HEAT KERNELS AND GREEN FUNCTIONS ON METRIC MEASURE SPACES

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Abstract. We prove that, in a setting of local Dirichlet forms on metric measure spaces, a two-sided sub-Gaussian estimate of the heat kernel is equivalent to the conjunction of the volume doubling propety, the elliptic Harnack inequality and a certain estimate of the capacity between concentric balls. The main technical tool is the equivalence between the capacity estimate and the estimate of a mean exit time in a ball, that uses two-sided estimates of a Green function in a ball.

Contents

1. Introduction

In this paper we are concerned with heat kernel estimates for regular local Dirichlet forms on metric measure spaces. The heat kernel is a surprising source of many phenomena in diverse

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areas of mathematics and science. There is a vast literature devoted to various aspects of heat kernels; see, for example, [2, 6, 13, 14, 15, 17, 32, 33, 34, 36, 37, 38, 39] for the Euclidean spaces or Riemannian manifolds, [8, 10, 24, 25] for tori or infinite graphs, [3, 5, 9, 27] for certain classes of fractals, and [12, 26, 28, 30, 31, 41, 19, 20] for metric spaces.

The purpose of this paper is to obtain equivalent conditions for two-sided sub-Gaussian estimates of the heat kernel for the full range of time and space variables. In the simplest case the sub-Gaussian estimate has the following form

$$p_t\left(x,y\right)\asymp \frac{C}{V\left(x,t^{1/\beta}\right)}\exp\left(-c\left(\frac{d^{\beta}(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right)$$

where $p_t\left(x,y\right)$ is the heat kernel in question, $d\left(x,y\right)$ is a metric, $V\left(x,r\right)$ is the volume function of a metric ball, and $\beta>1$ is a parameter that is called the walk dimension. One of our main results – Theorem 3.14, ensures that, under some simple assumptions about the volume function, such an estimate of the heat kernel is equivalent to the following two conditions: the uniform Harnack inequality for harmonic functions and the following estimate of the resistance between two concentric balls $B=B\left(x,r\right)$ and $KB=B\left(x,Kr\right)$:

$$\operatorname{res}\left(B,KB\right)\simeq\frac{r^{\beta}}{V\left(x,r\right)},\tag{1.1}$$

where K is a large fixed constant. On the other hand, such a sub-Gaussian estimate of the heat kernel is equivalent to a certain two-sided estimate of the Green function.

The main technical result of the paper is Theorem 3.12 that ensures the equivalence of the resistance condition (1.1) to a certain mean exit time estimate from a metric ball. To obtain Theorem 3.14, we then combine Theorem 3.12 with the results of [26] and [20].

In Section 2 we give necessary background material about abstract heat semigroups. In Section 3 we state the two above mentioned theorems, and prove Theorem 3.14 by using Theorem 3.12. The proof of Theorem 3.12 is postponed to Section 8 after we develop necessary tools for that.

In Section 5 we prove some properties of the Green operator, in particular, the existence of its kernel – the Green function, under the Harnack inequality. The most challenging result in this section is to obtain an annulus Harnack inequality for the Green function, without assuming any specific properties of the metric d, unlike previously known similar results [4], [25] where the geodesic property of the distance function was used. A desire to have the results for a general metric d is motivated by a number of applications. For example, the proof of the uniqueness of Brownian motion on Sierpinski carpet in [7] uses Theorem 3.14. Another possible application could be on self-similar fractals with a resistance metric.

In Section 6 we prove a representation formula for superharmonic functions via Riesz measures. This type of results is known in abstract Potential Theory [11], but in our setting those results are not directly applicable, and so we give an independent proof based on the heat semigroup.

In Section 7 we prove the pointwise estimates of the Green function using Harnack inequality and the resistance estimate. This type of estimates was known on graphs [25] and on smooth manifolds [17], but the present singular setting imposes certain difficulties that we overcome using the potential-theoretic tools from the previous sections.

In Section 8 we give the proof of Theorem 3.12 using all the machinery developed in the previous sections.

Appendix 9 contains some auxiliary properties of capacities and Dirichlet forms.

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Notation. The sign $\simeq$ below means that the ratio of the two sides is bounded from above and below by positive constants. The letters C,C',c,c' will always refer to positive constants, whose values are unimportant and may change at each occurrence. The sign $U\Subset\Omega$ means that U is precompact and $\overline{U}\subset\Omega$. For any bilinear form $\mathcal{E}\left(f,g\right)$ set $\mathcal{E}(f):=\mathcal{E}(f,f)$. If B is a ball of radius r then λB is the concentric ball with radius λr.

2. Heat semigroups

Throughout the paper, we assume that (M,d) is a locally compact separable metric space and μ is a Radon measure on M with full support. We refer to such a triple (M,d,μ) as a *metric measure space*.

Denote by

$$B(x,r)=\{y\in M: d(x,y)<r\}$$

the open metric ball of radius $r>0$ centered at x. We always assume that every ball $B(x,r)$ is precompact. In particular, the volume function

$$V(x,r):=\mu(B(x,r))$$

is finite and positive for all $x\in M$ and $r>0$.

Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form in $L^2(M,\mu)$. Recall that $(\mathcal{E},\mathcal{F})$ is *regular* if $\mathcal{F}\cap C_0(M)$ is dense both in $\mathcal{F}$ and in $C_0(M)$, where $C_0(M)$ is the space of all continuous functions with compact support in M, endowed with sup-norm. The form $(\mathcal{E},\mathcal{F})$ is *strongly local* if $\mathcal{E}(f,g)=0$ for any $f,g\in\mathcal{F}$ with compact supports, such that $f\equiv\text{const}$ in an open neighborhood of $\operatorname{supp} g$.

Let $\mathcal{L}$ be the generator of $\mathcal{E}$, that is, $\mathcal{L}$ is a self-adjoint and non-positive definite operator in $L^2(M,\mu)$ with the domain $\operatorname{dom}(\mathcal{L})$ that is dense in $\mathcal{F}$ and such that, for all $f\in\operatorname{dom}(\mathcal{L})$ and $g\in\mathcal{F}$,

$$\mathcal{E}(f,g)=-(\mathcal{L}f,g)\ ,$$

where $(\cdot,\cdot)$ is the inner product in $L^2(M,\mu)$. The associated *heat semigroup*

$$P_t=e^{t\mathcal{L}},\ t\geq 0,$$

is a family of contractive, strongly continuous, self-adjoint operators in $L^2(M,\mu)$ that satisfies the Markovian property (cf. [16]).

Recall that for any $f\in L^2(M,\mu)$, the function

$$t\mapsto\frac{1}{t}(f-P_tf,f)$$

is increasing as t is decreasing, and for any $f\in\mathcal{F}$,

$$\lim_{t\to 0+}\frac{1}{t}(f-P_tf,f)=\mathcal{E}(f).\tag{2.1}$$

The form $(\mathcal{E},\mathcal{F})$ is called *conservative* if $P_t1=1$ for every $t>0$. Unlike many other results about heat kernels of Dirichlet forms, we never assume explicitly the conservativeness of $(\mathcal{E},\mathcal{F})$, although it may follow from other hypotheses.

A family $\{p_t\}_{t>0}$ of non-negative $\mu\times\mu$-measurable functions on $M\times M$ is called the *heat kernel* of the form $(\mathcal{E},\mathcal{F})$ if p_t is the integral kernel of the operator P_t, that is, for any $t>0$ and for any $f\in L^2(M,\mu)$,

$$P_tf(x)=\int_M p_t(x,y)f(y)\,d\mu(y)\tag{2.2}$$

for μ-almost all $x\in M$.

For a non-empty open $\Omega\subset M$, let $\mathcal{F}(\Omega)$ be the closure of $\mathcal{F}\cap C_0(\Omega)$ in the norm of $\mathcal{F}$. It is known that if $(\mathcal{E},\mathcal{F})$ is regular, then $(\mathcal{E},\mathcal{F}(\Omega))$ is also a regular Dirichlet form in $L^2(\Omega,\mu)$. Denote by P_t^Ω the heat semigroup of $(\mathcal{E},\mathcal{F}(\Omega))$, and $\mathcal{L}^\Omega$ the generator of $(\mathcal{E},\mathcal{F}(\Omega))$.

Recall that for any regular Dirichlet form $(\mathcal{E},\mathcal{F})$, there is an associated *Hunt process*[1]. Denote by $X_t, t\geq 0$, the trajectories of a process and by $\mathbb{P}_x, x\in M$, the probability measure in the space of trajectories emanating from the point x. Denote by $\mathbb{E}_x$ the expectation of the probability measure $\mathbb{P}_x$. Then the relation between the Dirichlet form and the associated Hunt process is given by the following identity:

$$P_tf(x)=\mathbb{E}_xf(X_t),\tag{2.3}$$

[1]Loosely speaking, a *Hunt process* is a strong Markov process whose sample paths are right continuous and have left limit almost surely.

which holds for any bounded Borel function f, for every $t > 0$, and for μ-*almost all* $x \in M$ (note that $P_t f$ is a function from L^∞ and, hence, is defined up to a set of measure zero, whereas $\mathbb{E}_x f(X_t)$ is defined *pointwise* for all $x \in M$). By [16, Theorem 7.2.1, p.380], such a process always exists but, in general, is not unique. Let us fix one of such processes once and for all. If $(\mathcal{E}, \mathcal{F})$ is local, then the Hunt process X_t is a *diffusion*, that is, the sample path $t \mapsto X_t$ is continuous almost surely.

Example 2.1. Let M be a connected Riemannian manifold, d be the geodesic distance on M, μ be the Riemannian volume. Define the space

$$W^1 = \left\{ f \in L^2 : \nabla f \in L^2 \right\}$$

where $L^2 = L^2(M, \mu)$ and ∇f is the Riemannian gradient of f understood in the weak sense. For all $f, g \in W^1$, one defines the energy form

$$\mathcal{E}(f, g) = \int_M (\nabla f, \nabla g)\, d\mu.$$

Let $\mathcal{F}$ be the closure of $C_0^\infty(M)$ in W^1. Then $(\mathcal{E}, \mathcal{F})$ is a regular strongly local Dirichlet form in $L^2(M, \mu)$. The heat kernel admits (cf. [2]) the two-sided Gaussian bounds

$$p_t(x, y) \asymp \frac{C}{t^{n/2}} \exp\left(-\frac{|x-y|^2}{ct} \right).$$

Similar bounds hold also on some classes of Riemannian manifolds (see [18], [32]). Note that in the above examples the Dirichlet form is *local* and, hence, the corresponding Hunt process is a diffusion.

Example 2.2. On some classes of fractals the heat kernel is known to exist and to satisfy the following *sub-Gaussian* estimate:

$$p_t(x, y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-\left(\frac{d(x, y)}{ct^{1/\beta}} \right)^{\beta/(\beta-1)} \right), \tag{2.4}$$

for all $t > 0$ and $\mu \times \mu$-almost all $x, y \in M$. Here $d(x, y)$ is an appropriate distance function, and $\alpha > 0$ and $\beta > 1$ are some parameters that characterize the underlying space in question.

3. Description of the results

Let us introduce some definitions needed in this paper.

Definition 3.1. Let Ω be an open subset of M. We say that a function $u \in \mathcal{F}$ is *harmonic* in Ω if

$$\mathcal{E}(u, v) = 0 \text{ for any } v \in \mathcal{F}(\Omega).$$

A function $u \in \mathcal{F}$ is *superharmonic* in Ω if

$$\mathcal{E}(u, v) \geq 0 \text{ for any nonnegative } v \in \mathcal{F}(\Omega),$$

and is *subharmonic* in Ω if

$$\mathcal{E}(u, v) \leq 0 \text{ for any nonnegative } v \in \mathcal{F}(\Omega).$$

Definition 3.2. We say that the *elliptic Harnack inequality* (H) holds on M if, there exist constants $C_H > 1$ and $\delta \in (0, 1)$ such that, for any ball $B(x_0, r)$ in M and for any function $u \in \mathcal{F}$ that is harmonic and non-negative in $B(x_0, r)$, the following inequality is satisfied:

$$\operatorname*{esup}_{B(x_0, \delta r)} u \leq C_H \operatorname*{einf}_{B(x_0, \delta r)} u. \tag{H}$$

Let us emphasize that the constants C_H and δ are independent of the ball $B(x_0, r)$ and the function u.

Definition 3.3. We say that the *volume doubling* property (VD) holds if there exists a constant C_D such that, for all $x \in M$ and all $r > 0$

$$V(x, 2r) \leq C_D V(x, r). \tag{VD}$$

It is known that (VD) implies that, for all $x, y \in M$ and $0 < r \leq R$,

$$\frac{V(x,R)}{V(y,r)} \leq C_D \left(\frac{R + d(x,y)}{r} \right)^{\alpha}, \tag{3.1}$$

for some $\alpha > 0$ (see for example [20]).

Definition 3.4. We say that the *reverse volume doubling property* (RVD) holds if, there exist positive constants α' and c such that, for all $x \in M$ and $0 < r \leq R$,

$$\frac{V(x,R)}{V(x,r)} \geq c \left(\frac{R}{r} \right)^{\alpha'}. \tag{3.2}$$

Clearly, (RVD) implies that the space (M, d) is unbounded. On the other hand, if (M, d) is connected and unbounded then (VD) implies (RVD) (cf. [20]).

Let F be a continuous increasing bijection of $(0, \infty)$ onto itself, such that, for all $0 < r \leq R$,

$$C^{-1} \left(\frac{R}{r} \right)^{\beta} \leq \frac{F(R)}{F(r)} \leq C \left(\frac{R}{r} \right)^{\beta'}, \tag{3.3}$$

for some constants $1 < \beta \leq \beta'$ and $C > 1$. Consider the inverse function $\mathcal{R} = F^{-1}$. Obviously (3.3) implies that

$$C^{-1} \left(\frac{T}{t} \right)^{1/\beta'} \leq \frac{\mathcal{R}(T)}{\mathcal{R}(t)} \leq C \left(\frac{T}{t} \right)^{1/\beta} \tag{3.4}$$

for all $0 < t \leq T$.

Recall that a *cutoff function* ϕ of (A, Ω) means that $\phi \in \mathcal{F} \cap C_0(\Omega)$, $0 \leq \phi \leq 1$ in M, and $\phi = 1$ in a neighborhood of A. It is known that if $(\mathcal{E}, \mathcal{F})$ is regular, then for any open set $\Omega \subset M$ and any set $A \Subset \Omega$, there is a cutoff function of (A, Ω) (see [16, Lemma 1.4.2(ii), p.29]).

Definition 3.5. Let Ω be an open set in M and $A \Subset \Omega$ be a Borel set. Define the *capacity* $\mathrm{cap}(A, \Omega)$ by

$$\mathrm{cap}(A, \Omega) := \inf \left\{ \mathcal{E}(\varphi) : \varphi \text{ is a cutoff function of } (A, \Omega) \right\}. \tag{3.5}$$

It follows from the definition that the capacity $\mathrm{cap}(A, \Omega)$ is increasing in A, and decreasing in Ω, namely, if $A_1 \subset A_2, \Omega_1 \supset \Omega_2$, then $\mathrm{cap}(A_1, \Omega_1) \leq \mathrm{cap}(A_2, \Omega_2)$. Using the latter property, let us extend the definition of capacity as follows.

Definition 3.6. Let Ω be an open set in M and $A \subset \Omega$ be a Borel set. Define the capacity $\mathrm{cap}(A, \Omega)$ by

$$\mathrm{cap}(A, \Omega) = \lim_{n \to \infty} \mathrm{cap}(A \cap \Omega_n, \Omega) \tag{3.6}$$

where $\{\Omega_n\}$ is any increasing sequence of precompact open subsets of Ω exhausting Ω (in particular, $A \cap \Omega_n \Subset \Omega$).

Note that by the monotonicity property of the capacity, the limit in the right hand side of (3.6) exists (finite or infinite) and is independent of the choice of the exhausting sequence $\{\Omega_n\}$.

Definition 3.7. A function u in an open set $\Omega \subset M$ is called cap-*quasi-continuous* in Ω if, for every $\varepsilon > 0$, there exists an open set $U \subset \Omega$ such that u is continuous on $\Omega \setminus U$, and

$$\mathrm{cap}(U, \Omega) < \varepsilon. \tag{3.7}$$

By Lemma 9.1 that we prove in Appendix, for any open $\Omega \subset M$, any function $u \in \mathcal{F}(\Omega)$ admits a cap-quasi-continuous version $\widetilde{u}$. This result is analogous to [16, Theorems 2.1.3 (p.71) and 2.1.6 (p.74)] that deals with another definition of quasi-continuity, related to another notion of capacity [16, pp.69,74].

Next, define the *resistance* res (A,Ω) by

$$\operatorname{res}(A,\Omega)=\frac{1}{\operatorname{cap}(A,\Omega)}. \tag{3.8}$$

Definition 3.8. We say that the *resistance condition* (R_F) is satisfied if, there exist constants $K,C>1$ such that, for any ball B of radius $r>0$,

$$C^{-1}\frac{F(r)}{\mu(B)}\leq \operatorname{res}(B,KB)\leq C\frac{F(r)}{\mu(B)}, \tag{3.9}$$

where constants K and C are independent of the ball B. Equivalently, (3.9) can be written in the form

$$\operatorname{res}(B,KB)\simeq\frac{F(r)}{\mu(B)}. \tag{R_F}$$

We introduce the notions of the Green operator and the Green function.

Definition 3.9. For an open $\Omega\subset M$, a linear operator $G^\Omega:L^2(\Omega)\to\mathcal{F}(\Omega)$ is called a *Green operator* if, for any $\varphi\in\mathcal{F}(\Omega)$ and any $f\in L^2(\Omega)$,

$$\mathcal{E}(G^\Omega f,\varphi)=(f,\varphi). \tag{3.10}$$

If G^Ω admits an integral kernel g^Ω, that is,

$$G^\Omega f(x)=\int_\Omega g^\Omega(x,y)f(y)d\mu(y) \text{ for any } f\in L^2(\Omega), \tag{3.11}$$

then g^Ω is called a *Green function.*

We will address the existence and the properties of the Green operator G^Ω in Lemma 5.1. The issue of the Green function g^Ω is much more involved, and is one of the key topics in this paper (cf. Lemmas 5.2, 5.3, and 5.7).

For an open set $\Omega\subset M$, the function E^Ω is defined by

$$E^\Omega(x):=G^\Omega\mathbf{1}(x)\quad(x\in M), \tag{3.12}$$

namely, the function E^Ω is a unique weak solution of the following Poisson-type equation

$$-\mathcal{L}^\Omega E^\Omega=1, \tag{3.13}$$

provided that $\lambda_{\min}(\Omega)>0$.

It is known that

$$E^\Omega(x)=\mathbb{E}_x(\tau_\Omega)\ \text{ for } \mu\text{-a.a. } x\in M, \tag{3.14}$$

where τ_Ω is the *first exit time* of the Hunt process $\left\{\{X_t\}_{t\geq0},\{\mathbb{P}_x\}_{x\in M}\right\}$ associated with $(\mathcal{E},\mathcal{F})$, that is

$$\tau_\Omega=\inf\{t>0:X_t\notin\Omega\}, \tag{3.15}$$

where $X_t\notin\Omega$ means that either $X_t\in M\setminus\Omega$, or $X_t=\infty$. Clearly, if the Green function g^Ω exists, then

$$E^\Omega(x)=G^\Omega\mathbf{1}(x)=\int_\Omega g^\Omega(x,y)\,d\mu(y) \tag{3.16}$$

for μ-almost all $x\in M$.

Definition 3.10. We say that *condition* (E_F) holds if, there exist two constants $C>1$ and $\delta_1\in(0,1)$ such that, for any ball B of radius $r>0$,

$$\operatorname*{esup}_B E^B \leq CF(r), \tag{$E_F\leq$}$$

$$\operatorname*{einf}_{\delta_1 B} E^B \geq C^{-1}F(r). \tag{$E_F\geq$}$$

Next we introduce condition (G_F).

Definition 3.11. We say that *condition* (G_F) holds if, there exist constants $K > 1$ and $\dot{C} > 0$ such that, for any ball $B := B(x_0, R)$, the Green kernel g^B exists and is jointly continuous off the diagonal, and satisfies

$$g^B(x_0, y) \leq C \int_{K^{-1}d(x_0,y)}^{R} \frac{F(s)\,ds}{sV(x_0, s)} \text{ for all } y \in B \setminus \{x_0\}, \tag{$G_F \leq$}$$

$$g^B(x_0, y) \geq C^{-1} \int_{K^{-1}d(x_0,y)}^{R} \frac{F(s)\,ds}{sV(x_0, s)} \text{ for all } y \in K^{-1}B \setminus \{x_0\}. \tag{$G_F \geq$}$$

The following theorem is a key in our paper.

Theorem 3.12. *Let (M, d, μ) be a metric measure space, where all metric balls are precompact. Assume that $(\mathcal{E}, \mathcal{F})$ is a regular, strongly local Dirichlet form in $L^2(M, \mu)$. If (VD) and (RVD) are satisfied, then the following equivalences take place:*

$$(H) + (R_F) \Leftrightarrow (G_F) \Leftrightarrow (H) + (E_F).$$

Remark 3.13. Condition (RVD) is required only for proving the implication $(H) + (E_F) \Rightarrow (R_F \geq)$.

The proof of this theorem is quite involved, including numerous lemmas and propositions. We give the flowchart of the proof on the following diagram:

$$\begin{array}{ccccc}
 & L.5.7 & & & \\
 & \downarrow & & & \\
 & L.7.2 & \longleftarrow & L.7.1 & \\
 & \downarrow & & \uparrow & \\
L.7.3 \longrightarrow & L.7.4 & & L.6.5 & \\
 & \downarrow & \swarrow & & \nwarrow \\
 & \boxed{T.3.12} & & & L.6.2 \text{ and } L.6.4 \\
 & & \nwarrow & & \swarrow \\
 & & & L.8.2 &
\end{array}$$

Before stating the second theorem of this paper, we introduce more conditions.

(UE) *Upper estimate:* the heat kernel $p_t(x, y)$ exists, has a Hölder continuous in $x, y \in M$ version, and satisfies the following upper estimate

$$p_t(x, y) \leq \frac{C}{V(x, \mathcal{R}(t))} \exp\left(-\frac{1}{2} t \Phi\left(c\frac{d(x, y)}{t}\right)\right) \tag{UE}$$

for all $t > 0$ and all $x, y \in M$. Here c, C are positive constants, $\mathcal{R} : = F^{-1}$, and

$$\Phi(s) := \sup_{r>0} \left\{\frac{s}{r} - \frac{1}{F(r)}\right\}.$$

(NLE) *Near-diagonal lower estimate*: the heat kernel $p_t(x, y)$ exists, has a Hölder continuous in $x, y \in M$ version. and satisfies the lower estimate

$$p_t(x, y) \geq \frac{c}{V(x, \mathcal{R}(t))}, \tag{NLE}$$

for all $t > 0$ and all $x, y \in M$ such that $d(x, y) \leq \eta \mathcal{R}(t)$, where $\eta > 0$ is a sufficiently small constant.

Denote by (UE_{weak}) a modification of condition (UE) that is obtained by removing the Hölder continuity of $p_t(x, y)$ and by relaxing inequality (UE) to $\mu \times \mu$-almost all $x, y \in M$. In a similar way, we can define *condition* (NLE_{weak}).

Theorem 3.14. *Let (M,d,μ) be a metric measure space, where all metric balls are precompact. Assume that $(\mathcal{E},\mathcal{F})$ is a regular strongly local Dirichlet form in $L^2(M,\mu)$. Assume also that (VD) and (RVD) are satisfied. Then the following sets of conditions are equivalent:*

$$\begin{aligned}(H)+(E_F) &\Leftrightarrow (G_F)\Leftrightarrow (H)+(R_F)\\ &\Leftrightarrow (UE)+(NLE)\\ &\Leftrightarrow (UE_{weak})+(NLE_{weak}).\end{aligned}$$

Proof. The first line of equivalences is contained in Theorem 3.12. Denote by $(\widetilde{E}_F)$ the following condition:

$$\mathbb{E}_x\tau_{B(x,r)}\simeq F(r) \tag{$\widetilde{E}_F$}$$

for all $r>0$ and $x\in M\setminus\mathcal{N}$, where $\mathcal{N}$ is a properly exceptional set[2]. Let us show that the following implications take place:

$$\begin{array}{ccc} & \boxed{(UE)+(NLE)} & \\ & \Updownarrow & \\ & \boxed{(UE_{weak})+(NLE_{weak})} & \\ \Updownarrow & & \Uparrow \\ \boxed{(H)+(\widetilde{E}_F)} & \Rightarrow & \boxed{(H)+(E_F)}\end{array}$$

which contains the remaining equivalences in the statement of Theorem 3.14. Indeed, by [26, Theorem 7.4] we have the equivalences

$$(H)+(\widetilde{E}_F)\Leftrightarrow (UE_{weak})+(NLE_{weak})\Leftrightarrow (UE)+(NLE). \tag{3.17}$$

Let us verify that

$$(\widetilde{E}_F)\Rightarrow (E_F). \tag{3.18}$$

Indeed, let $B:=B(x_0,r)$ be any metric ball in M. For any $x\in B\setminus\mathcal{N}$ we have, using $(\widetilde{E}_F)$ and $B\subset B(x,2r)$, that

$$\mathbb{E}_x\tau_B\le \mathbb{E}_x\tau_{B(x,2r)}\le CF(2r)\le C'F(r).$$

Hence, it follows from (3.14) that

$$\operatorname*{esup}_B E^B=\operatorname*{esup}_{x\in B}\mathbb{E}_x\tau_B\le C'F(r),$$

thus proving $(E_F\le)$. On the other hand, for any $x\in\frac{1}{2}B\setminus\mathcal{N}$, we have, using $(\widetilde{E}_F)$ and $B(x,r/2)\subset B$, that

$$\mathbb{E}_x\tau_B\ge \mathbb{E}_x\tau_{B(x,r/2)}\ge C^{-1}F(r/2)\ge CF(r),$$

and thus,

$$\operatorname*{einf}_{B/2}E^B=\operatorname*{einf}_{x\in B/2}\mathbb{E}_x\tau_B\ge CF(r),$$

hence, proving $(E_F\ge)$ and as well as (3.18).

It remains to prove that

$$(H)+(E_F)\Rightarrow (UE_{weak})+(NLE_{weak}).$$

For that we use the proof of (3.17) in [26] and verify that the condition $(\widetilde{E}_F)$ in that proof can be replaced by a priori weaker condition (E_F). By [26, Theorem 3.11] we have

$$(H)+(E_F)\Rightarrow (FK),$$

where (FK) denotes a certain *Faber-Krahn type inequality* (see [26, Definition 3.9]). It follows from the inequality [22, (6.34)] that

$$(E_F)\Rightarrow (S_F),$$

[2]A set $\mathcal{N}\subset M$ is called properly exceptional if it is Borel, $\mu(\mathcal{N})=0$ and

$$\mathbb{P}_x\left(X_t\in\mathcal{N}\text{ or }X_{t-}\in\mathcal{N}\text{ for some }t\ge 0\right)=0$$

for all $x\in M\setminus\mathcal{N}$ (see [16, p.152 and Theorem 4.1.1 on p.155]).

where (S_F) stands for a *survival estimate* defined by [20, (5.23)]. By [20, Theorem 2.1] we have

$$(FK) + (S_F) \Rightarrow (UE_{weak}),$$

which implies

$$(H) + (E_F) \Rightarrow (UE_{weak}).$$

Arguing as in [26, Section 5.4], one obtains

$$(H) + (E_F) \Rightarrow (NLE_{weak}),$$

which finishes the proof. □

4. Maximum principles

We give three maximum principles, and the first two are for a subharmonic function on one open set, and the third is for a subharmonic function on the difference of two open sets. All of them will be used later on.

Lemma 4.1 (Maximum principle)**.** *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form in $L^2(M, \mu)$. Let $\Omega \subset M$ be open such that $\lambda_{\min}(\Omega) > 0$, and let $\Omega_1 \Subset \Omega$ be open. Assume that $u \geq 0$ in M.*

(1) *If u is subharmonic in Ω, then (see Fig. 1 below)*

$$\operatorname*{esup}_{\Omega} u \leq \operatorname*{esup}_{M \setminus \Omega_1} u. \tag{4.1}$$

Consequently, if in addition u vanishes outside Ω, then

$$\operatorname*{esup}_{\Omega} u = \operatorname*{esup}_{\Omega \setminus \Omega_1} u. \tag{4.2}$$

(2) *Assume in addition that $(\mathcal{E}, \mathcal{F})$ is strongly local, Ω is precompact, and that $u \in L^\infty(M)$. If u is subharmonic (resp. superharmonic) in Ω, then for any open $\Omega_2 \Supset \Omega$,*

$$\operatorname*{esup}_{\Omega} u \leq \operatorname*{esup}_{\Omega_2 \setminus \Omega_1} u, \tag{4.3}$$

$$\left(\textit{resp. } \operatorname*{einf}_{\Omega} u \geq \operatorname*{einf}_{\Omega_2 \setminus \Omega_1} u\right). \tag{4.4}$$

Moreover, if u is continuous in a neighborhood of $\partial\Omega$, the above inequalities can be replaced by

$$\operatorname*{esup}_{\overline{\Omega}} u = \sup_{\partial\Omega} u \tag{4.5}$$

$$\left(\textit{resp. } \operatorname*{einf}_{\overline{\Omega}} u = \inf_{\partial\Omega} u\right), \tag{4.6}$$

where $\partial\Omega = \overline{\Omega} \setminus \Omega$, the boundary of Ω.

Proof. (1). Assume that $\operatorname{esup}_{M\setminus\Omega_1} u$ is finite; otherwise (4.1) is automatically true. If (4.1) fails, there would have a finite positive number c such that

$$\operatorname*{esup}_{\Omega} u > c > \operatorname*{esup}_{M \setminus \Omega_1} u.$$

Since $c \geq 0$, the function $\varphi := (u-c)_+$ is a normal contraction of u ([16, p.5]), and thus, $\varphi \in \mathcal{F}$. Moreover, $\varphi \in \mathcal{F}(\Omega)$ since $(u-c)_+ = 0$ outside Ω_1. Using the subharmonicity of u and the Markov property of $(\mathcal{E}, \mathcal{F})$ (cf. [19, Lemma 4.3]), it follows that

$$\begin{aligned} 0 &\geq \mathcal{E}(u, \varphi) = \mathcal{E}(u, (u-c)_+) \\ &\geq \mathcal{E}((u-c)_+) \geq \lambda_{\min}(\Omega) \left\|(u-c)_+\right\|_2^2 > 0, \end{aligned} \tag{4.7}$$

a contradiction, thus proving (4.1).

If in addition $u = 0$ in $M \setminus \Omega$, we have

$$\operatorname*{esup}_{M \setminus \Omega_1} u = \operatorname*{esup}_{\Omega \setminus \Omega_1} u.$$

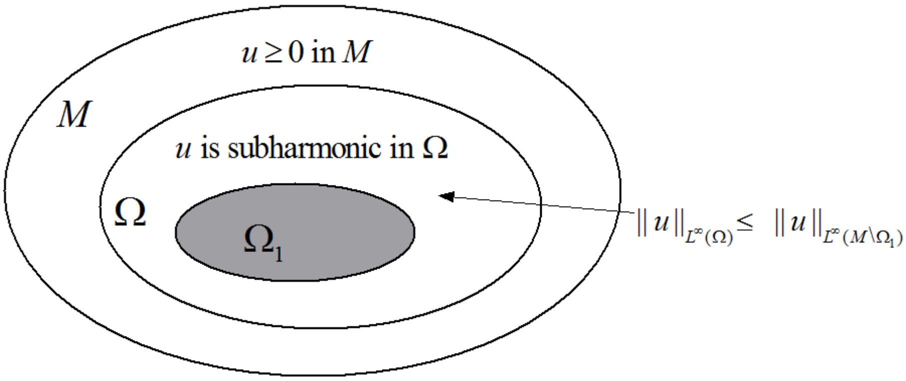

FIGURE 1. Maximum principle.

Hence, it follows from (4.1) that

$$\operatorname*{esup}_{\Omega\setminus\Omega_1} u \leq \operatorname*{esup}_{\Omega} u \leq \operatorname*{esup}_{\Omega\setminus\Omega_1} u,$$

showing (4.2).

(2). Let ψ be a cut-off function of $\left(\overline{\Omega}, \Omega_2\right)$. Since $u, \psi \in \mathcal{F} \cap L^\infty$, we see that $u\psi \in \mathcal{F} \cap L^\infty$. For any $\varphi \in \mathcal{F}(\Omega)$, observe that the product of the two functions $u(\psi - 1)$ and φ is equal to zero, and so (cf. [40, Prop. 4.1])

$$\mathcal{E}(u(\psi - 1), \varphi) = 0.$$

We first assume that u is subharmonic in Ω. It follows that

$$\mathcal{E}(u\psi, \varphi) = \mathcal{E}(u, \varphi) + \mathcal{E}(u(\psi - 1), \varphi) = \mathcal{E}(u, \varphi) \leq 0, \tag{4.8}$$

namely, the function $u\psi$ is also subharmonic in Ω. By (4.1), we have

$$\operatorname*{esup}_{\Omega} u = \operatorname*{esup}_{\Omega} (u\psi) \leq \operatorname*{esup}_{M\setminus\Omega_1} (u\psi) \leq \operatorname*{esup}_{\Omega_2\setminus\Omega_1} u,$$

proving (4.3).

We next assume that u is superharmonic in Ω. Similar to (4.8), the function $u\psi$ is also superharmonic in Ω. To show (4.4), consider the function $v := (a - u)\psi$, where $a := \operatorname{esup}_M u$. Then $v \geq 0$ in M, and is subharmonic in Ω since for any $\varphi \in \mathcal{F}(\Omega)$, using the strong locality of $(\mathcal{E}, \mathcal{F})$,

$$\mathcal{E}(v, \varphi) = a\mathcal{E}(\psi, \varphi) - \mathcal{E}(u\psi, \varphi) = -\mathcal{E}(u\psi, \varphi) \leq 0.$$

Hence, we see from (4.1) that

$$\operatorname*{esup}_{\Omega}(a - u) = \operatorname*{esup}_{\Omega} v \leq \operatorname*{esup}_{M\setminus\Omega_1} v \leq \operatorname*{esup}_{\Omega_2\setminus\Omega_1} (a - u),$$

proving (4.4).

Finally, if u is continuous in a neighborhood of $\partial\Omega$, we have that, letting $\Omega_2 \downarrow \Omega$,

$$\operatorname*{esup}_{\Omega_2\setminus\Omega_1} u \to \sup_{\overline{\Omega}\setminus\Omega_1} u.$$

Similarly, letting $\Omega_1 \uparrow \Omega$, we have

$$\sup_{\overline{\Omega}\setminus\Omega_1} u \to \sup_{\overline{\Omega}\setminus\Omega} u = \sup_{\partial\Omega} u.$$

Therefore, it follows from (4.3) that

$$\operatorname*{esup}_{\overline{\Omega}} u \leq \sup_{\partial\Omega} u,$$

which gives (4.5), by using the fact that $\sup_{\partial\Omega} u \le \operatorname{esup}_{\overline{\Omega}} u$ as $\partial\Omega \subset \overline{\Omega}$. The equality (4.6) can be proved similarly. $\square$

The second maximum principle is for a subharmonic function u where we do not know a-priori whether or not u keeps the same sign in the whole domain M, as required in the first maximum principle, although this function u turns out to be non-positive hereafter. This maximum principle will be used in the proof of Lemma 6.4 (b).

For an open $U \subset M$ and $u, v \in \mathcal{F}$, denote by

$$u \le v \mod \mathcal{F}(U), \ \ (\text{resp. } u = v \mod \mathcal{F}(U))$$

if there exists some $h \in \mathcal{F}(U)$ such that $u - v \le h$ in M (resp. $u - v = h$ in M).

Proposition 4.2. *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form. Let U be open such that $\lambda_{\min}(U) > 0$. If*

$$\begin{cases} u \text{ is subharmonic in } U, \\ \quad u \le 0 \mod \mathcal{F}(U), \end{cases} \tag{4.9}$$

then $u \le 0$ in U (and thus also in M).

Proof. Since $u \le 0 \mod \mathcal{F}(U)$, we have that $u_+ \in \mathcal{F}(U)$ (cf. [19, Lemma 4.4, p.114]). Since u is subharmonic in U, we have that, for any non-negative $\varphi \in \mathcal{F}(U)$,

$$\mathcal{E}(u, \varphi) \le 0.$$

Letting $\varphi = u_+$ and noting that

$$\mathcal{E}(u_+, u_-) = \lim_{t\to 0} \frac{1}{t}(u_+ - P_t u_+, u_-) = -\lim_{t\to 0} \frac{1}{t}(P_t u_+, u_-) \le 0,$$

we obtain that

$$\begin{aligned} 0 \ &\ge\ \mathcal{E}(u, u_+) = \mathcal{E}(u_+) - \mathcal{E}(u_-, u_+) \\ &\ge\ \mathcal{E}(u_+) \ge 0, \end{aligned}$$

and thus, $\mathcal{E}(u_+) = 0$. Therefore,

$$\|u_+\|^2_{L^2(U)} \le \frac{\mathcal{E}(u_+)}{\lambda_{\min}(U)} = 0,$$

which implies that $u \le 0$ in U. $\square$

Finally, we present a third maximum principle where the domain is the difference of two open sets. It will be used in the proofs of Lemmas 5.3 and 7.4.

Proposition 4.3. *Assume that $(\mathcal{E}, \mathcal{F})$ is regular, local. Let Ω be open such that $\lambda_{\min}(\Omega) > 0$, and let $A \subset \Omega$ be compact. Let $0 \le u \in \mathcal{F}(\Omega) \cap L^\infty$ and is subharmonic in $\Omega \setminus A$, and is continuous in some neighborhood of ∂U, for any open U with $A \Subset U \Subset \Omega$. Then,*

$$\operatorname*{esup}_{\Omega\setminus U} u = \sup_{\partial U} u. \tag{4.10}$$

Proof. Since we always have that $\operatorname{esup}_{\Omega\setminus U} u \ge \sup_{\partial U} u$, assume on the contrary that

$$m := \sup_{\partial U} u < \operatorname*{esup}_{\Omega\setminus U} u,$$

and we will deduce a contradiction.

Choose a small $\varepsilon > 0$ such that

$$\operatorname*{esup}_{\Omega\setminus U} u \ge m + \varepsilon. \tag{4.11}$$

Choose an open set V such that $A \subset V \subset U$, and

$$\sup_{U\setminus V} u \le m + \varepsilon/2.$$

Let φ be a cutoff function of (V,U). Consider the function

$$u^* := u - u\varphi.$$

Clearly, $u^* \in \mathcal{F} \cap L^\infty$, $u^*|_V = 0$, and

$$u^* \le u \le m + \varepsilon/2 \ \text{ in } U \setminus V.$$

Hence, the function $v := (u^* - (m+\varepsilon/2))_+$ satisfies that $v|_U = 0$. Since $v \in \mathcal{F}(\Omega)$, by Proposition 9.3 in Appendix, we have that $v \in \mathcal{F}(\Omega \setminus A)$.

On the other hand, using the locality of $(\mathcal{E},\mathcal{F})$ and the fact that $\varphi v = 0$, we have

$$\mathcal{E}(u\varphi, v) = 0.$$

Therefore, by the subharmonicity of u, we obtain

$$\mathcal{E}(u^*, v) = \mathcal{E}(u,v) - \mathcal{E}(u\varphi, v) \le 0.$$

It follows that

$$0 \ge \mathcal{E}(u^*, v) \ge \mathcal{E}(v) \ge \lambda_{\min}(\Omega) ||v||^2_{L^2(\Omega\setminus A)},$$

showing that $v = 0$ in $\Omega \setminus A$. Hence,

$$u^* \le m + \varepsilon/2 \ \text{ in } \Omega \setminus A,$$

in particular, we have that $u^* \le m + \varepsilon/2$ in $\Omega \setminus U$. But this is a contradiction by noting that $u^* = u$ in $\Omega \setminus U$ and using (4.11). □

5. Green operator and Green function

5.1. Green operator. We give the existence of the Green operator, and present its properties.

Lemma 5.1. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in $L^2(M,\mu)$, and let $\Omega \subset M$ be open such that $\lambda_{\min}(\Omega) > 0$. Let $\mathcal{L}^\Omega$ be the generator of $(\mathcal{E},\mathcal{F}(\Omega))$, and set $G^\Omega = (-\mathcal{L}^\Omega)^{-1}$, the inverse[3] of $-\mathcal{L}^\Omega$. Then the following statements are true.*

(1) $\left\|G^\Omega\right\| \le \lambda_{\min}(\Omega)^{-1}$, *that is, for any $f \in L^2(\Omega)$*

$$\left\|G^\Omega f\right\|_{L^2(\Omega)} \le \lambda_{\min}(\Omega)^{-1} \left\|f\right\|_{L^2(\Omega)}. \tag{5.1}$$

(2) *For any $f \in L^2(\Omega)$, we have that $G^\Omega f \in \mathcal{F}(\Omega)$, and*

$$\mathcal{E}(G^\Omega f, \varphi) = (f,\varphi) \ \textit{ for any } \varphi \in \mathcal{F}(\Omega). \tag{5.2}$$

(3) *For any $f \in L^2(\Omega)$,*

$$G^\Omega f = \int_0^\infty P_s^\Omega f \, ds. \tag{5.3}$$

(4) *G^Ω is non-negative definite: $G^\Omega f \ge 0$ if $f \ge 0$.*

Proof. (1). It is trivial since $\operatorname{spec}\left(G^\Omega\right) \subset [0, \lambda_{\min}(\Omega)^{-1}]$, and so $\left\|G^\Omega\right\| \le \lambda_{\min}(\Omega)^{-1}$.

(2). Let $u = G^\Omega f$. Then u lies in the domain of $\mathcal{L}^\Omega$, and hence, for any $\varphi \in \mathcal{F}(\Omega)$,

$$\mathcal{E}(G^\Omega f, \varphi) = \mathcal{E}(u,\varphi) = -\left(\mathcal{L}^\Omega u, \varphi\right) = (f,\varphi).$$

(3). Using the spectral resolution, we see that

$$P_s^\Omega f = \int_{\lambda_{\min}(\Omega)}^\infty e^{-s\lambda} dE_\lambda^\Omega f,$$

[3]Since $\lambda_1(\Omega) > 0$, the operator $-\mathcal{L}^\Omega$ has a bounded inverse in $L^2(\Omega,\mu)$.

and hence,

$$\begin{aligned}\int_0^\infty P_s^\Omega f\, ds &= \int_0^\infty \left(\int_{\lambda_{\min}(\Omega)}^\infty e^{-s\lambda} dE_\lambda^\Omega f\right) ds \\ &= \int_{\lambda_{\min}(\Omega)}^\infty \left(\int_0^\infty e^{-s\lambda} ds\right) dE_\lambda^\Omega f \\ &= \int_{\lambda_{\min}(\Omega)}^\infty \lambda^{-1} dE_\lambda^\Omega f = (-\mathcal{L}^\Omega)^{-1} f,\end{aligned}$$

showing (5.3).

(4). Finally, since $P_s^\Omega f \geq 0$ if $f \geq 0$ for any $s \geq 0$, we see from (5.3) that G^Ω is non-negative definite. □

5.2. **Harnack inequality and existence of Green function.** If condition (H) holds, we will show that the Green function g^Ω exists and is jointly continuous off diagonal.

Lemma 5.2. *Assume that $(\mathcal{E}, \mathcal{F})$ is strongly local, regular, and that conditions (H) and (VD) hold. Let $\Omega \subset M$ be open such that $\lambda_{\min}(\Omega) > 0$. Then there exists a function $g^\Omega(x, y)$ defined for $(x, y) \in \Omega \times \Omega \setminus diag$ with the following properties:*

1. *$G^\Omega f(x) = \int_\Omega g^\Omega(x, z) f(z) d\mu(z)$ for any $f \in L^2(\Omega)$ and a.e. $x \in \Omega$.*
2. *$g^\Omega(x, y) = g^\Omega(y, x) \geq 0$.*
3. *$g^\Omega(x, y)$ is jointly continuous in $(x, y) \in \Omega \times \Omega \setminus diag$.*
4. *For any ball B with $\overline{B} \subset \Omega$ and any $y \in \Omega \setminus B$,*

$$\sup_{x \in \delta B} g^\Omega(x, y) \leq C_H \inf_{x \in \delta B} g^\Omega(x, y), \tag{5.4}$$

where constants C_H, δ are the same as in condition (H).

Proof. The proof is quite long. We first show the existence of $g^\Omega(x, y)$ for $(x, y) \in \Omega \times \Omega \setminus diag$.

Fix a point $x \in \Omega$, a ball $B := B(x, R) \Subset \Omega$, and set $U = \Omega \setminus \overline{B}$. Let f be any non-negative function in $L^2(\Omega)$ that vanishes outside U. Then $G^\Omega f$ is harmonic in B because for any $\varphi \in \mathcal{F}(B)$,

$$\mathcal{E}(G^\Omega f, \varphi) = (f, \varphi) = 0.$$

Hence, by condition (H) and (5.1),

$$\begin{aligned}\operatorname*{esup}_{\delta B} G^\Omega f &\leq C_H \operatorname*{einf}_{\delta B} G^\Omega f \\ &\leq C_H \left(\frac{1}{\mu(\delta B)} \int_{\delta B} \left(G^\Omega f\right)^2 d\mu\right)^{1/2} \\ &\leq C_H \mu(\delta B)^{-1/2} \left\|G^\Omega f\right\|_{L^2(\Omega)} \\ &\leq C_H \mu(\delta B)^{-1/2} \lambda_{\min}(\Omega)^{-1} \|f\|_{L^2(\Omega)} = C_1(\Omega, B) \|f\|_{L^2(U)},\end{aligned} \tag{5.5}$$

where the constant $C_1(\Omega, B)$ is given by

$$C_1(\Omega, B) = \frac{C_H}{\lambda_{\min}(\Omega)\sqrt{\mu(\delta B)}}.$$

Since $(\mathcal{E},\mathcal{F})$ is strongly local, using (5.5) and the fact that $G^\Omega f\geq 0$, the harmonic function $G^\Omega f|_B$ satisfies the following oscillation property: for any ball $B(z,\rho)\subset\delta B$ and any $0<r\leq\rho$,

$$\begin{aligned}\operatorname*{Osc}_{B(z,r)} G^\Omega f \quad &: = \operatorname*{esup}_{B(z,r)} G^\Omega f-\operatorname*{einf}_{B(z,r)} G^\Omega f\\ &\leq 2\left(\frac{r}{\rho}\right)^\theta \operatorname*{Osc}_{B(z,\rho)} G^\Omega f\\ &\leq 2\left(\frac{r}{\rho}\right)^\theta \operatorname*{esup}_{\delta B} G^\Omega f \qquad (5.6)\\ &\leq 2C_1(\Omega,B)\left(\frac{r}{\rho}\right)^\theta \|f\|_{L^2(U)}, \qquad (5.7)\end{aligned}$$

where $\theta>0$ is a constant depending only on constants C_H,δ in condition (H), see [26, Lemma 5.2]. Thus the function $G^\Omega f$ admits a Hölder continuous version in δB, that will also be denoted by $G^\Omega f$.

It follows from (5.5) that

$$G^\Omega f(x)\leq C_1(\Omega,B)\|f\|_{L^2(U)}$$

so that the mapping $f\mapsto G^\Omega f(x)$ is a bounded linear functional on $L^2(U)$. By the Riesz representation theorem, there exists a unique $g_x^{\Omega,U}(\cdot)\in L^2(U)$ that is non-negative in U and such that

$$G^\Omega f(x)=\int_U g_x^{\Omega,U}(z)f(z)d\mu(z) \text{ for any } f\in L^2(U).$$

Let $\{B_k\}_{k\geq 1}$ be a shrinking sequence of balls centered at x such that $\cap B_k=\{x\}$, and let $U_k=\Omega\setminus\overline{B_k}$. Then we obtain a sequence of the functions g_x^{Ω,U_k} that is consistent in the sense that

$$\left.g_x^{\Omega,U_{k+1}}\right|_{U_k}=g_x^{\Omega,U_k}.$$

This allows us to define a function g_x^Ω on $\Omega\setminus\{x\}$ by

$$g_x^\Omega=g_x^{\Omega,U_k} \text{ on } U_k.$$

By construction, $g_x^\Omega\in L^2_{loc}(\Omega\setminus\{x\})$, is non-negative in $\Omega\setminus\{x\}$ and satisfies

$$G^\Omega f(x)=\int_\Omega g_x^\Omega(z)f(z)d\mu(z) \tag{5.8}$$

for any $f\in L^2(U_k)$ and $k\geq 1$.

We claim that (5.8) also holds for any $f\in L^2(\Omega)$, that is,

$$G^\Omega f(x)=\int_\Omega g_x^\Omega(z)f(z)d\mu(z) \text{ for any } f\in L^2(\Omega). \tag{5.9}$$

Indeed, set $f_k=f\mathbf{1}_{U_k}$ for any non-negative $f\in L^2(\Omega)$. Since (5.8) holds for f_k:

$$G^\Omega f_k(x)=\int_\Omega g_x^\Omega(z)f_k(z)d\mu(z), \tag{5.10}$$

we let $k\to\infty$ and obtain that

$$G^\Omega f_k\to G^\Omega f \text{ in } L^2(\Omega)$$

by using the monotone convergence theorem, because $f_k\overset{L^2(\Omega)}{\to} f$ and G^Ω is bounded in $L^2(\Omega)$ by (5.1). This proves our claim.

Observe that for any ball $A\Subset U$,

$$\left\|G^\Omega 1_{\delta A}\right\|_{L^\infty(\delta B)}\leq\frac{C_H\sqrt{\mu(\delta A)}}{\lambda_{\min}(\Omega)\sqrt{\mu(\delta B)}}, \tag{5.11}$$

since, taking $f=\mathbf{1}_{\delta A}$ in (5.5), we see that

$$\left\|G^{\Omega}\mathbf{1}_{\delta A}\right\|_{L^{\infty}(\delta B)}\leq C_1(\Omega,B)\left\|\mathbf{1}_{\delta A}\right\|_{L^2(\delta A)}=\frac{C_H}{\lambda_{\min}(\Omega)\sqrt{\mu(\delta B)}}\mu(\delta A)^{1/2}.$$

Let us show that $G^{\Omega}:L^1(\delta A)\to L^{\infty}(\delta B)$ is bounded, that is, for any $f\in L^1(\delta A)$,

$$\max_{\delta B}G^{\Omega}f\leq\frac{(C_H)^2}{\lambda_{\min}(\Omega)\sqrt{\mu(\delta B)\mu(\delta A)}}\left\|f\right\|_{L^1(\delta A)}. \tag{5.12}$$

Indeed, interchanging the balls A and B in (5.11), we obtain that

$$\left\|G^{\Omega}\mathbf{1}_{\delta B}\right\|_{L^{\infty}(\delta A)}\leq\frac{C_H\sqrt{\mu(\delta B)}}{\lambda_{\min}(\Omega)\sqrt{\mu(\delta A)}}. \tag{5.13}$$

Hence, for any non-negative $f\in L^1(\delta A)$,

$$\begin{aligned}\left\|G^{\Omega}f\right\|_{L^1(\delta B)} &= \left(G^{\Omega}f,\mathbf{1}_{\delta B}\right)=\left(f,G^{\Omega}\mathbf{1}_{\delta B}\right)\\ &\leq \left\|f\right\|_{L^1(\delta A)}\left\|G^{\Omega}\mathbf{1}_{\delta B}\right\|_{L^{\infty}(\delta A)}\\ &\leq \frac{C_H\sqrt{\mu(\delta B)}}{\lambda_{\min}(\Omega)\sqrt{\mu(\delta A)}}\left\|f\right\|_{L^1(\delta A)}.\end{aligned}$$

Therefore, using condition (H),

$$\begin{aligned}\max_{\delta B}G^{\Omega}f &\leq C_H\min_{\delta B}G^{\Omega}f\\ &\leq C_H\left(\frac{1}{\mu(\delta B)}\left\|G^{\Omega}f\right\|_{L^1(\delta B)}\right)\\ &\leq \frac{(C_H)^2}{\lambda_{\min}(\Omega)\sqrt{\mu(\delta B)\mu(\delta A)}}\left\|f\right\|_{L^1(\delta A)}, \qquad (5.14)\end{aligned}$$

proving (5.12).

Now for $y\in U$, let $\{\varepsilon_n\}_{n\geq1}$ be a decreasing sequence of positive numbers shrinking to 0 such that $A:=B(y,\varepsilon_1)\subset U$, see Figure 2.

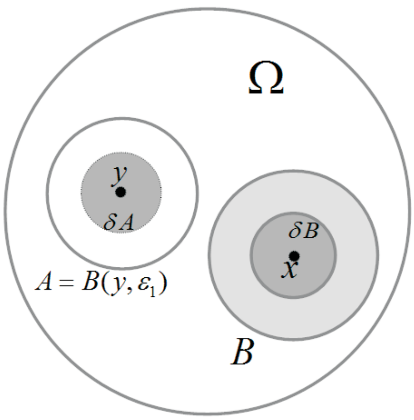

FIGURE 2. Domains A and B.

Let $u_{n,y}:=G^{\Omega}f_{n,y}$, where

$$f_{n,y}=\frac{1}{\mu(B(y,\varepsilon_n))}\mathbf{1}_{B(y,\varepsilon_n)},$$

such that $f_{n,y} \rightharpoonup \delta_y$ weakly in $C_0(M)$ as $n \to \infty$, where δ_y is the usual Dirac function concentrated at point y. It follows from (5.6) and (5.12) that for $B(z,\rho) \subset \delta B$ and $0 < r < \rho$,

$$\begin{aligned} \operatorname*{Osc}_{B(z,r)} u_{n,y} &\leq 2\left(\frac{r}{\rho}\right)^{\theta} \operatorname*{esup}_{\delta B} u_{n,y} \\ &\leq 2\left(\frac{r}{\rho}\right)^{\theta} \frac{(C_H)^2}{\lambda_{\min}(\Omega)\sqrt{\mu(\delta B)\mu(\delta A)}} \|f_{n,y}\|_{L^1(\delta A)} \\ &= 2\left(\frac{r}{\rho}\right)^{\theta} \frac{(C_H)^2}{\lambda_{\min}(\Omega)\sqrt{\mu(\delta B)\mu(\delta A)}}. \end{aligned} \tag{5.15}$$

Therefore, the sequence $\{u_{n,y}\}$ is uniformly bounded and equicontinuous in δB. By the Arzelà-Ascoli theorem, there exists a subsequence $\{u_{n_k,y}\}$ that is uniformly convergent in δB. In fact, the limit is g_y^Ω, that is,

$$g_y^\Omega(z) = \lim_{k\to\infty} G^\Omega f_{n_k,y}(z) \text{ uniformly for } z \in \delta B, \tag{5.16}$$

because, for any $\varphi \in C_0(\delta B)$, using (5.9),

$$\begin{aligned} (u_{n_k,y}, \varphi) &= \left(G^\Omega f_{n_k,y}, \varphi\right) = \left(f_{n_k,y}, G^\Omega \varphi\right) \\ &\to G^\Omega\varphi(y) = \left(g_y^\Omega, \varphi\right), \end{aligned}$$

and hence,

$$u_{n_k,y} \rightharpoonup g_y^\Omega \text{ weakly in } C_0(\delta B) \text{ as } k \to \infty.$$

We now define the function $g^\Omega(y,x)$ by

$$g^\Omega(y,x) := g_y^\Omega(x) = \lim_{k\to\infty} G^\Omega f_{n_k,y}(x) \geq 0$$

for almost all $(x,y) \in \Omega \times \Omega \setminus diag$.

We next show that such $g^\Omega(y,x)$ satisfies all the properties (1)-(4).

Indeed, property (1) is clear by (5.9). Property (2) follows by using (5.9),

$$\begin{aligned} g^\Omega(y,x) &= \lim_{k\to\infty} G^\Omega f_{n_k,y}(x) = \lim_{k\to\infty} \int_\Omega g_x^\Omega(z) f_{n_k,y}(z) d\mu(z) \\ &= g_x^\Omega(y) = g^\Omega(x,y). \end{aligned}$$

To show property (3), we have from (5.15) that, for any $0 < r < \delta R$,

$$\operatorname*{Osc}_{B(x,r)} G^\Omega f_{n_k,y} \leq 2\left(\frac{r}{\delta R}\right)^{\theta} \frac{(C_H)^2}{\lambda_{\min}(\Omega)\sqrt{\mu(\delta B)\mu(\delta A)}},$$

and hence, passing to the limit as $k \to \infty$,

$$\operatorname*{Osc}_{B(x,r)} g^\Omega(y,\cdot) \leq 2\left(\frac{r}{\delta R}\right)^{\theta} \frac{(C_H)^2}{\lambda_{\min}(\Omega)\sqrt{\mu(\delta B)\mu(\delta A)}}.$$

It follows that $g^\Omega(\cdot,y)$ is Hölder continuous in δB locally uniformly for $y \in U$, and thus, the function g^Ω is jointly continuous away from the diagonal.

More precisely, for any $x_1, y_1 \in \Omega$ and any $r_1, r_2 > 0$ such that $B(x_1,r_1) \cap B(y_1,r_2) = \emptyset$, and $B(x_1,r_1) \subset \Omega, B(y_1,r_2) \subset \Omega$, we have that

$$\begin{aligned} \left|g^\Omega(x_1,y_1) - g^\Omega(x_2,y_2)\right| \leq\ & \frac{2\delta^{-\theta}(C_H)^2}{\lambda_{\min}(\Omega)\sqrt{V(x_1,\delta r_1)V(y_1,\delta r_2)}} \left[\left(\frac{d(x_1,x_2)}{r_1}\right)^{\theta} \right. \\ & \left. + \left(\frac{d(y_1,y_2)}{r_2}\right)^{\theta}\right], \end{aligned} \tag{5.17}$$

where $x_2 \in B(x_1,\delta r_1)$ and $y_2 \in B(y_1,\delta r_2)$.

Finally, to show the property (4), let B be an arbitrary ball with $\overline{B} \subset \Omega$, and let $y \in B^c$. Note that $u_{n,y}$ satisfies condition (H) in δB uniformly for $n \geq 1$, that is,

$$\max_{\delta B} G^\Omega f_{n_k,y} \leq C_H \min_{\delta B} G^\Omega f_{n_k,y}. \tag{5.18}$$

Passing to the limit as $k \to \infty$, we obtain (5.4). □

The next is the maximum-minimum principle for the Green function $g^\Omega(x_0, \cdot)$. Since we do not know whether or not the function $g^\Omega(x_0, \cdot)$ belongs to $\mathcal{F}$, making it harmonic in $\Omega \setminus \{x_0\}$, we are not able to apply directly the maximum principles established before, as often did when M is a graph or a manifold.

Lemma 5.3. *Assume that all the hypotheses of Lemma 5.2 hold. If $x_0 \in U \Subset \Omega$, then*

$$\inf_{U \setminus \{x_0\}} g^\Omega(x_0, \cdot) = \inf_{\partial U} g^\Omega(x_0, \cdot), \tag{5.19}$$

$$\sup_{\Omega \setminus U} g^\Omega(x_0, \cdot) = \sup_{\partial U} g^\Omega(x_0, \cdot). \tag{5.20}$$

Proof. Let $\Omega_n \uparrow \Omega$ such that Ω_n is precompact open, $\Omega_n \supset U$ for each n. Let $U_k \downarrow \{x_0\}$ such that each U_k is open, and $U_1 \Subset U$. Let

$$u_k := G^\Omega f_{k,x_0}$$

where $f_{k,x_0} \rightharpoonup \delta_{x_0}$ weakly in $C(M)$ as $k \to \infty$, for example $f_{k,x_0} = \frac{1}{\mu(U_k)} 1_{U_k}$. By the proof of Lemma 5.2, the sequence $\{u_k\}_{k=1}^\infty$ converges uniformly to $g^\Omega(x_0, \cdot)$ on each compact subset of $\Omega \setminus \{x_0\}$, as $k \to \infty$.

We first prove (5.19). To do this, note that each $u_k = G^\Omega f_{k,x_0}$ is superharmonic in Ω (and in particular in U), since for any non-negative $\varphi \in \mathcal{F}(\Omega)$,

$$\mathcal{E}(u_k, \varphi) = (f_{k,x_0}, \varphi) \geq 0.$$

As U is precompact, we have from (4.6) that, for each k,

$$\operatorname*{einf}_{\overline{U}} u_k = \inf_{\partial U} u_k. \tag{5.21}$$

Clearly, for each n, we have that $\partial U \subset \overline{U} \setminus U_n \subset \overline{U}$, and thus

$$\operatorname*{einf}_{\overline{U}} u_k \leq \operatorname*{einf}_{\overline{U} \setminus U_n} u_k \leq \inf_{\partial U} u_k$$

for each k. Combining this with (5.21), we see

$$\operatorname*{einf}_{\overline{U} \setminus U_n} u_k = \inf_{\partial U} u_k.$$

Letting $k \to \infty$, we obtain that

$$\inf_{\overline{U} \setminus U_n} g^\Omega(x_0, \cdot) = \inf_{\partial U} g^\Omega(x_0, \cdot),$$

and then letting $n \to \infty$, we conclude that (5.19) holds.

We next show (5.20). In fact, since each u_k is harmonic in $\Omega \setminus \overline{U_1}$, it follows from Proposition 4.3 that

$$\sup_{\Omega_n \setminus U} u_k = \sup_{\partial U} u_k$$

for each n. Letting $k \to \infty$, we have that

$$\sup_{\Omega_n \setminus U} g^\Omega(x_0, \cdot) = \sup_{\partial U} g^\Omega(x_0, \cdot),$$

and then letting $n \to \infty$ and using the continuity of $g^\Omega(x_0, \cdot)$ off diagonal, we conclude that (5.20) holds. □

It is not hard to see that (5.4) is equivalent to the following: if

$$d(z_1, z_2) < \delta \left[d(x_0, z_1) \wedge d(x_0, z_2) \right], \tag{5.22}$$

for any points $x_0, z_1, z_2 \in \Omega$, then $g^\Omega(x_0, z_1) \simeq g^\Omega(x_0, z_2)$, that is,

$$C^{-1} g^\Omega(x_0, z_2) \le g^\Omega(x_0, z_1) \le C g^\Omega(x_0, z_2) \tag{5.23}$$

for some $C > 0$.

We introduce the Harnack inequality for the Green function g^Ω.

Definition 5.4. We say that the Green function g^Ω satisfies the Harnack inequality if g^Ω is jointly continuous off diagonal, and if there exist some (large) constants K, C such that for any ball $B = B(x_0, R)$ and for any precompact open set $\Omega \supset KB$,

$$\sup_{\partial B} g^\Omega(x_0, \cdot) \le C \inf_{\partial B} g^\Omega(x_0, \cdot), \tag{HG}$$

where C may depend on K, but both K and C are independent of the ball B and the set Ω.

We will show that (HG) is true if conditions (H) and (VD) hold. For doing his, we need the relatively connected property of balls.

Definition 5.5. *A metric space (M, d) is* relatively (ε, K)-ball-connected *if, for constants $\varepsilon \in (0, 1)$ and $K > 1$, there exists an integer $N = N(\varepsilon, K)$ such that for any ball $B(x_0, KR)$ and for any two points $x, y \in \overline{B(x_0, R)}$, there is a chain of balls $\{B_i\}_{i=0}^N$ of the same radius εR inside $B(x_0, KR)$ connecting x and y, that is,*

$$x \in B_0 \sim B_1 \sim B_2 \sim \cdots \sim B_N \ni y,$$

where $B_i \sim B_j$ means that $B_i \cap B_j \neq \emptyset$, see Figure 3.

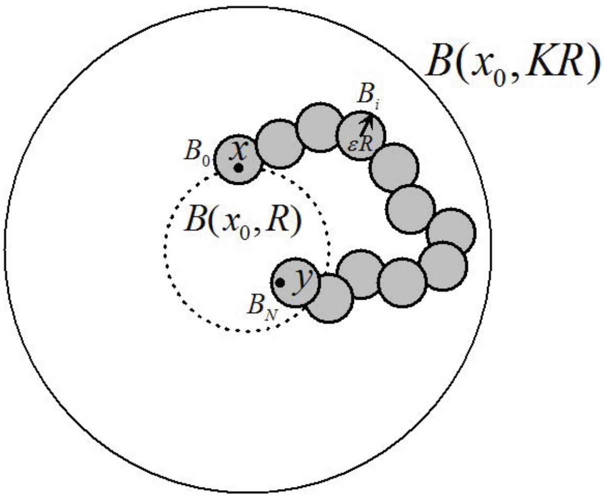

FIGURE 3. Balls $\{B_i\}_{i=0}^N$ connecting two points x and y.

We give a sufficient condition for the ball-connectedness.

Proposition 5.6. *Assume that $(\mathcal{E}, \mathcal{F})$ is a strongly local, regular Dirichlet form, and that conditions (H) and (VD) hold. Then (M, d) is relatively (ε, K)-ball-connected for any $\varepsilon \in (0, 1)$ and any $K > \delta^{-1}$, with the same δ as in condition (H).*

Proof. Fix $\varepsilon \in (0, 1)$ and $K > \delta^{-1}$, and let $B := B(x_0, R)$. For the ball $B(x_0, KR)$, by condition (VD), there exists a finite number of balls $\{B_i\}_{i=0}^N$ of the same radius εR that covers $B(x_0, KR)$, where N depends only on K, ε (cf. [29, Theorem 1.16, p.8]). It suffices to show that if $X_1, X_2 \in \{B_i\}$ and $X_j \cap \overline{B} \neq \emptyset$ $(j = 1, 2)$, then X_1 and X_2 can be connected by a chain of balls from $\{B_i\}$.

To see this, denote by Ω the union of all the balls in $\{B_i\}$ that can be connected to X_1. Clearly, the set Ω is open. We claim that Ω is also closed in $B(x_0, KR)$.

Indeed, for any point $y \in B(x_0, KR) \setminus \Omega$, there exists a ball X in $\{B_i\}$ such that $y \in X$. If X intersects one of the balls in Ω, then $X \subset \Omega$, which contradicts the fact that $y \notin \Omega$. Thus, X does not intersect any ball from Ω, that is $\Omega \cap X = \emptyset$, and y has an open neighborhood $X \cap B(x_0, KR)$ outside Ω. Therefore, the set $B(x_0, KR) \setminus \Omega$ is open, showing that Ω is closed in $B(x_0, KR)$.

Let $Y := B(x_0, \delta^{-1}R)$ so that $B \subset Y \subset B(x_0, KR)$, and let

$$A = \Omega \cap \overline{Y} = \overline{\Omega} \cap \overline{Y}.$$

Then A is compact. Let u be a cut-off function of (A, Ω). We will show that u is harmonic in Y. In fact, for any $\varphi \in \mathcal{F} \cap C_0(Y)$, we have that $\operatorname{supp}(u\varphi) \subset \Omega \cap \overline{Y} = A$ whilst $u \equiv 1$ in a neighborhood of A, see Figure 4. Hence, using the strong locality, we have that $\mathcal{E}(u, u\varphi) = 0$.

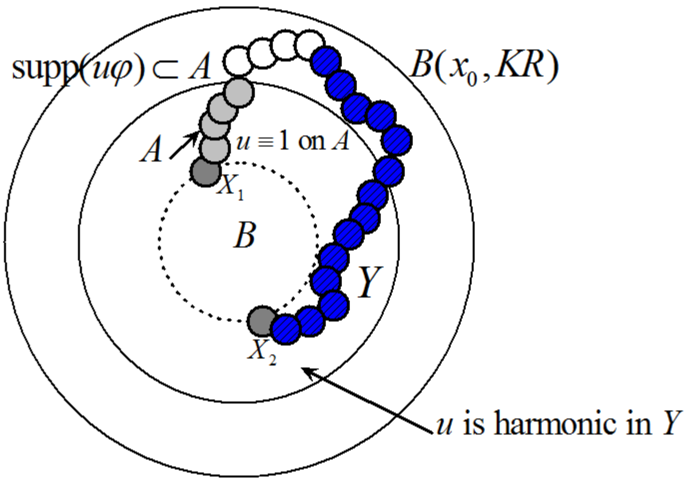

FIGURE 4. function u and domains A, Y.

Similarly, $\mathcal{E}(u, \varphi(1-u)) = 0$ because $\operatorname{supp}(\varphi(1-u)) \subset \overline{Y} \cap \overline{A^c} \subset \overline{Y} \cap \overline{\Omega^c}$ whilst $u = 0$ in $\overline{\Omega^c}$. Therefore,

$$\mathcal{E}(u, \varphi) = \mathcal{E}(u, u\varphi) + \mathcal{E}(u, \varphi(1-u)) = 0,$$

proving that u is harmonic in Y.

Hence, we can apply condition (H) for the non-negative harmonic function u for the pair (B, Y).

Let $x \in X_1 \cap \overline{B} \subset \Omega \cap \overline{Y} = A$. For any $y \in X_2 \cap \overline{B}$, we obtain

$$1 = u(x) \le C_H u(y),$$

which gives that $u(y) > 0$. Thus, $y \in \Omega$ since u is a cut-off function of (A, Ω) and $u = 0$ in Ω^c. Hence, $X_2 \cap \overline{B} \subset \Omega$, showing that X_2 can be connected to X_1 by a chain of balls in $\{B_i\}$. The proof is complete. □

The last part of the above proof was motivated by that in [26, Theorem 7.3(a)].

We next show that condition (HG) holds.

Lemma 5.7. *Assume that all the hypotheses in Lemma 5.2 are satisfied, then condition (HG) is true where $\dot{K} > \delta^{-1}$. Consequently, for any ball $KB \subset \Omega$ with center x_0,*

$$\sup_{\Omega \setminus B} g^{\Omega}(x_0, \cdot) \le C \inf_B g^{\Omega}(x_0, \cdot)$$

for some $C > 0$ independent of the ball B and Ω.

Proof. First observe that (M,d) is relatively ball-connected by using Proposition 5.6. Fix a ball $B := B(x_0,R)$, and let Ω be open such that $B(x_0,KR) \subset \Omega$. Since $g^\Omega(x_0,\cdot)$ is continuous on ∂B, let x and y be two points on ∂B such that

$$\begin{aligned} g^\Omega(x_0,x) &= \sup_{\partial B} g^\Omega(x_0,\cdot), \\ g^\Omega(x_0,y) &= \inf_{\partial B} g^\Omega(x_0,\cdot). \end{aligned}$$

We need to show that

$$g^\Omega(x_0,x) \le C g^\Omega(x_0,y). \tag{5.24}$$

Clearly, if $d(x,y) < \delta R$, then (5.24) with $C = C_H$ follows from (5.4). In the sequel, we assume that $d(x,y) \ge \delta R$.

Let $\varepsilon = \delta^3/4$, and let $\{B_i\}_{i=0}^N$ be any fixed chain of balls with the same radius εR in $B(x_0,KR)$ connecting x and y. Denote by $B_i := B(\xi_i,\varepsilon R)$, and note that

$$x \in B_0 \sim B_1 \sim B_2 \sim \cdots \sim B_N \ni y.$$

We will prove (5.24) according to the whereabouts of the centers $\{\xi_0,\xi_1,\xi_2,\cdots,\xi_N\}$ of the balls $\{B_i\}_{i=0}^N$. We distinguish two cases.

Case 1: $d(x_0,\xi_i) > \delta R$ for each i (see Figure 5).

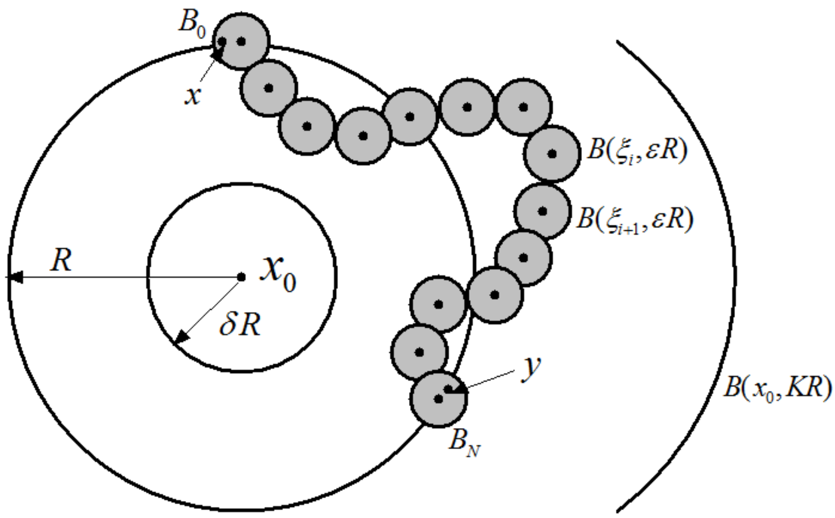

FIGURE 5. The point x_0 lies outside each of the balls $B(\xi_i,\delta R)$.

Consider the function $g^\Omega(x_0,\cdot)$. For $i = 0,\cdots,N-1$, note that

$$\begin{aligned} d(\xi_i,\xi_{i+1}) &< 2\varepsilon R = \delta^3 R/2 < \delta(\delta R) \\ &< \delta \min\left\{d(x_0,\xi_i), d(x_0,\xi_{i+1})\right\}. \end{aligned}$$

Applying (5.23), we obtain that $g^\Omega(x_0,\xi_i) \simeq g^\Omega(x_0,\xi_{i+1})$, and thus,

$$g^\Omega(x_0,\xi_0) \simeq g^\Omega(x_0,\xi_N).$$

Also we have

$$\begin{aligned} g^\Omega(x_0,x) &\simeq g^\Omega(x_0,\xi_0), \\ g^\Omega(x_0,\xi_N) &\simeq g^\Omega(x_0,y). \end{aligned}$$

Therefore, we conclude that

$$g^\Omega(x_0,x) \simeq g^\Omega(x_0,y),$$

proving (5.24).

Case 2: $d(x_0, \xi_i) \le \delta R$ for some i.

Let $x' := \xi_k$ be the point from $\{\xi_0, \xi_1, \cdots, \xi_N\}$ such that all the centers $\xi_0, \xi_1, \xi_2, \cdots, \xi_k$ lie outside $B(x_0, \delta R)$ whilst the next center ξ_{k+1} lies inside $B(x_0, \delta R)$. Denote by $x'' := \xi_{k+1}$ (see Fig. 6).

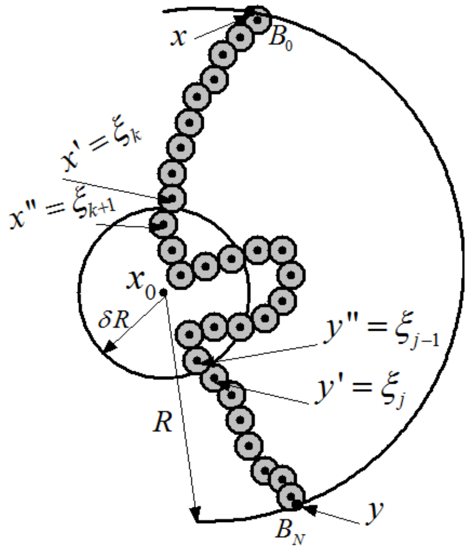

FIGURE 6. The points x', x'' and y', y''.

At the same time, let $y' := \xi_j$ be the point from $\{\xi_0, \xi_1, \cdots, \xi_N\}$ such that ξ_{j-1} lies inside $B(x_0, \delta R)$ whilst all the next centers $\xi_j, \xi_{j+1}, \cdots, \xi_N$ lie outside $B(x_0, \delta R)$. Denote by $y'' := \xi_{j-1}$. At this stage, we do not care about any ball with the center in $\left\{\xi_{k+2}, \xi_{k+3}, \cdots, \xi_{j-2}\right\}$ if any.

We further distinguish three cases.

Case (2a): There exists a point η from $\left\{y', \xi_{j+1}, \cdots, \xi_N\right\}$ such that

$$d(x', \eta) \le \frac{2\delta^2}{3} R.$$

(See Fig 7).

By Case 1, we have already proved that

$$\begin{aligned} g^\Omega(x_0, x') &\simeq g^\Omega(x_0, x), \\ g^\Omega(x_0, \eta) &\simeq g^\Omega(x_0, y). \end{aligned} \tag{5.25}$$

On the other hand, consider the function $g^\Omega(x_0, \cdot)$. Since

$$d(x', \eta) \le \frac{2\delta^2}{3} R < \delta^2 R < \delta \min\left\{d(x_0, x'), d(x_0, \eta)\right\},$$

we see by (5.23) that

$$g^\Omega(x_0, x') \simeq g^\Omega(x_0, \eta),$$

which combines with (5.25) to show that (5.24) also holds.

Case (2b): There exists a point ξ from $\left\{\xi_0, \xi_1, \xi_2, \cdots, \xi_{k-1}, x'\right\}$ such that

$$d(y', \xi) \le \frac{2\delta^2}{3} R.$$

In this case, we can similarly prove that (5.24) holds, as we did in Case (2a).

Case (2c): $d(x', z) > \frac{2\delta^2}{3} R$ for all $z \in \left\{y', \xi_{j+1}, \cdots, \xi_N\right\}$, and $d(y', z) > \frac{2\delta^2}{3} R$ for all $z \in \left\{\xi_0, \xi_1, \xi_2, \cdots, \xi_{k-1}, x'\right\}$ (see Fig. 6).

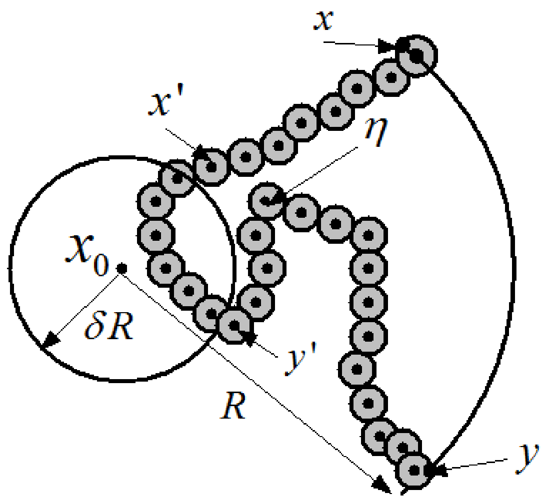

FIGURE 7. The points x' and η are close.

Consider the function $g^{\Omega}(x', \cdot)$. For each $k = j, j+1, \cdots, N-1$, we see that

$$\begin{aligned} d(\xi_k, \xi_{k+1}) &\leq 2\varepsilon R = \frac{\delta^3 R}{2} < \delta\left(\frac{2\delta^2}{3}R\right) \\ &< \delta \min\left\{d(x', \xi_k), d(x', \xi_{k+1})\right\}. \end{aligned}$$

Applying (5.23), we have that $g^{\Omega}(x', \xi_k) \simeq g^{\Omega}(x', \xi_{k+1})$, and so

$$g^{\Omega}(x', y') \simeq g^{\Omega}(x', y). \tag{5.26}$$

On the other hand, consider the function $g^{\Omega}(y, \cdot)$. Since

$$\begin{aligned} d(y, x'') &> d(y, x_0) - d(x_0, x'') > R - \delta R > \delta^2 R, \\ d(y, x') &> d(y, x_0) - d(x_0, x'') - d(x'', x') \\ &> R - \delta R - 2\varepsilon R > \delta^2 R, \end{aligned}$$

we see that

$$\begin{aligned} d(x'', x') &< 2\varepsilon R = \frac{\delta^3 R}{2} < \delta(\delta^2 R) \\ &< \delta \min\left\{d(y, x'), d(y, x'')\right\}. \end{aligned}$$

Thus, we have by (5.23) that

$$g^{\Omega}(y, x') \simeq g^{\Omega}(y, x''). \tag{5.27}$$

Also noting that $d(x'', x_0) < \delta R$, and $d(y, x_0) = R$, we apply (5.4) to obtain that

$$g^{\Omega}(y, x'') \simeq g^{\Omega}(y, x_0). \tag{5.28}$$

Therefore, as $g^{\Omega}(x', y) = g^{\Omega}(y, x')$, it follows from (5.26)-(5.28) that

$$g^{\Omega}(x', y') \simeq g^{\Omega}(y, x_0).$$

Similarly, we obtain that

$$g^{\Omega}(x', y') \simeq g^{\Omega}(x, x_0).$$

Therefore, we conclude that (5.24) also holds. □

6. Some Potential theory

6.1. **Riesz measures associated with superharmonic functions.** For any open $\Omega \subset M$, we show that any non-negative superharmonic function $f \in \mathcal{F}(\Omega)$ admits a regular Borel measure ν_f such that f can be expressed as an integral of the Green function g^Ω with respect to ν_f. This measure ν_f is called a *Riesz measure* associated with f. Recall that for the classical case, F. Riesz proved this theorem, now called the Riesz decomposition theorem (cf. [1, T.4.4.1, p.105, and Def. 4.3.4, p.102]).

Lemma 6.1. *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form. Let $\Omega \subset M$ be non-empty open, and let $f \in \mathcal{F}$ in M.*

- (a) *If f is superharmonic in Ω and if either one of the following two conditions is satisfied:*
  - (1) *$f \geq 0$ in M;*
  - (2) *$f \in \mathcal{F}(\Omega)$ (f being not necessarily non-negative in M);*

  then $P_t^\Omega f \leq f$ in Ω for all $t > 0$.
- (b) *If $P_t^\Omega f \leq f$ in Ω for all $t > 0$ and $f \in \mathcal{F}(\Omega)$, then f is superharmonic in Ω.*

Consequently, when $\Omega = M$, any non-negative function f is superharmonic in M if and only if $P_t f \leq f$ for all $t > 0$

Proof. (a). The function $u(t, \cdot) := P_t^\Omega f - f$ is a *weak subsolution* of the heat equation in $\mathbb{R}_+ \times \Omega$ (cf. [19, Example 4.10, p.117]), and satisfies the initial condition

$$u_+(t, \cdot) \overset{L^2(\Omega)}{\longrightarrow} 0 \text{ as } t \to 0.$$

We need to verify the boundary condition

$$u_+(t, \cdot) \in \mathcal{F}(\Omega). \tag{6.1}$$

If $f \geq 0$ in M, then $u(t, \cdot) = P_t^\Omega f - f \leq P_t^\Omega f$ in M, and thus, by [19, Lemma 4.4], condition (6.1) is true. If $f \in \mathcal{F}(\Omega)$, so is $u(t, \cdot)$, and (6.1) is also true. In both cases, using the parabolic maximum principle (see [19, Prop. 4.11, p.117]), we obtain that $u \leq 0$ in $(0, \infty) \times \Omega$, that is, $P_t^\Omega f \leq f$ in Ω for all $t > 0$.

(b). Assume now that $P_t^\Omega f \leq f \in \mathcal{F}(\Omega)$. Then, for any non-negative function $\varphi \in \mathcal{F}(\Omega)$,

$$\mathcal{E}(f, \varphi) = \lim_{t \to 0} \left(\frac{f - P_t^\Omega f}{t}, \varphi \right) \geq 0,$$

which means that f is superharmonic in Ω. □

We will show that the Riesz measure exists for any non-negative superharmonic function.

Lemma 6.2. *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form, and let $\Omega \subset M$ be an open set. Let $f \in \mathcal{F}(\Omega)$ be a non-negative superharmonic function in Ω.*

- (a) *Then there is a regular Borel measure ν_f on Ω such that*

$$\frac{f - P_t^\Omega f}{t} \rightharpoonup \nu_f \text{ as } t \to 0, \tag{6.2}$$

  where the convergence is weak in $C_0(\Omega)$. Moreover, measure ν_f does not charge any open set where f is harmonic.
- (b) *Assume further that $\lambda_{\min}(\Omega) > 0$ and that the Green function g^Ω exists and is jointly continuous off diagonal. Assume in addition that the function f is bounded in Ω and harmonic in $U = \Omega \setminus S$ where $S \subset \Omega$ is a compact set. Then*

$$f(x) = \int_S g^\Omega(x, y)\, d\nu_f(y) \tag{6.3}$$

  for μ-a.a. $x \in \Omega$.

It follows from (6.2) that, for any $\varphi \in \mathcal{F} \cap C_0(\Omega)$,

$$\mathcal{E}(f, \varphi) = \int_\Omega \varphi d\nu_f. \tag{6.4}$$

Recall that if $f \in \operatorname{dom} \mathcal{L}^\Omega$, then $\mathcal{E}(f, \varphi) = \left(-\mathcal{L}^\Omega f, \varphi\right)$. Hence, the identity (6.4) allows to define

$$-\mathcal{L}^\Omega f := \nu_f \tag{6.5}$$

for any non-negative superharmonic function $f \in \mathcal{F}(\Omega)$.

Proof. (a) For any $t > 0$ and $\varphi \in C_0(\Omega)$, set

$$\mathcal{E}_t(f, \varphi) := \left(\frac{f - P_t^\Omega f}{t}, \varphi\right)$$

so that $\varphi \mapsto \mathcal{E}_t(f, \varphi)$ is a linear functional in $C_0(\Omega)$. Let us show that $\lim_{t\to 0} \mathcal{E}_t(f, \varphi)$ exists for all $\varphi \in C_0(\Omega)$. Fix a precompact open set $V \subset \Omega$ and we shall prove that $\lim_{t\to 0} \mathcal{E}_t(f, \varphi)$ exists for all $\varphi \in C_0(V)$ (which will imply the same for all $\varphi \in C_0(\Omega)$). Let ψ be a cutoff function of $\left(\overline{V}, \Omega\right)$. Then, as $t \to 0$,

$$\begin{aligned} \left\| \frac{f - P_t^\Omega f}{t} \right\|_{L^1(V)} &\le \int_\Omega \frac{f - P_t^\Omega f}{t} \psi d\mu \\ &= \mathcal{E}_t(f, \psi) \to \mathcal{E}(f, \psi). \end{aligned}$$

It follows that, for sufficiently small $t > 0$ and for all $\varphi \in C_0(V)$,

$$\begin{aligned} |\mathcal{E}_t(f, \varphi)| &\le \left\| \frac{f - P_t^\Omega f}{t} \right\|_{L^1(V)} \sup|\varphi| \\ &\le [\mathcal{E}(f, \psi) + 1] \sup|\varphi|, \end{aligned}$$

that is, $\mathcal{E}_t(f, \varphi)$ is a bounded linear functional of $\varphi \in C_0(V)$, and the norm of this functional is bounded uniformly in t. Since $\lim_{t\to 0} \mathcal{E}_t(f, \varphi)$ exists (and is equal to $\mathcal{E}(f, \varphi)$) for all $\varphi \in \mathcal{F}$, in particular, for $\varphi \in \mathcal{F} \cap C_0(V)$, and the latter set is dense in $C_0(V)$ by the regularity of $(\mathcal{E}, \mathcal{F})$, it follows that $\lim_{t\to 0} \mathcal{E}_t(f, \varphi)$ exists for all $\varphi \in C_0(V)$.

Since $\mathcal{E}_t(f, \varphi) \ge 0$ for non-negative φ, the $\lim_{t\to 0} \mathcal{E}_t(f, \varphi)$ is a non-negative linear functional on $C_0(\Omega)$. By the Riesz representation theorem, the functional $\lim_{t\to 0} \mathcal{E}_t(f, \varphi)$ determines a regular Borel measure ν_f on Ω, so that

$$\lim_{t\to 0} \mathcal{E}_t(f, \varphi) = \int_\Omega \varphi d\nu_f \text{ for all } \varphi \in C_0(\Omega). \tag{6.6}$$

If f is harmonic in an open set U, then $\mathcal{E}(f, \varphi) = 0$ for all $\varphi \in \mathcal{F}(U)$. It follows that $\mathcal{E}_t(f, \varphi) \to 0$ as $t \to 0$ for all $\varphi \in \mathcal{F} \cap C_0(U)$, and hence,

$$\int_U \varphi d\nu_f = 0$$

for all such φ. Since $\mathcal{F} \cap C_0(U)$ is dense in $C_0(U)$, we conclude that $\nu_f = 0$ on U.

(b) Since g^Ω is jointly continuous off diagonal and measure μ is non-atomic, we see that $g^\Omega(x, y)$ is measurable with respect to $d\nu_f(y)d\mu(x)$, as the measure of the diagonal is zero. Then the integral

$$\int_M \int_M g^\Omega(x, y)\varphi(x) d\nu_f(y) d\mu(x)$$

is defined for all $\varphi \in C_0(M)$, and hence, by Fubini's theorem, the integral

$$\int_M \left[\int_M g^\Omega(x, y)\varphi(x) d\mu(x)\right] d\nu_f(y)$$

is also defined. Therefore, the function

$$G^\Omega \varphi = \int_M g^\Omega(x, y)\varphi(x) d\mu(x)$$

is ν_f-measurable.

Let us prove that, for any fixed non-negative $\varphi \in C_0(\Omega)$,

$$\mathcal{E}\left(f, G^{\Omega}\varphi\right) = \int_S G^{\Omega}\varphi \, d\nu_f. \tag{6.7}$$

Observe first that

$$\left\|G^{\Omega} f\right\|_{\infty} \leq \frac{1}{\lambda_{\min}(\Omega)} \|f\|_{\infty},$$

that is, G^{Ω} is a bounded operator in $L^{\infty}(\Omega)$ (see (8.20) below, or [26, Lemma 3.2]). Hence, the function $u := G^{\Omega}\varphi$ is a non-negative bounded function on Ω. Recall that, by (5.2),

$$\mathcal{E}(f, u) = \mathcal{E}\left(f, G^{\Omega}\varphi\right) = (f, \varphi). \tag{6.8}$$

Let ψ_1 be a cutoff function of S in some small neighborhood of S. Let V be a precompact open neighborhood of $\operatorname{supp}\psi_1$. By Lemma 9.1 from Appendix, the function u is cap-quasi-continuous in Ω, and, hence, in V. That is, for any $\varepsilon > 0$, there is an open set $E \subset V$ such that $\operatorname{cap}(E, V) < \varepsilon/2$, and u is continuous in $V \setminus E$. By the properties of capacity we have also

$$\operatorname{cap}(E, \Omega) < \varepsilon/2.$$

Since $E \Subset \Omega$, there exists a cutoff function ψ_2 of (E, Ω) such that $\mathcal{E}(\psi_2) < \varepsilon$ (see Fig. 8).

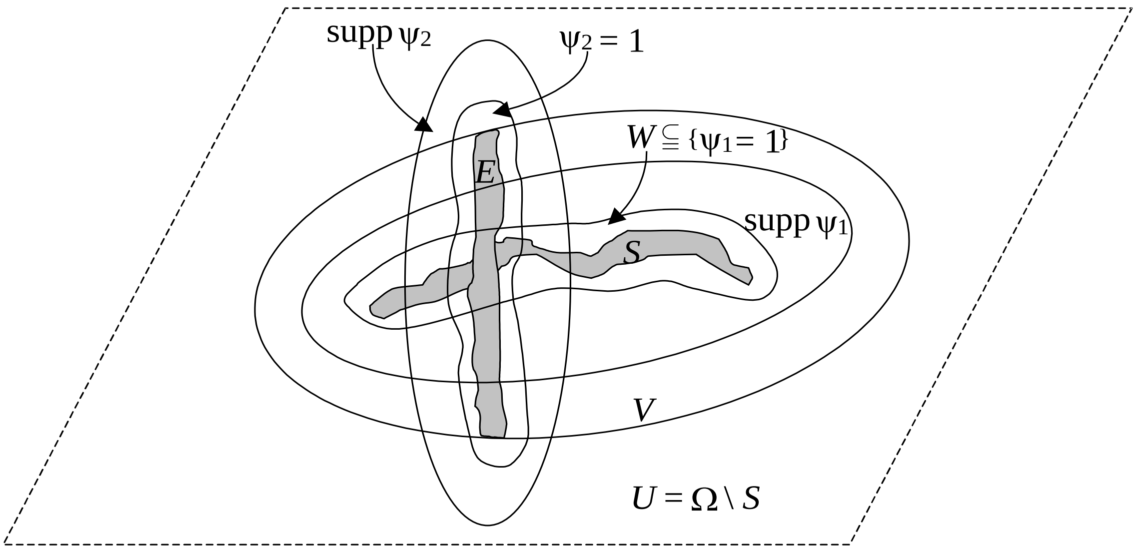

FIGURE 8. Functions ψ_1 and ψ_2

Since $u \in \mathcal{F} \cap L^{\infty}$, we see that the following three functions are also in $\mathcal{F} \cap L^{\infty}$:

$$u_1 := u\psi_1(1 - \psi_2), \quad u_2 = u\psi_1\psi_2, \quad u_3 = u(1 - \psi_1).$$

Note that $u_1 + u_2 + u_3 = u$ in M. Let us investigate the terms in (6.7) separately for each of the functions u_i.

By construction, u_1 has compact support and is continuous in Ω. Indeed, u_1 vanishes on the sets $\{\psi_1 = 0\}$ and $\{\psi_2 = 1\}$ while on $\{\psi_1 > 0\} \cap \{\psi_2 < 1\}$ the function u is continuous. By (6.4), we have

$$\mathcal{E}(f, u_1) = \int_{\Omega} u_1 d\nu_f = \int_S u(1 - \psi_2) \, d\nu_f,$$

where we have used the fact that $\nu_f(S^c) = 0$ and $\psi_1 \equiv 1$ on S. It follows that

$$\left|\mathcal{E}(f, u_1) - \int_S u d\nu_f\right| \leq \|u\|_{\infty} \int_S \psi_2 d\nu_f = \|u\|_{\infty} \mathcal{E}(f, \psi_2).$$

Next, we have

$$\begin{aligned} |\mathcal{E}(f,u_2)| &= \lim_{t\to 0}\left(\frac{f-P_t^{\Omega}f}{t}, u\psi_1\psi_2\right) \\ &\le \|u\|_\infty \lim_{t\to 0}\left(\frac{f-P_t^{\Omega}f}{t}, \psi_2\right) = \|u_\infty\| \, \mathcal{E}(f,\psi_2). \end{aligned}$$

The function u_3 vanishes in an open neighborhood W of S (where $\psi_1 = 1$), we have that $u_3 \in \mathcal{F}(U)$ by using Proposition 9.3 in Appendix. Since f is harmonic in U, we obtain

$$\mathcal{E}(f,u_3) = 0.$$

Adding up the above estimates of $\mathcal{E}(f,u_i)$ and using the fact that

$$\mathcal{E}(f,\psi_2) \le \mathcal{E}(f)^{1/2}\mathcal{E}(\psi_2)^{1/2} \le \mathcal{E}(f)^{1/2}\varepsilon^{1/2},$$

we obtain

$$\left|\mathcal{E}(f,u) - \int_S u d\nu_f\right| \le 2\|u_\infty\| \, \mathcal{E}(f)^{1/2}\varepsilon^{1/2}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that (6.7) holds.

Finally, for any $0 \le \varphi \in C_0(\Omega)$, we have that, using (6.8) and (6.7),

$$\begin{aligned} \int_\Omega f(x)\varphi(x)\,d\mu(x) &= \mathcal{E}(f,u) = \int_S G^{\Omega}\varphi(y)\,d\nu_f(y) \\ &= \int_S\left(\int_\Omega g^{\Omega}(y,x)\varphi(x)\,d\mu(x)\right)d\nu_f(y) \\ &= \int_\Omega\left(\int_S g^{\Omega}(x,y)\,d\nu_f(y)\right)\varphi(x)\,d\mu(x), \end{aligned}$$

showing that (6.3) holds for μ-a.a. $x \in \Omega$. □

The following example says that for some superharmonic function f, the associated Riesz measure ν_f may coincide with the measure μ, that is, $\nu_f = \mu$.

Example 6.3. Let $f = E^{\Omega}\mathbf{1}_\Omega$ be the weak solution of (3.13). Then $0 \le f \in \mathcal{F}(\Omega)$, and is superharmonic in Ω since for any $0 \le \varphi \in \mathcal{F}(\Omega)$,

$$\mathcal{E}(f,\varphi) = \mathcal{E}(E^{\Omega}\mathbf{1}_\Omega,\varphi) = \int_\Omega \varphi d\mu \ge 0.$$

Hence, this function admits a Riesz measure ν_f, which actually is equal to μ, since for any $\varphi \in \mathcal{F}\cap C_0(\Omega)$,

$$\int_\Omega \varphi d\mu = \mathcal{E}(f,\varphi) = \int_\Omega \varphi d\nu_f,$$

and then use the fact that the space $\mathcal{F}\cap C_0(\Omega)$ is dense in $C_0(\Omega)$.

6.2. **Reduced function.** We introduce a reduced function $\widehat{u}$ of $u \in \mathcal{F}\cap L^\infty$ with respect to (A,Ω). Roughly speaking, a reduced function $\widehat{u}$ of (A,Ω) is the one that is obtained by cutting off u such that $\widehat{u} = u$ in A, and $\widehat{u}$ is harmonic in $\Omega\setminus A$, and $\widehat{u} \in \mathcal{F}(\Omega)$ (so that $\widehat{u}$ vanishes outside Ω).

Lemma 6.4. *Assume that $(\mathcal{E},\mathcal{F})$ is regular, and let $\Omega \subset M$ be precompact with $\lambda_{\min}(\Omega) > 0$. Let A be a compact subset of Ω and set $U = \Omega\setminus A$. Fix a function $u \in \mathcal{F}\cap L^\infty$ and fix a cutoff function ψ of (A,Ω), and let $f \in \mathcal{F}$ be the solution to the weak Dirichlet problem in U:*

$$\begin{cases} f \text{ is harmonic in } U, \\ f = u\psi \mod \mathcal{F}(U). \end{cases} \tag{6.9}$$

Define the function $\widehat{u}$ on M (see Fig. 9) by

$$\widehat{u} = \begin{cases} u & \text{in } A, \\ f & \text{in } A^c. \end{cases} \tag{6.10}$$

(*a*) *Then* $\widehat{u} \in \mathcal{F}(\Omega)$.
(*b*) *If in addition* $u \geq 0$ *in* M *and* u *is superharmonic in* Ω*, then* $\widehat{u}$ *is also superharmonic in* Ω*, and* $0 \leq \widehat{u} \leq u$ *in* M*.*

The above function $\widehat{u}$ is called a *reduced function* of u with respect to (A, Ω). For example, the capacitory potential of (A, Ω) is a reduced function of any cutoff function of $(\overline{\Omega}, M)$, see Proposition 9.2 in Appendix.

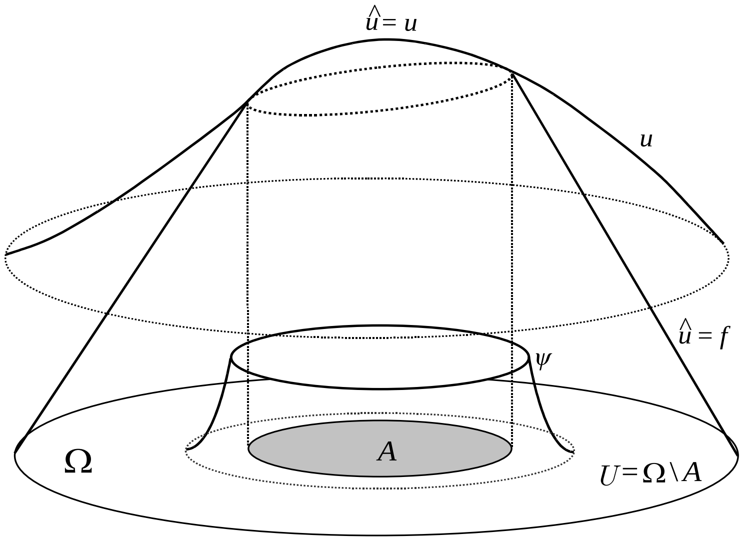

FIGURE 9. Functions u and $\widehat{u}$.

Proof. (*a*) We have $u\psi \in \mathcal{F} \cap L^{\infty}$, and the Dirichlet problem (6.9) has a unique weak solution (cf. [26, Lemma 7.1]). It follows from (6.9) that

$$v := u\psi - f \in \mathcal{F}(U).$$

Let us verify that $\widehat{u} = f$ in M, that is,

$$\widehat{u} = u\psi - v \text{ in } M. \tag{6.11}$$

Indeed, in A we have

$$\widehat{u} = u = u\psi - v$$

because $\psi \equiv 1$ and $v \equiv 0$ in A, and in A^c we have

$$\widehat{u} = f = u\psi - v$$

by the definition of v. Since $u\psi \in \mathcal{F}(\Omega)$ and $v \in \mathcal{F}(U) \subset \mathcal{F}(\Omega)$, it follows from (6.11) that $\widehat{u} \in \mathcal{F}(\Omega)$.

(*b*) Since $u\psi \geq 0$ and $\lambda_{\min}(U) \geq \lambda_{\min}(\Omega) > 0$, we have by the maximum principle (cf. Proposition 4.2) that $f \geq 0$ in M and, hence, $\widehat{u} \geq 0$ in M. The function $f - u$ is obviously subharmonic in U. Since $f - u \leq f - u\psi$ in M and $f - u\psi = 0 \mod \mathcal{F}(U)$, we have

$$f - u \leq 0 \mod \mathcal{F}(U).$$

Hence, using the maximum principle again, we obtain that $f - u \leq 0$ in M. Therefore, $\widehat{u} \leq u$ in M.

It remains to show that $\widehat{u}$ is superharmonic in Ω. By Lemma 6.1(*b*), we need to show that

$$P_t^{\Omega}\widehat{u} \leq \widehat{u} \text{ for any } t > 0. \tag{6.12}$$

Indeed, we have that in A,

$$P_t^{\Omega}\widehat{u} \leq P_t^{\Omega}u \leq u = \widehat{u}. \tag{6.13}$$

To prove (6.12) in U, observe that $w(t,\cdot) := P_t^\Omega \widehat{u} - \widehat{u}$ obviously is a weak subsolution of the heat equation in $\mathbb{R}_+ \times U$, and satisfies the initial condition

$$w_+(t,\cdot) \stackrel{L^2(U)}{\longrightarrow} 0 \text{ as } t \to 0.$$

We claim that the boundary condition

$$w_+(t,\cdot) \in \mathcal{F}(U) \tag{6.14}$$

also holds. To see this, note that, using part (a) and (6.11),

$$\begin{aligned} P_t^\Omega \widehat{u} - \widehat{u} &\le P_t^\Omega u - \widehat{u} \le u - (u\psi - v) \\ &= (1-\psi)\, u + v \text{ in } M. \end{aligned} \tag{6.15}$$

The function $h := (1-\psi)\, u$ vanishes in an open neighborhood of A, and thus, by Proposition 9.3 in Appendix, we see that $h \in \mathcal{F}(U)$. As $v \in \mathcal{F}(U)$, it follows from (6.15) that

$$w(t,\cdot) \le h + v \in \mathcal{F}(U),$$

thus proving our claim (6.14) by using Lemma 4.4 in [19].

Finally, using the parabolic maximum principle (see [19, Prop. 4.11, p.117]), we conclude that $w \le 0$ in $\mathbb{R}_+ \times U$. This finishes the proof. □

6.3. **Capacitory measure.**

We prove here some properties of the *capacitory measure* (also called the *equilibrium measure*).

Lemma 6.5. *Assume that $(\mathcal{E},\mathcal{F})$ is a strongly local, regular Dirichlet form. Let Ω, U be precompact open subset of M such that $U \Subset \Omega$. Assume that $\lambda_{\min}(\Omega) > 0$ and that the Green function g^Ω exists and is jointly continuous off diagonal. Let u_p be the capacitory potential of (U,Ω). Then there exists a regular Borel measure ν_p supported on ∂U such that*

$$\nu_p(\partial U) = \mathrm{cap}(U,\Omega) \tag{6.16}$$

and

$$u_p(x) = \int_{\partial U} g^\Omega(x,y)\, d\nu_p(y) \quad \text{for all } x \in \Omega \setminus \partial U, \tag{6.17}$$

In particular, we have

$$\int_{\partial U} g^\Omega(x,y)\, d\nu_p(y) = 1 \text{ for all } x \in U. \tag{6.18}$$

Proof. The capacitory potential satisfies the following properties: $u_p \in \mathcal{F}(\Omega), 0 \le u_p \le 1$ in Ω, $u_p|_U = 1$,

$$\mathcal{E}(u_p) = \mathrm{cap}(U,\Omega), \tag{6.19}$$

and u_p is harmonic in $\Omega \setminus \overline{U}$. Note that u_p is a reduced function of any cutoff function of $\left(\overline{\Omega}, M\right)$, and is superharmonic in Ω (cf. Proposition 9.2 in Appendix).

We claim that, for any two precompact open subsets U_1, U_2 of Ω with $U_1 \Subset U \Subset U_2$, the potential function u_p is harmonic in $\Omega \setminus S$ where $S := \overline{U_2} \setminus U_1$.

Indeed, for any $0 \le \varphi \in \mathcal{F}(\Omega \setminus S)$, by Proposition 9.4 in Appendix, we can decompose $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 \in \mathcal{F}(U), \varphi_2 \in \mathcal{F}\left(\Omega \setminus \overline{U}\right)$. Therefore, as u_p is harmonic in $\Omega \setminus \overline{U}$ and $(\mathcal{E},\mathcal{F})$ is strongly local, it follows that

$$\begin{aligned} \mathcal{E}(u_p,\varphi) &= \mathcal{E}(u_p,\varphi_1+\varphi_2) = \mathcal{E}(u_p,\varphi_1) + \mathcal{E}(u_p,\varphi_2) \\ &= \mathcal{E}(u_p,\varphi_1) = 0, \end{aligned}$$

thus proving our claim.

Therefore, by Lemma 6.2, there exists a regular Borel measure ν_p associated with u_p as in (6.2), and ν_p is supported on $S = \overline{U_2} \setminus U_1$ for any $U_1 \Subset U \Subset U_2$.

On the other hand, let $\{u_k\}_{k=1}^{\infty}$ be a minimizing sequence of u_p, that is, each u_k is a cutoff function of $\left(\overline{U},\Omega\right)$, and $\mathcal{E}\left(u_k\right)\to\mathcal{E}(u_p)$. By (6.4),

$$\mathcal{E}\left(u_p,u_k\right)=\int_S u_k d\nu_p.$$

Since $u_k=1$ in a neighborhood of $\overline{U}$, and $0\le u_p\le 1$ in M, we see that

$$\nu_p(\partial U)\le\nu_p(\overline{U}\setminus U_1)\le\int_S u_k d\nu_p\le\nu_p(\overline{U}_2\setminus U_1),$$

and hence,

$$\nu_p(\partial U)\le\mathcal{E}\left(u_p,u_k\right)\le\nu_p(\overline{U}_2\setminus U_1).$$

Letting $k\to\infty$ and then using (6.19), it follows that, for any $U_1\Subset U\Subset U_2$,

$$\nu_p(\partial U)\le\operatorname{cap}\left(U,\Omega\right)\le\nu_p(\overline{U}_2\setminus U_1).$$

By the regularity of ν_p, the measure $\nu_p(\overline{U}_2\setminus U_1)\to\nu_p(\partial U)$ as $U_1\uparrow U$ and $U_2\downarrow U$. Therefore, we conclude that

$$\operatorname{cap}\left(U,\Omega\right)=\mathcal{E}\left(u_p\right)=\nu_p(\partial U),$$

thus proving (6.16).

Finally, if g^{Ω} exists and is jointly continuous off diagonal, then (6.17) follows directly from (6.3). □

For any point $x_0\in\Omega$ and any $c>0$, consider the set

$$A_c(x_0):=\left\{y\in\Omega:g^{\Omega}\left(x_0,y\right)>c\right\}. \tag{6.20}$$

We look at the capacity $\operatorname{cap}(A_c(x_0),\Omega)$.

Proposition 6.6. *Assume that $(\mathcal{E},\mathcal{F})$ is regular, strongly local, and let $\Omega\subset M$ be precompact open such that $\lambda_{\min}(\Omega)>0$. Assume that the Green function g^{Ω} exists and is jointly continuous off diagonal. For any $c>0$, if $x_0\in A_c(x_0)$ and if $A_c(x_0)\Subset\Omega$, then*

$$\operatorname{cap}(A_c(x_0),\Omega)=\frac{1}{c}. \tag{6.21}$$

Proof. Since g^{Ω} is jointly continuous off diagonal, the set $U:=A_c(x_0)$ is an open subset of Ω, and the boundary

$$\partial U=\partial A_c(x_0)=\left\{y\in\Omega:g^{\Omega}\left(x_0,y\right)=c\right\}.$$

As $x_0\in U$, it follows from (6.18) that

$$1=\int_{\partial U}g^{\Omega}\left(x_0,y\right)d\nu_p\left(y\right)=c\nu_p\left(\partial U\right).$$

Combines this with (6.16), we obtain

$$\operatorname{cap}(U,\Omega)=\nu_p\left(\partial U\right)=\frac{1}{c}.$$

This finishes the proof. □

7. Resistance

7.1. **Green function and resistance.** The following lemma gives a two-sided estimate of the resistance $\operatorname{res}\left(U,\Omega\right)$ in terms of the values of the Green function g^{Ω} on the boundary ∂U.

Lemma 7.1. *Assume that $(\mathcal{E},\mathcal{F})$ is regular and strongly local, and that conditions (H) and (VD) hold. Let $\Omega\subset M$ be open such that $\lambda_{\min}(\Omega)>0$. If $x_0\in U\Subset\Omega$, we have that*

$$\inf_{\partial U}g^{\Omega}\left(x_0,\cdot\right)\le\operatorname{res}\left(U,\Omega\right)\le\sup_{\partial U}g^{\Omega}\left(x_0,\cdot\right). \tag{7.1}$$

Proof. Let $A_c(x_0)$ be defined as in (6.20), and let

$$\begin{aligned} a &:= \sup_{\partial U} g^{\Omega}(x_0,\cdot), \\ b &:= \inf_{\partial U} g^{\Omega}(x_0,\cdot). \end{aligned}$$

Since $g^{\Omega}(x_0,\cdot)$ is non-negative and jointly continuous off diagonal, we see that

$$0 \le b \le a < \infty.$$

Note that $a>0$; otherwise $g^{\Omega}(x_0,\cdot) \equiv 0$ on ∂U, and thus, using (6.18), we have

$$1 = \int_{\partial U} g^{\Omega}(x_0,y)\, d\nu_p(y) = 0,$$

where ν_p is the capacitory measure for $\mathrm{cap}(U,\Omega)$, leading to a contradiction.

Note that if $b=0$, the first inequality in (7.1) is clear, and the second one can be proved in a similar way as below. In the sequel, assume that $b>0$. Let $\varepsilon>0$ be arbitrarily small.

We first show

$$\inf_{\partial U} g^{\Omega}(x_0,\cdot) \le \mathrm{res}(U,\Omega). \tag{7.2}$$

Indeed, by Lemma 5.3, we see that

$$\inf_{\overline{U}} g^{\Omega}(x_0,\cdot) = \inf_{\partial U} g^{\Omega}(x_0,\cdot) = b > b-\varepsilon > 0,$$

and thus $\overline{U} \subset A_{b-\varepsilon}(x_0)$. Since $g^{\Omega}(x_0,\cdot)$ is continuous in $\Omega \setminus \{x_0\}$, we can choose an open set U_1 such that $U \subset U_1 \Subset \Omega$, and

$$g^{\Omega}(x_0,x) \ge b-\varepsilon \text{ for any } x \in \partial U_1$$

where ∂U_1 is contained in a neighborhood of ∂U. Let

$$A'_{b-\varepsilon}(x_0) = U_1 \cap A_{b-\varepsilon}(x_0).$$

Then, we see that $x_0 \in U \subset A'_{b-\varepsilon}(x_0) \Subset \Omega$, and for any $y \in \partial\left(A'_{b-\varepsilon}(x_0)\right)$,

$$g^{\Omega}(x_0,y) \ge b-\varepsilon.$$

It follows from (6.18) and (6.16) that

$$\begin{aligned} 1 &= \int_{\partial\left(A'_{b-\varepsilon}(x_0)\right)} g^{\Omega}(x_0,y)\, d\nu_b(y) \\ &\ge (b-\varepsilon)\,\nu_b\left(\partial\left(A'_{b-\varepsilon}(x_0)\right)\right) = (b-\varepsilon)\,\mathrm{cap}(A'_{b-\varepsilon}(x_0),\Omega), \end{aligned}$$

where ν_b is the capacitory measure for $\mathrm{cap}(A'_{b-\varepsilon}(x_0),\Omega)$. Therefore,

$$\mathrm{cap}(U,\Omega) \le \mathrm{cap}(A'_{b-\varepsilon}(x_0),\Omega) \le \frac{1}{b-\varepsilon},$$

that is, $b-\varepsilon \le \mathrm{res}(U,\Omega)$, proving (7.2).

We next show the second inequality in (7.1), namely,

$$\mathrm{res}(U,\Omega) \le \sup_{\partial U} g^{\Omega}(x_0,\cdot). \tag{7.3}$$

Indeed, by Lemma 5.3, we see that

$$\sup_{\Omega\setminus U} g^{\Omega}(x_0,\cdot) = \sup_{\partial U} g^{\Omega}(x_0,\cdot) = a,$$

and thus $A_a(x_0) \subset \overline{U}$, and

$$\mathrm{cap}(A_a(x_0),\Omega) \le \mathrm{cap}(\overline{U},\Omega).$$

If $x_0 \in A_a(x_0) \subset \overline{U} \Subset \Omega$, using Proposition 6.6, we have

$$\mathrm{cap}(A_a(x_0),\Omega) = \frac{1}{a}, \tag{7.4}$$

thus proving (7.3).

If $x_0 \notin A_a(x_0)$, by definition of $A_a(x_0)$, we have that

$$g^{\Omega}(x_0, x_0) \leq a < a + \varepsilon.$$

Using the continuity of $g^{\Omega}(x_0, \cdot)$, we can choose a neighborhood N_{x_0} of x_0 such that $N_{x_0} \subset U$, and

$$g^{\Omega}(x_0, x) \leq a + \varepsilon \text{ for any } x \in N_{x_0}.$$

Denote by the set

$$A'_a(x_0) := A_a(x_0) \cup N_{x_0}.$$

Then, we see that $x_0 \in N_{x_0} \subset A'_a(x_0) \subset \overline{U} \Subset \Omega$, and for any $y \in \partial A'_a(x_0)$,

$$g^{\Omega}(x_0, y) \leq a + \varepsilon.$$

It follows from (6.18) and (6.16) that

$$\begin{aligned} 1 &= \int_{\partial A'_a(x_0)} g^{\Omega}(x_0, y)\, d\nu_a(y) \\ &\leq (a+\varepsilon)\, \nu_a\left(\partial A'_a(x_0)\right) = (a+\varepsilon)\operatorname{cap}(A'_a(x_0), \Omega), \end{aligned}$$

where ν_a is the capacitory measure for $\operatorname{cap}(A'_a(x_0), \Omega)$. Therefore,

$$\begin{aligned} \operatorname{cap}(U, \Omega) &= \operatorname{cap}(\overline{U}, \Omega) \geq \operatorname{cap}(A'_a(x_0), \Omega) \\ &\geq \frac{1}{a+\varepsilon}, \end{aligned}$$

that is, $a + \varepsilon \geq \operatorname{res}(U, \Omega)$, thus proving (7.3).

Finally, combining (7.2) and (7.3), we finish the proof. □

As a consequence of Lemma 7.1, we have the following.

Lemma 7.2. *Assume that $(\mathcal{E}, \mathcal{F})$ is regular and strongly local, and that conditions (H) and (VD) hold. If Ω is a precompact open set containing a ball KB where $B = B(x_0, R)$ and $K > \delta^{-1}$, and such that $\lambda_{\min}(\Omega) > 0$, then*

$$\inf_{\partial B} g^{\Omega}(x_0, \cdot) \simeq \operatorname{res}(B, \Omega) \simeq \sup_{\partial B} g^{\Omega}(x_0, \cdot). \tag{7.5}$$

Proof. Since condition (HG) holds, we see that

$$\inf_{\partial B} g^{\Omega}(x_0, \cdot) \simeq \sup_{\partial B} g^{\Omega}(x_0, \cdot).$$

Using (7.1), we obtain the desired. □

We next estimate the sum of a finite number of resistances. For this, we need the following lemma.

Lemma 7.3. *Assume that $(\mathcal{E}, \mathcal{F})$ is regular. For any two open sets Ω_1, Ω_2 in M such that $\Omega_1 \Subset \Omega_2$ and $\lambda_{\min}(\Omega_1) > 0$, and for any non-negative $f \in L^2(\Omega_2)$, we have*

$$\operatorname*{esup}_{\Omega_2}\left(G^{\Omega_2} f - G^{\Omega_1} f\right) \leq \operatorname*{esup}_{\Omega_2 \setminus U} G^{\Omega_2} f \tag{7.6}$$

where U is any open subset with $U \Subset \Omega_1$. If $G^{\Omega_2} f$ is continuous in a neighborhood of $\partial \Omega_1$, then

$$\operatorname*{esup}_{\Omega_2}\left(G^{\Omega_2} f - G^{\Omega_1} f\right) = \operatorname*{esup}_{\Omega_2 \setminus \Omega_1} G^{\Omega_2} f. \tag{7.7}$$

Proof. Let $u := G^{\Omega_2} f - G^{\Omega_1} f$. Then $u \geq 0$ in M, and is harmonic in Ω_1 since for any $\varphi \in \mathcal{F}(\Omega_1)$,

$$\mathcal{E}(u, \varphi) = \mathcal{E}\left(G^{\Omega_2} f - G^{\Omega_1} f, \varphi\right) = (f, \varphi) - (f, \varphi) = 0.$$

Therefore, for any $U \Subset \Omega_1$, by the maximum principle (4.1), we have

$$\operatorname*{esup}_{\Omega_1} u \leq \operatorname*{esup}_{M \setminus U} u = \operatorname*{esup}_{\Omega_2 \setminus U} u.$$

As $u \le G^{\Omega_2} f$ in M, we see that

$$\operatorname*{esup}_{\Omega_2 \setminus U} u \le \operatorname*{esup}_{\Omega_2 \setminus U} G^{\Omega_2} f.$$

Hence, it follows that

$$\operatorname*{esup}_{\Omega_1} u \le \operatorname*{esup}_{\Omega_2 \setminus U} G^{\Omega_2} f, \tag{7.8}$$

which implies that, using the fact that $\Omega_2 \setminus \Omega_1 \subset \Omega_2 \setminus U$,

$$\begin{aligned} \operatorname*{esup}_{\Omega_2} u &= \operatorname*{esup}_{\Omega_1} u \vee \operatorname*{esup}_{\Omega_2 \setminus \Omega_1} u = \operatorname*{esup}_{\Omega_1} u \vee \operatorname*{esup}_{\Omega_2 \setminus \Omega_1} G^{\Omega_2} f \\ &\le \operatorname*{esup}_{\Omega_2 \setminus U} G^{\Omega_2} f \vee \operatorname*{esup}_{\Omega_2 \setminus \Omega_1} G^{\Omega_2} f = \operatorname*{esup}_{\Omega_2 \setminus U} G^{\Omega_2} f, \end{aligned}$$

proving (7.6).

If $G^{\Omega_2} f$ is continuous in a neighborhood of $\partial\Omega_1$, we let $U \uparrow \Omega_1$ in (7.6) and obtain

$$\operatorname*{esup}_{\Omega_2} u \le \operatorname*{esup}_{\Omega_2 \setminus \Omega_1} G^{\Omega_2} f.$$

On the other hand, it is obvious that

$$\operatorname*{esup}_{\Omega_2} u \ge \operatorname*{esup}_{\Omega_2 \setminus \Omega_1} u = \operatorname*{esup}_{\Omega_2 \setminus \Omega_1} G^{\Omega_2} f.$$

Thus, we conclude that (7.7) holds. □

Lemma 7.4. *Assume that $(\mathcal{E}, \mathcal{F})$ is regular and strongly local, and that conditions (H) and (VD) hold. Fix a ball $B(x_0, R)$ and set $B_n = K^n B$ for $n = 0, 1, 2, \cdots$, where $K > \delta^{-1}$. For all $n > m \ge 0$, if $\lambda_{\min}(B_n) > 0$ then*

$$\sup_{\partial B_m} g^{B_n}(x_0, \cdot) \simeq \sum_{k=m}^{n-1} \operatorname{res}(B_k, B_{k+1}) \simeq \inf_{\partial B_m} g^{B_n}(x_0, \cdot). \tag{7.9}$$

Proof. For each $k \ge 0$, let us show that for any $y \in M \setminus \{x_0\}$,

$$g^{B_{k+1}}(x_0, y) - g^{B_k}(x_0, y) \le \sup_{B_{k+1} \setminus B_k} g^{B_{k+1}}(x_0, \cdot). \tag{7.10}$$

Indeed, note that (7.10) trivially holds for any $y \notin B_k$. We will prove (7.10) for any $y \in B_k \setminus \{x_0\}$.

To do this, we have from (7.6) that, for any concentric ball $B' \Subset B_k$,

$$\operatorname*{esup}_{B_{k+1}} \left(G^{B_{k+1}} f - G^{B_k} f \right) \le \operatorname*{esup}_{B_{k+1} \setminus B'} G^{B_{k+1}} f, \tag{7.11}$$

and thus for any fixed point $y \in B_k \setminus \{x_0\}$,

$$G^{B_{k+1}} f(y) - G^{B_k} f(y) \le \operatorname*{esup}_{B_{k+1} \setminus B'} G^{B_{k+1}} f. \tag{7.12}$$

Choose $f = f_{n,x_0} \rightharpoonup \delta_{x_0}$ weakly in $C(M)$ as $n \to \infty$. The function $G^{B_{k+1}} f_{n,x_0}$ is harmonic in $B_{k+1} \setminus B'$ since f_{n,x_0} vanishes in a small neighborhood of x_0. Hence, using the maximum principle (4.10),

$$\operatorname*{esup}_{B_{k+1} \setminus B'} G^{B_{k+1}} f_{n,x_0} = \operatorname*{esup}_{\partial B'} G^{B_{k+1}} f_{n,x_0}.$$

As $G^{B_{k+1}} f_{n,x_0}$ is continuous in $B_{k+1} \setminus B'$, letting $B' \uparrow B_k$, we obtain from (7.12) that

$$G^{B_{k+1}} f_{n,x_0}(y) - G^{B_k} f_{n,x_0}(y) \le \sup_{\partial B_k} G^{B_{k+1}} f_{n,x_0} \tag{7.13}$$

By (5.16), we have already shown that, as $n \to \infty$,

$$\begin{aligned} G^{B_k} f_{n,x_0}(y) &\to g^{B_k}(x_0, y), \\ G^{B_{k+1}} f_{n,x_0}(y) &\to g^{B_{k+1}}(x_0, y), \end{aligned}$$

and at the same time,

$$G^{B_{k+1}} f_{n,x_0}(\cdot) \to g^{B_{k+1}}(x_0,\cdot)$$

uniformly in the compact subset ∂B_k.

Therefore, passing to the limit as $n \to \infty$ in (7.13), we obtain that

$$g^{B_{k+1}}(x_0,y) - g^{B_k}(x_0,y) \le \sup_{\partial B_k} g^{B_{k+1}}(x_0,\cdot) \le \sup_{B_{k+1}\setminus B_k} g^{B_{k+1}}(x_0,\cdot),$$

thus showing that (7.10) holds for any $y \in B_k \setminus \{x_0\}$.

It follows from (7.10) that, using (5.20) and (7.5),

$$\begin{aligned} g^{B_{k+1}}(x_0,y) - g^{B_k}(x_0,y) &\le \sup_{B_{k+1}\setminus B_k} g^{B_{k+1}}(x_0,\cdot) \\ &= \sup_{\partial B_k} g^{B_{k+1}}(x_0,\cdot) \le C_1 \operatorname{res}(B_k, B_{k+1}), \end{aligned}$$

for some $C_1 > 0$. Adding up k from $m+1$ to $n-1$, we obtain that for all $y \in M \setminus \{x_0\}$,

$$g^{B_n}(x_0,y) - g^{B_{m+1}}(x_0,y) \le C_1 \sum_{k=m+1}^{n-1} \operatorname{res}(B_k, B_{k+1}). \tag{7.14}$$

On the other hand, using (7.5) again, we have

$$\sup_{\partial B_m} g^{B_{m+1}}(x_0,\cdot) \simeq \operatorname{res}(B_m, B_{m+1}). \tag{7.15}$$

Therefore, combining (7.14) and (7.15), we conclude that

$$\sup_{\partial B_m} g^{B_n}(x_0,\cdot) \le C_1 \sum_{k=m}^{n-1} \operatorname{res}(B_k, B_{k+1}). \tag{7.16}$$

We next show that

$$\sum_{k=m}^{n-1} \operatorname{res}(B_k, B_{k+1}) \le C_2 \inf_{\partial B_m} g^{B_n}(x_0,\cdot), \tag{7.17}$$

for some $C_2 > 0$. Indeed, since $(\mathcal{E}, \mathcal{F})$ is strongly local, we have (cf. [23, Lemma 2.5, p.157]) that

$$\sum_{k=m}^{n-1} \operatorname{res}(B_k, B_{k+1}) \le \operatorname{res}(B_m, B_n).$$

Using (7.5), we have

$$\operatorname{res}(B_m, B_n) \simeq \inf_{\partial B_m} g^{B_n}(x_0,\cdot).$$

Therefore, we obtain (7.17).

Finally, combining (7.16) and (7.17), we obtain (7.9). □

7.2. **Estimates of the Green function.** We give upper estimate of the Green function under conditions (H) and $(E_F \le)$.

Theorem 7.5. *Assume that $(\mathcal{E}, \mathcal{F})$ is regular and strongly local, and that conditions (H), (VD) and $(E_F \le)$ all hold. Then, for any ball $B := B(x_0, R)$, the Green kernel g^B exists and satisfies the following estimate: for all $y \in B \setminus \{x_0\}$,*

$$g^B(x_0,y) \le C \int_{r/4}^{R} \frac{F(s)\,ds}{sV(x_0,s)}, \tag{$G_F \le$}$$

where $r = d(x_0,y)$ and constant $C > 0$ is independent of the ball B.

Proof. Fix a point $y \in B \setminus \{x_0\}$. Let $r := d(x_0, y)$, and let n be an integer such that

$$2^{-n}R \le r < 2^{-(n-1)}R.$$

For $k = 0, 1, \cdots, n$, let

$$r_k := 2^{-k}R \;\text{ and }\; B_k := B(x_0, r_k).$$

Let $0 \le f \in L^2$. Note that for $U \subset \Omega$,

$$\begin{aligned}
\operatorname*{einf}_U G^\Omega f &\le \frac{1}{\mu(U)} \int_U G^\Omega f \, d\mu \\
&= \frac{1}{\mu(U)} \int_U \left(\int_\Omega g^\Omega(x,y) f(y) d\mu(y) \right) d\mu(x) \\
&= \frac{1}{\mu(U)} \int_\Omega \left(\int_U g^\Omega(x,y) d\mu(x) \right) f(y) d\mu(y) \\
&\le \frac{\|E^\Omega\|_{L^\infty(\Omega)}}{\mu(U)} \|f\|_{L^1(\Omega)} . \qquad (7.18)
\end{aligned}$$

Since the function $G^{B_k} f - G^{B_{k+1}} f$ is harmonic in B_{k+1} for each k, we have by (H) that it is Hölder continuous in δB_{k+1} and, using (7.18), (VD) and $(E_F \le)$,

$$\begin{aligned}
\sup_{\delta B_{k+1}} \left(G^{B_k} f - G^{B_{k+1}} f \right) &\le C_H \inf_{\delta B_{k+1}} \left(G^{B_k} f - G^{B_{k+1}} f \right) \\
&\le C_H \inf_{\delta B_{k+1}} G^{B_k} f \\
&\le C_H \frac{\|E^{B_k}\|_{L^\infty(B_k)}}{\mu(\delta B_{k+1})} \|f\|_1 \\
&\le C \frac{F(r_k)}{\mu(B_k)} \|f\|_1 .
\end{aligned}$$

Therefore, for $k = 0, 1, \cdots, n$,

$$G^{B_k} f(x_0) - G^{B_{k+1}} f(x_0) \le C \frac{F(r_k)}{\mu(B_k)} \|f\|_1 .$$

Choosing $f = f_{n,y} \longrightarrow \delta_y$ weakly in $C_0(M)$, and using the facts that $G^{B_{n+1}} f_{n,y} \equiv 0$ and $\|f_{n,y}\|_1 = 1$, we obtain that

$$\begin{aligned}
G^B f_{n,y}(x_0) &= \sum_{k=0}^n \left[G^{B_k} f_{n,y}(x_0) - G^{B_{k+1}} f_{n,y}(x_0) \right] \\
&\le C \sum_{k=0}^n \frac{F(r_k)}{\mu(B_k)}.
\end{aligned}$$

Hence, letting $n \to \infty$ and using (5.16), we have

$$g^B(x_0, y) \le C \sum_{k=0}^n \frac{F(r_k)}{\mu(B_k)}. \qquad (7.19)$$

On the other hand, as both F and $V(x_0,\cdot)$ are non-decreasing and $r/4 < 2^{-(n+1)}R = r_{n+1}$, the integral

$$\begin{aligned}\int_{r/4}^{R} \frac{F(s)\,ds}{sV(x_0,s)} &\geq \sum_{k=0}^{n} \int_{r_{k+1}}^{r_k} \frac{F(s)\,ds}{sV(x_0,s)} \\ &\geq \sum_{k=0}^{n} \frac{F(r_{k+1})}{V(x_0,r_k)} \int_{r_{k+1}}^{r_k} \frac{ds}{s} = \ln 2 \sum_{k=0}^{n} \frac{F(r_{k+1})}{V(x_0,r_k)} &\quad (7.20) \\ &\geq C' \sum_{k=0}^{n} \frac{F(r_k)}{V(x_0,r_k)} \quad (\text{using } (3.3)\ ). &\quad (7.21)\end{aligned}$$

Combining (7.19) and (7.21), we obtain $(G_F \leq)$. □

7.3. **Continuity of $G^{\Omega}f$.** We investigate the continuity of the function $G^{\Omega}f$. Before doing this, we need the following general result.

Proposition 7.6. *Assume that conditions (3.3) and (VD) hold, and let $0 < \lambda, \lambda_1 \leq 1$ and $B := B(x_0,R)$. For any $t \geq 0$, let*

$$f(t) := \int_{\lambda t}^{R} \frac{F(s)\,ds}{sV(x_0,s)}.$$

Then, we have

$$C_1F(R) \leq \int_{\lambda_1 B} f(d(x_0,y))d\mu(y) \leq C_2F(R), \tag{7.22}$$

where constants C_1, C_2 are independent of the ball B, but may depend on λ, λ_1. If further condition (3.2) holds, then

$$\int_{\lambda_1 B} f(d(x_0,y))d\mu(y) \leq C(\lambda)\left[\lambda_1^{\alpha'} \ln\frac{1}{\lambda_1} + \lambda_1^{\alpha'} + \lambda_1{}^{\beta}\right] F(R), \tag{7.23}$$

where $C(\lambda)$ is independent of λ_1 and R.

Proof. Since f is non-increasing, we have that

$$\int_{\lambda_1 B} f(d(x_0,y))d\mu(y) \geq f(\lambda_1 R)V(x_0,\lambda_1 R). \tag{7.24}$$

As the functions F and $V(x_0,\cdot)$ are non-decreasing, we see that, using (3.3),

$$\begin{aligned}f(\lambda_1 R) &= \int_{\lambda\lambda_1 R}^{R} \frac{F(s)\,ds}{sV(x_0,s)} \\ &\geq \frac{F(\lambda\lambda_1 R)}{V(x_0,R)} \int_{\lambda\lambda_1 R}^{R} \frac{ds}{s} \geq C'(\lambda,\lambda_1)\frac{F(R)}{V(x_0,R)}. &\quad (7.25)\end{aligned}$$

Therefore, it follows from (7.24), (7.25) that, using (VD) again,

$$\begin{aligned}\int_{\lambda_1 B} f(d(x_0,y))d\mu(y) &\geq f(\lambda_1 R)V(x_0,\lambda_1 R) \\ &\geq C'(\lambda,\lambda_1)\frac{V(x_0,\lambda_1 R)F(R)}{V(x_0,R)} \\ &\geq C^{-1}F(R),\end{aligned}$$

thus proving the first inequality in (7.22).

We next show the second inequality in (7.22). Indeed, we have that

$$\begin{aligned}\int_{\lambda_1 B} f(d(x_0,y))d\mu(y) &= \int_0^{\lambda_1 R} f(t)dV(x_0,t)\\ &= f(t)V(x_0,t)|_0^{\lambda_1 R} - \int_0^{\lambda_1 R} V(x_0,t)f'(t)dt\\ &\leq f(\lambda_1 R)V(x_0,\lambda_1 R) - \int_0^{\lambda_1 R} V(x_0,t)f'(t)dt. \qquad (7.26)\end{aligned}$$

By (3.1), we see that

$$\begin{aligned}\frac{1}{V(x_0,\lambda\lambda_1 R)} &= \frac{1}{V(x_0,R)}\frac{V(x_0,R)}{V(x_0,\lambda\lambda_1 R)}\\ &\leq C_D\left(\frac{1}{\lambda\lambda_1}\right)^{\alpha}\frac{1}{V(x_0,R)},\end{aligned}$$

and hence,

$$\begin{aligned}f(\lambda_1 R) &= \int_{\lambda\lambda_1 R}^{R}\frac{F(s)\,ds}{sV(x_0,s)} \leq \frac{F(R)}{V(x_0,\lambda\lambda_1 R)}\int_{\lambda\lambda_1 R}^{R}\frac{ds}{s}\\ &= \frac{F(R)}{V(x_0,\lambda\lambda_1 R)}\ln\frac{1}{\lambda\lambda_1}\\ &\leq C_D\left(\frac{1}{\lambda\lambda_1}\right)^{\alpha}\left(\ln\frac{1}{\lambda\lambda_1}\right)\frac{F(R)}{V(x_0,R)}. \qquad (7.27)\end{aligned}$$

On the other hand, using (3.1) and (3.3),

$$\begin{aligned}0 &\leq -\int_0^{\lambda_1 R} V(x_0,t)f'(t)dt = \int_0^{\lambda_1 R} V(x_0,t)\frac{F(\lambda t)}{tV(x_0,\lambda t)}dt\\ &\leq C(\lambda)\int_0^{\lambda_1 R}\frac{F(\lambda t)}{t}dt = C(\lambda)F(R)\int_0^{\lambda_1 R}\frac{F(\lambda t)}{F(R)}\frac{dt}{t}\\ &\leq C'(\lambda)F(R)\int_0^{\lambda_1 R}\left(\frac{\lambda t}{R}\right)^{\beta}\frac{dt}{t} = C(\lambda)\lambda_1^{\beta}F(R). \qquad (7.28)\end{aligned}$$

Therefore, it follows from (7.26), (7.27) and (7.28) that

$$\begin{aligned}\int_{\lambda_1 B} f(d(x_0,y))d\mu(y) &\leq C_D\left(\frac{1}{\lambda\lambda_1}\right)^{\alpha}\left(\ln\frac{1}{\lambda\lambda_1}\right)\frac{V(x_0,\lambda_1 R)F(R)}{V(x_0,R)}\\ &\quad + C(\lambda)\lambda_1^{\beta}F(R) \qquad (7.29)\\ &\leq C(\lambda,\lambda_1)F(R).\end{aligned}$$

Finally, it remains to show (7.23). Note that

$$\begin{aligned}f(\lambda_1 R)V(x_0,\lambda_1 R) &= V(x_0,\lambda_1 R)\int_{\lambda\lambda_1 R}^{R}\frac{F(s)\,ds}{sV(x_0,s)}\\ &= V(x_0,\lambda_1 R)\left\{\int_{\lambda\lambda_1 R}^{\lambda_1 R}\frac{F(s)\,ds}{sV(x_0,s)} + \int_{\lambda_1 R}^{R}\frac{F(s)\,ds}{sV(x_0,s)}\right\}. \qquad (7.30)\end{aligned}$$

By the monotonicity of F and $V(x_0,\cdot)$, the first term

$$\begin{aligned}V(x_0,\lambda_1 R)\int_{\lambda\lambda_1 R}^{\lambda_1 R}\frac{F(s)\,ds}{sV(x_0,s)} &\leq F(\lambda_1 R)\frac{V(x_0,\lambda_1 R)}{V(x_0,\lambda\lambda_1 R)}\int_{\lambda\lambda_1 R}^{\lambda_1 R}\frac{ds}{s}\\ &\leq C(\lambda)F(\lambda_1 R) = C(\lambda)F(R)\frac{F(\lambda_1 R)}{F(R)}\\ &\leq C'(\lambda)F(R)(\lambda_1)^{\beta} \quad \text{(using (3.3))}. \qquad (7.31)\end{aligned}$$

Similarly, using (3.3) and (3.2), the second term

$$\begin{aligned} V(x_0,\lambda_1 R)\int_{\lambda_1 R}^{R}\frac{F(s)\,ds}{sV(x_0,s)} &= F(R)\int_{\lambda_1 R}^{R}\frac{F(s)}{F(R)}\frac{V(x_0,\lambda_1 R)}{V(x_0,s)}\frac{ds}{s} \\ &\le cF(R)\int_{\lambda_1 R}^{R}\left(\frac{s}{R}\right)^{\beta}\left(\frac{\lambda_1 R}{s}\right)^{\alpha'}\frac{ds}{s} \\ &= cF(R)\,(\lambda_1)^{\alpha'}\int_{\lambda_1}^{1}s^{\beta-\alpha'-1}ds. \end{aligned}$$

If $\beta=\alpha'$, we have

$$\int_{\lambda_1}^{1}s^{\beta-\alpha'-1}ds=\ln\frac{1}{\lambda_1},$$

and if $\beta\neq\alpha'$, we have

$$\int_{\lambda_1}^{1}s^{\beta-\alpha'-1}ds=\frac{1}{\beta-\alpha'}\left(1-(\lambda_1)^{\beta-\alpha'}\right).$$

Hence, the second term

$$V(x_0,\lambda_1 R)\int_{\lambda_1 R}^{R}\frac{F(s)\,ds}{sV(x_0,s)}\le cF(R)\left(\lambda_1^{\alpha'}\ln\frac{1}{\lambda_1}+\lambda_1^{\alpha'}+\lambda_1{}^{\beta}\right). \tag{7.32}$$

Therefore, it follows from (7.30), (7.31), (7.32) that

$$f(\lambda_1 R)V(x_0,\lambda_1 R)\le C'(\lambda)F(R)\left(\lambda_1^{\alpha'}\ln\frac{1}{\lambda_1}+\lambda_1^{\alpha'}+\lambda_1{}^{\beta}\right). \tag{7.33}$$

Combining (7.26), (7.33), and (7.28), we arrive at (7.23). □

Lemma 7.7. *Assume that $(\mathcal{E},\mathcal{F})$ is regular and strongly local, and that conditions (H), (VD), (RVD) and $(E_F\le)$ all hold. Let Ω be a bounded open subset of M with $\lambda_{\min}(\Omega)>0$, and let $f\in L^\infty(\Omega)$. Then, the function*

$$G^\Omega f(x)=\int_\Omega g^\Omega(x,y)f(y)d\mu(y)$$

is continuous for $x\in\Omega$. In particular, the function $E^\Omega=G^\Omega\mathbf{1}_\Omega$ is continuos in Ω.

Proof. Without loss of generality, assume that $\|f\|_\infty\le 1$. Fix a point $x_0\in\Omega$, and let $R>0,\rho\ge 1$ such that

$$B:=B(x_0,R)\Subset\Omega\subset B(x_0,\rho R).$$

Let $\{x_k\}_{k=1}^\infty\subset B$ such that $x_k\to x_0$ as $k\to\infty$. Let $\eta>0$ be small, and let $d(x_k,x_0)<\delta(\eta R)$ for any $k\ge 1$, where δ is the same as in (H). Then,

$$\begin{aligned} |G^\Omega f(x_k) &- G^\Omega f(x_0)|=\left|\int_\Omega g^\Omega(x_k,y)f(y)d\mu(y)-\int_\Omega g^\Omega(x_0,y)f(y)d\mu(y)\right| \\ &\le \int_{B(x_0,\eta R)}g^\Omega(x_k,y)d\mu(y)+\int_{B(x_0,\eta R)}g^\Omega(x_0,y)d\mu(y) \qquad (7.34) \\ &\quad +\int_{\Omega\setminus B(x_0,\eta R)}\left|g^\Omega(x_k,y)-g^\Omega(x_0,y)\right|d\mu(y). \qquad (7.35) \end{aligned}$$

We claim that

$$\lim_{k\to\infty}\int_{\Omega\setminus B(x_0,\eta R)}\left|g^\Omega(x_k,y)-g^\Omega(x_0,y)\right|d\mu(y)=0. \tag{7.36}$$

Indeed, as g^Ω is jointly continuous off diagonal, we have that, for any $y\in\Omega\setminus B(x_0,\eta R)$,

$$\lim_{k\to\infty}g^\Omega(x_k,y)=g^\Omega(x_0,y).$$

Noting that $x_k\in B(x_0,\delta(\eta R))$ for all $k\ge 1$, it follows from (5.4) that, for any $y\in\Omega\setminus B(x_0,\eta R)$,

$$g^\Omega(x_k,y)\le C_H g^\Omega(x_0,y).$$

By condition ($E_F \leq$), the function $g^{\Omega}(x_0, \cdot)$ is integrable in Ω, that is,

$$\int_{\Omega} g^{\Omega}(x_0, y) d\mu(y) = E^{\Omega}(x_0) \leq CF(R).$$

Therefore, applying the dominated convergence theorem,

$$\lim_{k\to\infty} \int_{\Omega \setminus B(x_0,\eta R)} g^{\Omega}(x_k, y) d\mu(y) = \int_{\Omega \setminus B(x_0,\eta R)} g^{\Omega}(x_0, y) d\mu(y),$$

proving our claim.

We next estimate the two terms in (7.34). It is enough to consider the first term. The second one is treated similarly. Now fix $k \geq 1$, and let

$$f(t) = \int_{t/4}^{2\rho R} \frac{F(s)\, ds}{sV(x_k, s)}.$$

By Theorem 7.5, we have that, using that fact that $\Omega \subset B(x_0, \rho R) \subset B_1 := B(x_k, 2\rho R)$,

$$\begin{aligned} \int_{B(x_0,\eta R)} g^{\Omega}(x_k, y) d\mu(y) &\leq \int_{B(x_k,2\eta R)} g^{B_1}(x_k, y) d\mu(y) \\ &\leq C \int_{B(x_k,2\eta R)} f(d(x_k, y))\, d\mu(y). \end{aligned}$$

Using (7.23) with $\lambda_1 = \eta/\rho, \lambda = \frac{1}{4}$ and with R, x_0 being replaced by $2\rho R$, x_k respectively, we obtain that

$$\begin{aligned} \int_{B(x_k,2\eta R)} f(d(x_k, y))\, d\mu(y) &\leq C\left(\eta^{\alpha'} \ln \frac{1}{\eta} + \eta^{\alpha'} + \eta^{\beta}\right) F(2\rho R) \\ &= o(\eta). \end{aligned} \tag{7.37}$$

Therefore, it follows from (7.34), (7.35), (7.36) and (7.37) that

$$\lim_{k\to\infty} \left|G^{\Omega} f(x_k) - G^{\Omega} f(x_0)\right| \leq 2o(\eta),$$

thus proving the continuity of $G^{\Omega} f$. □

Remark 7.8. Under the hypotheses of Lemma 7.7, the essential supremum and essential infimum in conditions ($E_F \leq$) and ($E_F \geq$) in Definition 3.10 can be replaced by supremum and infimum, respectively.

8. Proof of Theorem 3.12

8.1. Implication $(H) + (R_F) \Rightarrow (G_F)$.

Proof. Let $B := B(x_0, R)$ and choose $K > 4 \vee \delta^{-1}$. We split the proof into two steps.

Step 1. We prove the lower bound ($G_F \geq$): there exists some $C > 0$ such that for all $y \in K^{-1}B \setminus \{x_0\}$,

$$g^B(x_0, y) \geq C^{-1} \int_{K^{-1}r}^{R} \frac{F(s)\, ds}{sV(x_0, s)}, \quad r = d(x_0, y). \tag{$G_F \geq$}$$

Indeed, choose the integer $n > 1$ such that

$$K^{-n-1}R \leq r < K^{-n}R, \tag{8.1}$$

and for $i \geq 0$, set

$$r_i := K^{-i}R \text{ and } B_i := B(x_0, r_i). \tag{8.2}$$

As $K^{-1}r \geq K^{-n-2}R$, similar to (7.20), we have that

$$\begin{aligned}\int_{K^{-1}r}^{R}\frac{F(s)\,ds}{sV(x_0,s)} &\leq \sum_{i=0}^{n+1}\int_{r_{i+1}}^{r_i}\frac{F(s)\,ds}{sV(x_0,s)} \leq \ln K\sum_{i=0}^{n+1}\frac{F(r_i)}{V(x_0,r_{i+1})} \\ &\leq C\sum_{i=0}^{n+1}\frac{F(r_{i+1})}{V(x_0,r_{i+1})} \quad \text{(by (3.3)).} \end{aligned} \tag{8.3}$$

The last two terms for $i=n$ and $i=n+1$ in the sum can be bounded by the term $\frac{F(r_n)}{V(x_0,r_n)}$, since we have that, using (3.3) and (VD),

$$\begin{aligned}\frac{F(r_{n+1})}{V(x_0,r_{n+1})} &= \frac{F(r_n)}{V(x_0,r_n)}\cdot\frac{F(r_{n+1})}{F(r_n)}\cdot\frac{V(x_0,r_n)}{V(x_0,r_{n+1})} \\ &\leq C\frac{F(r_n)}{V(x_0,r_n)},\end{aligned}$$

and a similar bound for the other term:

$$\frac{F(r_{n+2})}{V(x_0,r_{n+2})} \leq C\frac{F(r_n)}{V(x_0,r_n)}.$$

Hence, it follows from (8.3) that

$$\begin{aligned}\int_{K^{-1}r}^{R}\frac{F(s)\,ds}{sV(x_0,s)} &\leq C'\sum_{i=0}^{n-1}\frac{F(r_{i+1})}{V(x_0,r_{i+1})} \\ &\leq C'\sum_{i=0}^{n-1}\mathrm{res}(B_{i+1},B_i) \quad \text{(by condition } (R_F\geq)\text{)} \\ &\leq C''\inf_{\partial B_n} g^B(x_0,\cdot) \quad \text{(by (7.9)).}\end{aligned} \tag{8.4}$$

On the other hand, using the fact that $y\in B_n\setminus B_{n+1}$, we have from (5.19) that

$$g^B(x_0,y) \geq \inf_{B_n} g^B(x_0,\cdot) = \inf_{\partial B_n} g^B(x_0,\cdot).$$

This combines with (8.4) to prove that $(G_F\geq)$ holds.

Step 2. We prove the upper bound $(G_F\leq)$: there exists some $C>0$ such that for all $y\in B\setminus\{x_0\}$,

$$g^B(x_0,y) \leq C\int_{K^{-1}r}^{R}\frac{F(s)\,ds}{sV(x_0,s)}, \quad r=d(x_0,y). \tag{$G_F\leq$}$$

Fix $y\in B\setminus\{x_0\}$, and set $r=d(x_0,y)$ as before.

Case (a) when $y\in K^{-1}B\setminus\{x_0\}$. Let n, r_i and B_i be respectively defined as in (8.1), (8.2). It follows that

$$\begin{aligned}g^B(x_0,y) &\leq \sup_{B\setminus B_{n+1}} g^B(x_0,\cdot) = \sup_{\partial B_{n+1}} g^B(x_0,\cdot) \quad \text{(by (5.20))} \\ &\leq C\sum_{i=0}^{n}\mathrm{res}(B_{i+1},B_i) \quad \text{(by (7.9))} \\ &\leq C'\sum_{i=0}^{n}\frac{F(r_{i+1})}{V(x_0,r_{i+1})} \quad \text{(by condition } (R_F\leq)\text{).}\end{aligned} \tag{8.5}$$

Therefore, using (7.20), we obtain $(G_F\leq)$.

Case (b) when $y\in B\setminus K^{-1}B$. We want to derive $(E_F\leq)$. If so, we are done by using Theorem 7.5.

Let $x\in B$. We see that

$$B\subset B(x,2R) := B'.$$

It follows that, using (5.20),

$$\begin{aligned} E^B(x) &= \int_B g^B(x,y)\,d\mu(y) \le \int_{B'} g^{B'}(x,y)\,d\mu(y) \\ &= \int_{\delta B'} g^{B'}(x,y)\,d\mu(y) + \int_{B'\setminus\delta B'} g^{B'}(x,y)\,d\mu(y) \\ &\le \int_{\delta B'} g^{B'}(x,y)\,d\mu(y) + \sup_{\partial(\delta B')} g^{B'}(x,\cdot)\mu(B'). \end{aligned} \tag{8.6}$$

By (7.5) and (3.3),

$$\sup_{\partial(\delta B')} g^{B'}(x,\cdot) \simeq \operatorname{res}\left(\delta B', B'\right) \le C\frac{F(2\delta R)}{\mu\left(\delta B'\right)} \le C'\frac{F(R)}{\mu\left(B'\right)},$$

and hence,

$$\sup_{\partial(\delta B')} g^{B'}(x,\cdot)\mu(B') \le C'F(R). \tag{8.7}$$

It remains to estimate the integral on the right-hand side of (8.6). Indeed, by Case (a), we have that for $y \in \delta B'$,

$$g^{B'}(x,y) \le C\int_{K^{-1}r}^{2R} \frac{F(s)\,ds}{sV(x,s)}.$$

Therefore, by Proposition 7.6 where $f(t) = \int_{K^{-1}r}^{2R} \frac{F(s)ds}{sV(x,s)}$, we obtain

$$\begin{aligned} \int_{\delta B'} g^{B'}(x,y)\,d\mu(y) &\le C\int_{\delta B'} f\left(d(x,y)\right)d\mu(y) \\ &\le C'F(2R) \le CF(R). \end{aligned} \tag{8.8}$$

Finally, adding up (8.8) and (8.7), we prove that condition $(E_F \le)$ holds.

This finishes the proof. □

8.2. Equivalence $(HG') \Leftrightarrow (H)$.

We introduce an alternative Harnack inequality, denoted by (HG'), for the Green function g^B on a ball B, and will show that $(HG') \iff (H)$ by using Lemmas 6.4 and 6.2.

Definition 8.1 (**Condition** (HG'))**.** We say that condition (HG') holds if, for any ball B in M, the Green function g^B exists and is jointly continuous off diagonal, and for any $y \in \overline{B_1}\setminus B_2$ with some balls $B_1 = \rho_1 B, B_2 = \rho_2 B$ $(0 < \rho_2 < \rho_1 < 1)$,

$$\operatorname*{esup}_{\delta' B_2} g^B(\cdot,y) \le C_H' \operatorname*{einf}_{\delta' B_2} g^B(\cdot,y), \tag{HG'}$$

where $C_H' \ge 1$ and $\delta' \in (0,1)$ are independent of B and y, but δ' may depend on ρ_2,ρ_1, and C_H' on δ',ρ_2,ρ_1.

We now show the implication $(HG') \iff (H)$.

Lemma 8.2. *Assume that $(\mathcal{E},\mathcal{F})$ is a local, regular Dirichlet form, and that $\lambda_{\min}(B) > 0$ for any ball B in M. Then,*

$$\left(HG'\right) \Rightarrow (H).$$

If in addition $(\mathcal{E},\mathcal{F})$ is strongly local and (VD) holds, then

$$\left(HG'\right) \Leftrightarrow (H). \tag{8.9}$$

Proof. Fix a ball B in M, and let $u \in L^\infty(M)$ be non-negative in B and be harmonic in B. We need to show that

$$\operatorname*{esup}_{\delta B} u \le C_H \operatorname*{einf}_{\delta B} u \tag{8.10}$$

for some constants $C_H \geq 1$ and $\delta \in (0,1)$, which will imply condition (H). It suffices to prove (8.10) assuming in addition that $u \in L^\infty(M)$, because then the Harnack inequality for arbitrary u follows by the argument in [26, p.1280 (proof of Theorem 7.4)].

Assuming in the sequel that $u \in L^\infty$, we split the proof into four steps. Let B_1 and B_2 be the same as in condition (HG').

Step 1. We cut off the function u such that it becomes non-negative globally in M, but still in $\mathcal{F}$. For doing this, let ϕ be a cutoff function of (B_1, B). Let

$$u_1 := u\phi.$$

This function u_1 will do. Indeed, it is easy to see that $u_1 \geq 0$ in M (noting that u_1 vanishes outside B), and $u_1 \in \mathcal{F} \cap L^\infty$.

Let us further show that u_1 is harmonic in B_1. Indeed, let $\varphi \in \mathcal{F}(B_1)$. We have that $\mathcal{E}(u,\varphi) = 0$ by the harmonicity of u. Noting that $u(\phi - 1) \equiv 0$ in a neighborhood of B_1, we see that $\mathcal{E}(u(\phi-1),\varphi) = 0$ by the locality of $(\mathcal{E},\mathcal{F})$. Hence,

$$\mathcal{E}(u_1,\varphi) = \mathcal{E}(u\phi,\varphi) = \mathcal{E}(u(\phi-1),\varphi) + \mathcal{E}(u,\varphi) = 0,$$

showing that u_1 is harmonic in B_1.

Step 2. Let $\widehat{u_1}$ be a reduced function of u_1 with respect to $\left(\overline{B_2}, B_1\right)$, as defined in Lemma 6.4, that is

$$\begin{cases} \widehat{u_1} \in \mathcal{F}(B_1), \\ \widehat{u_1} \text{ is superharmonic in } B_1, \\ \widehat{u_1} = u_1 \text{ in } B_2. \end{cases} \tag{8.11}$$

Let us show that $\widehat{u_1}$ is harmonic in B_2. Indeed, let $\varphi \in \mathcal{F}(B_2)$. By Step 1, the function u_1 is harmonic in B_1, and thus, $\mathcal{E}(u_1,\varphi) = 0$. Since the function $\widehat{u_1} - u_1$ vanishes in B_2, by the locality of $(\mathcal{E},\mathcal{F})$,

$$\mathcal{E}(\widehat{u_1} - u_1, \varphi) = 0.$$

Hence, we conclude that

$$\mathcal{E}(\widehat{u_1},\varphi) = \mathcal{E}(u_1,\varphi) + \mathcal{E}(\widehat{u_1} - u_1,\varphi) = 0,$$

proving that $\widehat{u_1}$ is harmonic in B_2.

Step 3. Let $S = \overline{B_1} \setminus B_2$. By Step 2, the function $\widehat{u_1}$ is harmonic in $B_2 = B_1 \setminus S$. Since $\lambda_{\min}(B_1) > 0$, it follows by Lemma 6.2 that

$$\widehat{u_1}(x) = \int_S g^{B_1}(x,y)\, d\nu(y) \text{ for } \mu\text{-a.a. all } x \in B_2,$$

where $\nu := \nu_{\widehat{u_1}}$ is a regular Borel measure determined as in (6.2) whose support is contained in S. By condition (GH'), for any $x_1, x_2 \in \delta' B_2$ and for any $y \in \overline{B_1} \setminus B_2 = S$,

$$g^{B_1}(x_1,y) \leq C'_H\, g^{B_1}(x_2,y).$$

Therefore, we conclude that, for almost all $x_1, x_2 \in \delta' B_2$,

$$\begin{aligned} u(x_1) &= \widehat{u_1}(x_1) = \int_S g^{B_1}(x_1,y)\, d\nu(y) \\ &\leq C'_H \int_S g^{B_1}(x_2,y)\, d\nu(y) = C'_H\, \widehat{u_1}(x_2) = C'_H\, u(x_2). \end{aligned}$$

Setting $C_H = C'_H$ and choosing $\delta > 0$ such that $\delta B = \delta' B_2$, that is, $\delta = \rho_2 \delta'$, we obtain (8.10).

Finally, by Lemma 5.2, the opposite implication $(H) \Rightarrow (HG')$ is clear. Indeed, we may choose $\rho_1 = \frac{3}{4}, \rho_2 = \frac{1}{2}$ and $\delta' = \delta, C'_H = C_H$, and then apply (5.4). Hence, the equivalence (8.9) does hold. This finishes the proof of (8.10) for bounded u and, hence, the entire proof. $\square$

8.3. **Implication** $(G_F) \Rightarrow (H) + (E_F)$.

Proof. Fix a ball $B := B(x_0, R)$. Let K be the same as in condition (G_F). We split the proof into three steps.

Step 1. $(G_F) \Rightarrow (H)$. By Lemma 8.2, it suffices to prove that $(G_F) \Rightarrow (HG')$. Choose $\delta' = \frac{3}{4}$ and

$$B_1 := (4K)^{-1}B \text{ and } B_2 := (6K)^{-1}B.$$

We need to show that there exists a constant $C = C(K) > 0$ such that, for all $x_1, x_2 \in \delta' B_2 = (8K)^{-1}B$ and all $y \in B_1 \setminus B_2$,

$$C^{-1}g^B(x_1, y) \leq g^B(x_2, y) \leq Cg^B(x_1, y). \tag{8.12}$$

Let us prove the first inequality in (8.12).

For $i = 1, 2$, we have that

$$\begin{aligned} d(y, x_i) &\leq d(y, x_0) + d(x_0, x_i) \\ &< (4K)^{-1}R + (8K)^{-1}R = 3(8K)^{-1}R, \end{aligned}$$

and that

$$\begin{aligned} d(y, x_i) &\geq d(y, x_0) - d(x_0, x_i) \\ &> (6K)^{-1}R - (8K)^{-1}R = (24K)^{-1}R. \end{aligned}$$

As $B \subset B(y, 2R)$, we have by $(G_F \leq)$ that

$$\begin{aligned} g^B(x_1, y) &\leq g^{B(y,2R)}(x_1, y) = g^{B(y,2R)}(y, x_1) \\ &\leq C_1 \int_{K^{-1}d(y,x_1)}^{2R} \frac{F(s)\,ds}{sV(y,s)} \\ &\leq C_1 \int_{K^{-1}(24K)^{-1}R}^{2R} \frac{F(s)\,ds}{sV(y,s)} \\ &\leq C_2 \frac{F(R)}{V(y,R)} \quad \text{(similar to (7.27)).} \end{aligned} \tag{8.13}$$

On the other hand, as $B(y, R/2) \subset B$, we have by $(G_F \geq)$ that, using the fact that $d(y, x_2) \leq 3(8K)^{-1}R < K^{-1}(R/2)$,

$$\begin{aligned} g^B(x_2, y) &\geq g^{B(y,R/2)}(x_2, y) = g^{B(y,R/2)}(y, x_2) \\ &\geq C_3 \int_{K^{-1}d(y,x_2)}^{R/2} \frac{F(s)\,ds}{sV(y,s)} \\ &\geq C_3 \int_{K^{-2}R/2}^{R/2} \frac{F(s)\,ds}{sV(y,s)} \\ &\geq C_4 \frac{F(R)}{V(y,R)} \quad \text{(similar to (7.25)).} \end{aligned} \tag{8.14}$$

Combining (8.13) and (8.14), we obtain the the first inequality in (8.12).

The second inequality also holds by interchanging x_1 and x_2. Hence, condition (HG') holds.

Step 2. $(G_F) \Rightarrow (E_F)$. We first show that, for some $C_2 > 0$,

$$\sup_{x \in B} E^B(x) \leq C_2 F(R). \tag{8.15}$$

Indeed, for $x \in B$, we have that $B \subset B(x, 2R)$, and thus

$$\begin{aligned} E^B(x) &= \int_B g^B(x,y)\, d\mu(y) \\ &\le \int_B g^{B(x,2R)}(x,y)\, d\mu(y) \\ &\le \int_B \left[C \int_{K^{-1}d(x,y)}^{2R} \frac{F(s)\, ds}{sV(x,s)} \right] d\mu(y) \quad (\text{using } (G_F \le)) \\ &\le C_2 F(R) \quad (\text{using } (7.22)), \end{aligned}$$

thus proving (8.15).

We next show the opposite inequality, that is, for some $C_1 > 0$,

$$\inf_{x \in \delta B} E^B(x) \ge C_1 F(R), \tag{8.16}$$

where $\delta = K^{-1}$. Indeed, fix $x \in \delta B$, and let $B' := B(x, (1-\delta)R)$. Then $B' \subset B$, and thus

$$\begin{aligned} E^B(x) &= \int_B g^B(x,y)\, d\mu(y) \\ &\ge \int_B g^{B'}(x,y)\, d\mu(y) \\ &\ge \int_{K^{-1}B'} \left[C^{-1} \int_{K^{-1}d(x,y)}^{(1-\delta)R} \frac{F(s)\, ds}{sV(x,s)} \right] d\mu(y) \quad (\text{using } (G_F \ge)) \\ &\ge C_1 F(R) \quad (\text{using } (7.22)), \end{aligned}$$

thus proving (8.16). □

8.4. **Implication** $(H) + (E_F) \Rightarrow (H) + (R_F)$**.** We need the following two lemmas.

Lemma 8.3. *Assume that* $(\mathcal{E}, \mathcal{F})$ *is regular. Then, for any two open subsets* $U \Subset \Omega$ *of* M *such that* $\lambda_{\min}(\Omega) > 0$*, we have*

$$\operatorname{res}(U, \Omega) \le \frac{\|E^\Omega\|_\infty}{\mu(U)}. \tag{8.17}$$

Proof. Let u_p be the capacitory potential of (U, Ω), that is, $u_p \in \mathcal{F}(\Omega)$, $u_p|_U = 1$, and

$$\mathcal{E}(u_p) = \operatorname{cap}(U, \Omega).$$

It follows that $\|u_p\|_2^2 \ge \mu(U)$, and

$$\lambda_{\min}(\Omega) \le \frac{\mathcal{E}(u_p)}{\|u_p\|_2^2} \le \frac{\operatorname{cap}(U,\Omega)}{\mu(U)}, \tag{8.18}$$

showing that

$$\operatorname{res}(U, \Omega) \le \frac{1}{\mu(U)\, \lambda_{\min}(\Omega)}. \tag{8.19}$$

On the other hand, we claim that

$$\frac{1}{\lambda_{\min}(\Omega)} \le \|E^\Omega\|_\infty. \tag{8.20}$$

Let u_e be a non-negative minimizing function for the first eigenvalue

$$\lambda_{\min}(\Omega) = \inf_{u \in \mathcal{F}(\Omega) \setminus \{0\}} \frac{\mathcal{E}(u)}{\|u\|_2^2},$$

(such a function u_e exists since $\lambda_{\min}(\Omega) > 0$), that is, $0 \le u_e \in \mathcal{F}(\Omega)$ and

$$\mathcal{E}(u_e, \varphi) = \lambda_{\min}(\Omega) \int_\Omega u_e \varphi\, d\mu \quad \text{for any } \varphi \in \mathcal{F}(\Omega).$$

In particular, taking $\varphi = G^{\Omega}\mathbf{1}_{\Omega} = E^{\Omega}$, we have

$$\mathcal{E}\left(u_e, G^{\Omega}\mathbf{1}_{\Omega}\right) = \lambda_{\min}(\Omega) \int_{\Omega} u_e \left(G^{\Omega}\mathbf{1}_{\Omega}\right) d\mu.$$

Observing that

$$\mathcal{E}\left(u_e, G^{\Omega}\mathbf{1}_{\Omega}\right) = \int_{\Omega} u_e d\mu,$$

it follows that

$$\begin{aligned} \lambda_{\min}(\Omega) &= \frac{\int_{\Omega} u_e d\mu}{\int_{\Omega} u_e \left(G^{\Omega}\mathbf{1}_{\Omega}\right) d\mu} \\ &\geq \frac{\int_{\Omega} u_e d\mu}{\|G^{\Omega}\mathbf{1}_{\Omega}\|_{\infty} \int_{\Omega} u_e d\mu} = \frac{1}{\|G^{\Omega}\mathbf{1}_{\Omega}\|_{\infty}}, \end{aligned}$$

proving our claim.

Finally, combining (8.19) and (8.20), we finish the proof. □

Lemma 8.4. *Assume that $(\mathcal{E}, \mathcal{F})$ is regular, strongly local. Let $\Omega \subset M$ be open with $\lambda_{\min}(\Omega) > 0$, and assume that the Green function g^{Ω} exists and is jointly continuous off diagonal. Then, for any open set $U \Subset \Omega$,*

$$\operatorname{res}(U, \Omega) \geq \frac{\left(\inf_{\partial U} E^{\Omega}\right)^2}{\mu(\Omega)\left\|E^{\Omega}\right\|_{\infty}}. \tag{8.21}$$

Proof. Let u_p be the capacitory potential of (U, Ω). By (8.18),

$$\lambda_{\min}(\Omega) \leq \frac{\operatorname{cap}(U, \Omega)}{\|u_p\|_2^2}.$$

We see that, using the Cauchy-Schwarz inequality,

$$\|u_p\|_2^2 = \int_{\Omega} u_p^2 d\mu \geq \frac{\left(\int_{\Omega} u_p d\mu\right)^2}{\mu(\Omega)}.$$

Note that by Lemma 6.5, for all $x \in \Omega \setminus \partial U$,

$$u_p(x) = \int_{\partial U} g^{\Omega}(x, y)\, d\nu_p(y)$$

where ν_p is the equilibrium measure of (U, Ω) supported on ∂U. Hence,

$$\begin{aligned} \int_{\Omega} u_p(x)\, d\mu(x) &= \int_{\partial U} \int_{\Omega} g^{\Omega}(x, y)\, d\mu(x)\, d\nu_p(y) \\ &= \int_{\partial U} E^{\Omega}(y) d\nu_p(y) \geq \nu_p(\partial U) \inf_{\partial U} E^{\Omega} \\ &= \operatorname{cap}(U, \Omega) \inf_{\partial U} E^{\Omega}, \end{aligned}$$

whence, it follows that

$$\begin{aligned} \lambda_{\min}(\Omega) &\leq \frac{\operatorname{cap}(U, \Omega) \mu(\Omega)}{[\operatorname{cap}(U, \Omega) \inf_{\partial U} E^{\Omega}]^2} \\ &= \frac{\operatorname{res}(U, \Omega)\, \mu(\Omega)}{\left(\inf_{\partial U} E^{\Omega}\right)^2}. \end{aligned}$$

Substituting (8.20) into this inequality, we obtain (8.21). □

We now turn to the proof.

Proof of $(H)+(E_F) \Rightarrow (H)+(R_F)$. Fix a ball $B := B(x_0, R)$. We split the proof into two steps.

Step 1. $(H)+(E_F) \Rightarrow (R_F \leq)$.

Indeed, this easily follows from (8.17): for any $\delta \in (0,1)$,

$$\operatorname{res}(\delta B, B) \leq \frac{\left\|E^B\right\|_\infty}{\mu(\delta B)} \leq C\frac{F(R)}{\mu(B)}.$$

Step 2. $(H)+(E_F) \Rightarrow (R_F \geq)$.

Let $0<\delta<\delta_1$ where δ_1 is the same as in condition $(E_F \geq)$, and let

$$U = \delta B \text{ and } \Omega = B.$$

Note that by Lemma 7.7, the function E^B is continuous in B. Hence, by condition $(E_F \geq)$, we have that

$$\inf_{\partial U} E^\Omega \geq \inf_U E^\Omega = \operatorname*{einf}_U E^\Omega \geq C^{-1}F(R).$$

Therefore, using (8.21) and condition $(E_F \leq)$, we conclude that

$$\begin{aligned}\operatorname{res}(\delta B, B) &\geq \frac{\left(\inf_{\partial U} E^\Omega\right)^2}{\mu(U)\left\|E^\Omega\right\|_\infty} \\ &\geq \frac{\left[C^{-1}F(R)\right]^2}{\mu(U)\left[CF(R)\right]} \geq C'\frac{F(R)}{\mu(B)},\end{aligned}$$

thus proving condition $(R_F \geq)$, as desired. □

9. Appendix

9.1. **Capacity.** Recall that the capacity $\operatorname{cap}(A,\Omega)$ as well as the notion of a cap-quasi-continuous function are defined in Section 3. It easily follows from the definition (3.5) that, for any two Borel sets $A, B \Subset \Omega$,

$$\operatorname{cap}(A\cup B,\Omega) \leq \operatorname{cap}(A,\Omega)+\operatorname{cap}(B,\Omega). \tag{9.1}$$

It follows from (9.1) and (3.6) that, for any sequence $\{A_i\}_{i=1}^\infty$ of precompact open subsets of Ω,

$$\operatorname{cap}(\cup_{n=1}^\infty A_i,\Omega) = \lim_{k\to\infty}\operatorname{cap}(\cup_{n=1}^k A_i,\Omega) \leq \sum_{i=1}^\infty \operatorname{cap}(A_i,\Omega). \tag{9.2}$$

Lemma 9.1. *Assume that* $(\mathcal{E},\mathcal{F})$ *is a regular Dirichlet form and* Ω *is an open subset of* M. *Then, each function* $u \in \mathcal{F}(\Omega)$ *admits a* cap-*quasi-continuous version.*

Proof. We adapt the proof of [16, Thm 2.1.3 (p.71)] to our capacity. We first show that, for each $u \in \mathcal{F}\cap C_0(\Omega)$ and each $\lambda>0$,

$$\operatorname{cap}(G,\Omega) \leq \frac{4}{\lambda^2}\mathcal{E}(u), \tag{9.3}$$

where

$$G := \{x\in\Omega : |u(x)|>\lambda\}.$$

Indeed, let

$$G' := \left\{x\in\Omega : |u(x)| > \frac{\lambda}{2}\right\}.$$

Then both G and G' are open and precompact in Ω, because u is continuous in Ω with compact support. Also, we have

$$\overline{G} = \{x\in\Omega : |u(x)|\geq\lambda\} \subset G'.$$

Set

$$\varphi := \frac{u}{\lambda/2}\wedge 1.$$

Clearly, $\varphi \in \mathcal{F} \cap C_0(\Omega)$, and $\varphi = 1$ on G', and hence, it is a test function for $\operatorname{cap}(G,\Omega)$, that is

$$\operatorname{cap}(G,\Omega) \leq \mathcal{E}(\varphi) \leq \frac{4}{\lambda^2}\mathcal{E}(u),$$

thus proving (9.3).

For each $u \in \mathcal{F}(\Omega)$, by the regularity of $(\mathcal{E},\mathcal{F})$, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset \mathcal{F} \cap C_0(\Omega)$ such that $u_n \xrightarrow{\mathcal{F}} u$ as $n \to \infty$. Without loss of generality, we can assume that, for any $l \geq 1$,

$$\mathcal{E}(u_{l+1} - u_l) \leq 2^{-3l}. \tag{9.4}$$

Set

$$\begin{aligned} G_l &= \left\{x \in \Omega : |u_{l+1}(x) - u_l(x)| > 2^{-l}\right\}, \\ F_k &= \Omega \setminus \left(\cup_{l=k}^{\infty} G_l\right) = \cap_{l=k}^{\infty}(\Omega \setminus G_l). \end{aligned}$$

Note that each G_l is a precompact open subset of Ω. Fix some $k \geq 1$. For any $x \in F_k$ and any $l \geq k$, we have

$$|u_{l+1}(x) - u_l(x)| \leq 2^{-l}.$$

It follows that the sequence $\{u_l(x)\}$ is Cauchy in $C(F_k)$ and, hence, it converges uniformly to a continuous function on F_k. Let

$$\widetilde{u}(x) = \lim_{l\to\infty} u_l(x).$$

Then $\widetilde{u}$ is defined on $\cup_{k=1}^{\infty} F_k$, and $\widetilde{u}|_{F_k}$ is continuous for each $k \geq 1$. Moreover, using (9.2), (9.3) and (9.4), we obtain

$$\begin{aligned} \operatorname{cap}(\Omega \setminus F_k, \Omega) &\leq \sum_{l=k}^{\infty} \operatorname{cap}(G_l,\Omega) \leq \sum_{l=k}^{\infty} \frac{4}{2^{-2l}}\mathcal{E}(u_{l+1} - u_l) \\ &\leq \sum_{l=k}^{\infty} \frac{4}{2^{-2l}} \cdot 2^{-3l} = 8 \cdot 2^{-k}. \end{aligned}$$

We conclude that $\widetilde{u}$ is continuous on F_k, the set $\Omega \setminus F_k = {F_k}^c$ is open, and $\operatorname{cap}(\Omega \setminus F_k, \Omega) \leq 8 \cdot 2^{-k}$. Since $\widetilde{u} = u$ μ-a.e., we conclude that $\widetilde{u}$ is a cap-quasi-continuous version of u in Ω. □

The next proposition shows that the capacitory potential u_p of (A,Ω) exists for any compact subset A. In the classical potential theory, this issue is called the *equilibrium problem* or the *Robin problem* (cf. [35, p.189]). It turns out that the capacitory potential u_p of (A,Ω) is a reduced function of any cutoff function of $(\overline{\Omega}, M)$ for any precompact open Ω with $\lambda_{\min}(\Omega) > 0$.

Proposition 9.2. *Assume that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form. Let $\Omega \subset M$ be precompact open such that $\lambda_{\min}(\Omega) > 0$, and let ψ be any cutoff function of $(\overline{\Omega}, M)$ and let A be a compact subset of Ω. Then the capacitory potential u_p of (A,Ω) is a reduced function of ψ w.r.t. (A,Ω). If in addition $(\mathcal{E},\mathcal{F})$ is strongly local, then u_p is superharmonic in Ω.*

Proof. Let u_p be the capacitory potential of (A,Ω). By the standard approach, there exists a minimizing sequence $\{u_k\}_{k=1}^{\infty}$ of cutoff functions of (A,Ω) such that $u_k \xrightarrow{\mathcal{F}} u_p$ as $k \to \infty$, and

$$\mathcal{E}(u_p) = \operatorname{cap}(A,\Omega),$$

and moreover, the function $u_p \in \mathcal{F}(\Omega), 0 \leq u_p \leq 1$ in Ω, and $u_p|_A = 1$. Note that this potential u_p is unique. Also u_p is harmonic in $U = \Omega \setminus A$, since for any $\varphi \in \mathcal{F} \cap C_0(U)$ and any number a, each function $u_k + a\varphi$ for $k \geq 1$ is a cutoff function of (A,Ω), and thus

$$\begin{aligned} \operatorname{cap}(A,\Omega) &\leq \mathcal{E}(u_k + a\varphi) = \mathcal{E}(u_k) + 2a\mathcal{E}(u_k,\varphi) + a^2\mathcal{E}(\varphi) \\ &\to \mathcal{E}(u_p) + 2a\mathcal{E}(u_p,\varphi) + a^2\mathcal{E}(\varphi), \end{aligned}$$

which implies that $2a\mathcal{E}(u_p,\varphi) + a^2\mathcal{E}(\varphi) \geq 0$, showing that $\mathcal{E}(u_p,\varphi) = 0$.

Since ψ is a cutoff function of $(\overline{\Omega}, M)$, it is straightforward to verify that u_p is a reduced function of ψ.

Finally, if $(\mathcal{E},\mathcal{F})$ is strongly local, the cutoff function ψ is harmonic (in particular superharmonic) in Ω. Therefore, we obtain from Lemma 6.4 that u_p is superharmonic in Ω. □

9.2. **Functions in $\mathcal{F}(\Omega\setminus A)$.** The following give a sufficient condition for a function belonging to the space $\mathcal{F}(\Omega\setminus A)$, and it can be viewed as a supplement of Proposition 2.8 in [21].

Proposition 9.3. *Assume that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form. Let $\Omega\subset M$ be open, and let $S\subset\Omega$ be compact. If $v\in\mathcal{F}(\Omega)$ vanishes in a neighborhood V of A, then $v\in\mathcal{F}(\Omega\setminus A)$.*

Proof. Note that $v=v_+-v_-$, and $v_+=v_-=0$ in V, and that $v_+,v_-\in\mathcal{F}$. It suffices to assume that $v\geq 0$ in Ω. We can also assume that v is bounded because otherwise consider a sequence $v_k:=v\wedge k$ that tends to v in $\mathcal{F}$-norm as $k\to\infty$ by [16, Theorem 1.4.2(iii), p.28]; if we already know that $v_k\in\mathcal{F}(\Omega\setminus A)$ then we can conclude that also $v\in\mathcal{F}(\Omega\setminus A)$. Hence, we can assume in the sequel that v is non-negative and bounded in M, say $0\leq v\leq 1$.

Let φ be a cut-off function of (A,V). Let $\{v_k\}_{k=1}^{\infty}$ be a sequence of functions from $\mathcal{F}\cap C_0(\Omega)$ such that $v_k\xrightarrow{\mathcal{F}}v$ as $k\to\infty$. Consider

$$u_k:=v_k-v_k\wedge\varphi.$$

Note that each $u_k\in\mathcal{F}\cap C_0(\Omega)$, $u_k=0$ in A, and hence, the support of u_k is outside a neighborhood of A, that is,

$$u_k\in\mathcal{F}\cap C_0(\Omega\setminus A).$$

We claim that $\{u_k\}$ converges to v weakly in $\mathcal{F}$:

$$u_k\overset{\mathcal{F}}{\rightharpoonup}v \text{ as } k\to\infty.$$

Indeed, as $v\geq 0$ and $v_k\xrightarrow{\mathcal{F}}v$, we have by [16, Theorem 1.4.2(v), p.28] that $|v_k-\varphi|\overset{\mathcal{F}}{\rightharpoonup}|v-\varphi|$, as $k\to\infty$. It follows that

$$\begin{aligned} v_k\wedge\varphi &= \frac{1}{2}\left[v_k+\varphi-|v_k-\varphi|\right] \\ &\overset{\mathcal{F}}{\rightharpoonup} \frac{1}{2}\left[v+\varphi-|v-\varphi|\right]=v\wedge\varphi, \end{aligned}$$

and hence, $u_k=v_k-v_k\wedge\varphi\overset{\mathcal{F}}{\rightharpoonup}v-v\wedge\varphi=v$, proving our claim.

Since $u_k\in\mathcal{F}\cap C_0(\Omega\setminus A)$, we conclude that $v\in\mathcal{F}(\Omega\setminus A)$. □

As a conclusion of this subsection, we will give a decomposition of a function $u=u_1+u_2\in\mathcal{F}(U\cup V)$ such that $u_1\in\mathcal{F}(U_1),u_2\in\mathcal{F}(V_1)$ for any disjoint neighborhoods U_1,V_1 of U,V respectively.

Proposition 9.4. *Assume that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form. Let U,V be two precompact open subsets of M such that their closures $\overline{U},\overline{V}$ are disjoint. If $u\in\mathcal{F}(U\cup V)\cap L^\infty(M)$, we can decompose $u=u_1+u_2$, where $u_1\in\mathcal{F}(U_1),u_2\in\mathcal{F}(V_1)$, and where U_1,V_1 are any respective neighborhoods of U,V with disjoint closures $\overline{U_1},\overline{V_1}$.*

Proof. Let ϕ be a cutoff function of (U,U_1). Since $u\in\mathcal{F}\cap L^\infty$, we see also that $u_1:=u\phi\in\mathcal{F}\cap L^\infty$. We show that $u_1\in\mathcal{F}(U_1)$. In fact, since the support of u_1 is contained in the set

$$\operatorname{supp}(u)\cap\operatorname{supp}(\phi)\subsetneqq\overline{U\cup V}\cap\overline{U_1}=\overline{U}\subset U_1.$$

Hence, as U is precompact, we obtain by [21, Prop. 2.8, p.2620] that $u_1\in\mathcal{F}(U_1)$.

To show that $u_2:=(1-\phi)u\in\mathcal{F}(V_1)$, observe that the support of u_2 is contained in the set

$$\operatorname{supp}(u)\cap\operatorname{supp}(1-\phi)\subsetneqq\overline{U\cup V}\cap\overline{M\setminus U}=\overline{V}\subset V_1.$$

Hence, using [21, Prop. 2.8, p.2620] again, we have that $u_2\in\mathcal{F}(V_1)$.

Finally, note that

$$u=u\phi+(1-\phi)u=u_1+u_2.$$

We finish the proof. □

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