, T) - E(R_k, T_k) \leq \frac{\text{const}}{R^2}$. If $T_k = 0$ for some k, then $E(R_k, T_k) = 0$ and, letting $R \to \infty$, we get $E(\infty, T) = 0$.

Such a T_k will surely exist (and the proof will be complete) if

$$\sum_{k=1}^{\infty}\frac{1}{h(P_k)} = \infty. \tag{3.14}$$

We shall therefore choose the sequence P_k so that, besides (3.12), it also satisfies (3.14). To that end we first determine a function $P(k)$ of a continuous argument k from the differential equation

$$\frac{d}{dk}P(k) = P(k) + \frac{F(P(k))}{F'(P(k))},\quad P(0) = R. \tag{3.15}$$

The function $P(k)$ has the following properties.

1. $$F(P(k+s)) \geqslant e^{s/2}F(e^{s/2}P(k)). \tag{3.16}$$

Inded, it follows from (3.15) that $\frac{d}{dk}P(k) \geq P(k)$, whence $P(k + s) \geq e^sP(k)$. It also follows from (3.15) that

$$\frac{d}{dk}F(P(k)) = F'(P(k))\left(P + \frac{F(P)}{F'(P)}\right) \geqslant F(P(k)),$$

whence $F(P(k + s)) \geq e^sF(P(k))$. Combining these two inequalities, we obtain (3.16).

2. $$\int^{\infty}\frac{dk}{h(P(k))}=\infty. \tag{3.17}$$

Indeed, by (3.15) the substitution $R = P(k)$ transforms the integral (3.17) to (3.10).

3. There exists $c\in(0, 1)$ such that

$$\sum_{k=1}^{\infty}\frac{1}{h(P(ck))}=\infty. \tag{3.18}$$

It is generally true that if $f(R)$ is a positive continuous function and $\int^{\infty}f(R)\,dR=\infty$, then there exists $c\in(0, 1)$ such that $\sum_{k=1}^{\infty}f(ck)=\infty$. For monotone f this is obvious; for arbitrary f the statement appeared as a problem in a correspondence contest held by the Moscow State University Faculty of Mechanics and Mathematics during the 1974/75 school year. The proof was published in the collection Problems of Student Mathematics Olympiads [in Russian], Moscow State University Publishing House (1987).

Thus, we determine a suitable number c in accordance with property 3 and define $P_k = P(ck)$, $a = e^{c/2}$. It is easy to see that (3.12) follows from (3.16) and (3.14) from (3.18).

This completes the proof of the theorem.

Remark. Since h is not necessarily monotone, (3.10) and (3.11) can be combined in a single formula:

$$\int^{\infty}\frac{F(R)^2\,dR}{\left(R+\frac{F}{F'}\right)\ln V(2R)}=\infty.$$

In particular, if $Q = 1$ we obtain a condition for stochastic completeness [2]:

$$\int^{\infty}\frac{R\,dR}{\ln V(R)}=\infty.$$

COROLLARY 3.2. Let $v(R)$, $f(R)$ be monotone increasing positive functions such that

$$\int^{\infty}\frac{dR}{f(R)}=\infty, \tag{3.19}$$

and assume that certain regularity conditions hold.

Suppose that for sufficiently large R the following conditions hold on a manifold M with Lipschitzian exhaustion function $r(x)$:

$$V(R)\leqslant Cv(R),$$
$$\min_{S_R}Q\geqslant C^{-1}\frac{v'}{vf}, \tag{3.20}$$

then Liouville's Theorem is true for equation (0.1) (with condition (1.1)).

Remarks. 1. The regularity conditions are as follows (where $w = \ln v$):

$$0<\frac{Rw'}{w}\leqslant C,$$
$$(fw')'\geqslant 0,$$

where $C > 0$ is arbitrary. The first condition implies an upper bound $v(R) \leq \exp R^C$.

The second condition does not actually impose essential restrictions on f; the standard functions $f(r) = r$; $r \ln r$; $r \ln r \ln\ln r$ and so on usually satisfy it.

2. If $v(R)$ also satisfies the condition

$$\frac{v'''v'}{(v'')^2}\geqslant c>0$$

(from which it follows that $v(R) \geq \text{const}\ R^{2+\varepsilon}$, $\varepsilon = \frac{c}{1-c}$), then conditions (3.19) and (3.20) imply the best possible restrictions on Q for which Liouville's Theorem is still valid.

Indeed, if M is a model for which $V(R) = v(R)$, then for any function $f(R)$ not satisfying (3.19) the Schrödinger equation with potential $Q = \frac{v'}{vf}$ has a nontrivial bounded solution (since, as is readily proved, inequality (2.5) is then true).

Example. In R^n condition (3.20) yields $Q \geq \frac{\text{const}}{Rf(R)}$, in particular $Q \geq \frac{\text{const}}{R^2 \ln R}$; in Lobachevskii space, similarly $Q \geq \frac{\text{const}}{R \ln R}$.

The proof of Corollary 3.2 amounts to verifying conditions (3.10) and (3.11) for the function $h = \text{const}\ \frac{w'f}{w}$.

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