Analysis on fractal spaces and heat kernels

Alexander Grigor'yan
University of Bielefeld and YMSC

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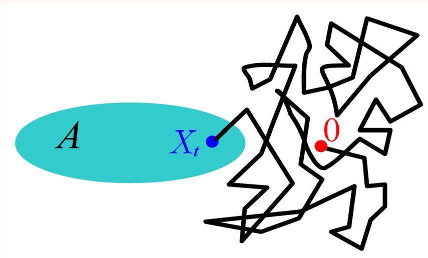
Classical heat kernel

The *heat kernel* in $\mathbb{R}^n$ is the fundamental solution of the heat equation $\partial_t u = \Delta u$:

$$p_t(x) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right).$$

This function is also called the Gauss-Weierstrass function. Some applications:

- Solving the Cauchy problem: $u(t,\cdot) = p_t * f$.
- Mollification of functions: $p_t * f \to f$ as $t \to 0$ locally uniformly provided $f \in C_b(\mathbb{R})$.
- Proof of Sobolev embedding theorems.
- $p_t(x)$ is the Gauss distribution in $\mathbb{R}^n$ and the transition density of Brownian motion:
  $$\mathbb{P}(X_t \in A) = \int_A p_t(x)dx.$$



- Approximation of the Dirichlet integral: for any $f \in W^{1,2}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |\nabla f|^2 dx = \lim_{t\to 0} \frac{1}{2t} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} p_t(x-y)\,|f(x)-f(y)|^2 dxdy.$$

Heat kernels on manifolds

Let M be a complete non-compact Riemannian manifold. The Laplace-Beltrami operator Δ on M possesses *the heat equation*: the smallest positive fundamental solution $p_t(x, y)$ of the heat equation $\partial_t u = \Delta u$ that is a smooth function of $t > 0$ and $x, y \in M$.

On any manifold, the heat kernel satisfies the following *Varadhan asymptotics*:

$$\log p_t(x, y) \sim -\frac{d^2(x, y)}{4t} \quad \text{as } t \to 0+,$$

where d is the geodesic distance. It also satisfies the upper bound of *Davies-Gaffney*: for any disjoint measurable subsets A and B of M,

$$\int_A \int_B p_t(x, y) d\mu(x)\mu(y) \leq \sqrt{\mu(A)\mu(B)} \exp\left(-\frac{d^2(A, B)}{4t}\right)$$

where μ is the Riemannian measure.

If $Ricci_M \geq 0$ then the heat kernel admits two sided pointwise Gaussian estimates *of Li–Yau*:

$$p_t\left(x, y\right) \asymp \frac{C}{\mu(B\left(x, \sqrt{t}\right))} \exp\left(-c\frac{d^2\left(x, y\right)}{t}\right),$$

where $B(x, r)$ denotes geodesic ball. We see that the heat kernel on the Euclidean spaces and manifolds exhibits the following space/time scaling: *time=distance*^{2}.

Analysis on metric spaces: integration

Since the time of Newton and Leibniz, mathematical analysis consists of differentiation and integration. By Lebesgue, integration amounts to construction of a measure.

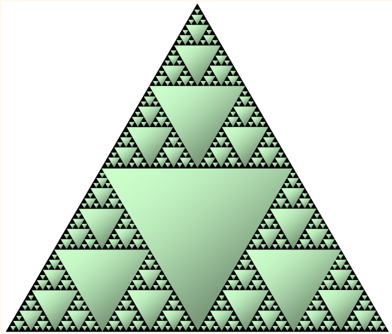
Let (M, d) be a metric space and μ be a Borel measure on M. We always assume that M is *α-regular*, that is, for any metric ball $B(x,r) := \{y \in M : d(x,y) < r\}$ of radius $r < r_0$,

$$\mu\left(B\left(x,r\right)\right) \simeq r^{\alpha}, \tag{1}$$

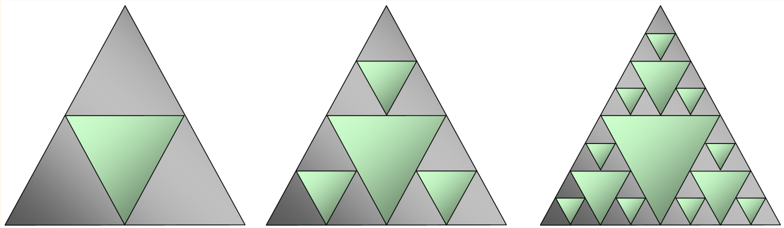
where $\alpha > 0$. It follows from (1) that α is the Hausdorff dimension of M and $\mathcal{H}_\alpha \simeq \mu$. Hence, in some sense, α is a numerical characteristic of the integral calculus on M.

α-regular spaces with fractional α are usually called *fractals*. Fractals first appeared in mathematics as curious examples that initially served as counterexamples to illustrate various theorems (like the *Cantor set*).

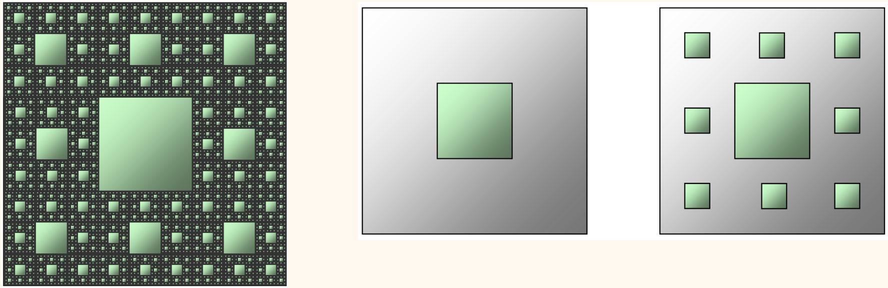
Here is a connected fractal set – *Sierpinski gasket*:



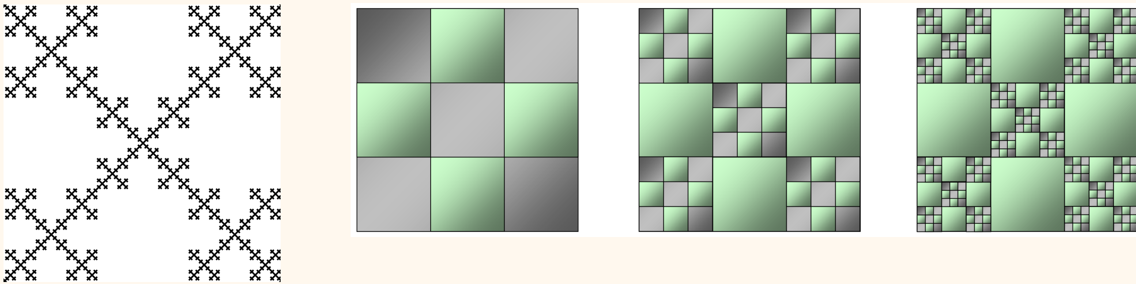
Three steps of construction of SG: $\alpha = \frac{\log 3}{\log 2} \approx 1.58.$



Sierpinski carpet and two steps of construction of SC: $\alpha = \frac{\log 8}{\log 3} \approx 1.89.$



Vicsek snowflake and three steps of construction of VS: $\alpha = \frac{\log 5}{\log 3} \approx 1.46,$



Analysis on metric spaces: differentiation

On certain metric spaces, including fractal spaces, it is possible to construct a *Laplace-type* operator, by means of the theory of Dirichlet forms (Beurling–Deny and Fukushima).

A *Dirichlet form* in $L^2(M,\mu)$ is a pair $(\mathcal{E},\mathcal{F})$ where $\mathcal{F}$ is dense subspace of $L^2(M,\mu)$ and $\mathcal{E}$ is a symmetric bilinear form on $\mathcal{F}$ with the following properties:

- It is *positive definite*, that is, $\mathcal{E}(f,f) \geq 0$ for all $f \in \mathcal{F}$.
- It is *closed*, that is, $\mathcal{F}$ is complete with respect to the norm

$$\int_M f^2 d\mu + \mathcal{E}(f,f).$$

- It is *Markovian*, that is, if $f \in \mathcal{F}$ then $\widetilde{f} := \min(f_+, 1) \in \mathcal{F}$ and $\mathcal{E}(\widetilde{f},\widetilde{f}) \leq \mathcal{E}(f,f)$.

Any Dirichlet form has the generator: a positive definite self-adjoint operator $\mathcal{L}$ in $L^2(M,\mu)$ with domain $\mathrm{dom}(\mathcal{L}) \subset \mathcal{F}$ such that $(\mathcal{L}f,g) = \mathcal{E}(f,g)$ for all $f \in \mathrm{dom}(\mathcal{L})$ and $g \in \mathcal{F}$.

For example, the Dirichlet integral

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^n} |\nabla f|^2 \, dx \tag{2}$$

is the quadratic part of a Dirichlet form $(\mathcal{E},\mathcal{F})$ with domain $\mathcal{F} = W_2^1(\mathbb{R}^n)$. Its generator is $\mathcal{L} = -\Delta$ with $\mathrm{dom}(\mathcal{L}) = W_2^2(\mathbb{R}^n)$.

Another example of a Dirichlet form in $\mathbb{R}^n$:

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(f(x)-f(y))^2}{|x-y|^{n+s}} dxdy, \tag{3}$$

where $s \in (0,2)$ and $\mathcal{F} = B^{s/2}_{2,2}(\mathbb{R}^n)$. Its generator is $\mathcal{L} = (-\Delta)^{s/2}$.

The generator $\mathcal{L}$ of any Dirichlet form determines the *heat semigroup* $\{e^{-t\mathcal{L}}\}_{t\geq 0}$ in $L^2(M,\mu)$.

If the operator $e^{-t\mathcal{L}}$ for $t>0$ is an integral operator:

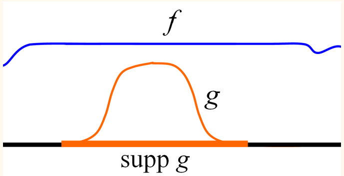
$$e^{-t\mathcal{L}} f(x) = \int_M p_t(x,y) f(y) d\mu(y) \quad \text{for all } f \in L^2,$$

then its integral kernel $p_t(x,y)$ (that is ≥ 0) is called the *heat kernel* of $\mathcal{L}$.

A Dirichlet form $(\mathcal{E},\mathcal{F})$ is called *strongly local* if $\mathcal{E}(f,g)=0$ whenever

$f = \text{const}$ in a neighborhood of $\operatorname{supp} g$.

For example, the Dirichlet form (2) is strongly local, while the Dirichlet form (3) is non-local.



The local Dirichlet form (2) with the generator $\mathcal{L} = -\Delta$ has the heat kernel

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right). \tag{4}$$

The non-local Dirichlet form (3) with the generator $\mathcal{L} = (-\Delta)^{s/2}$ has the heat kernel that admits the following estimate:

$$p_t(x,y) \simeq \frac{1}{t^{n/s}} \left(1 + \frac{|x-y|}{t^{1/s}}\right)^{-(n+s)}. \tag{5}$$

In the special case $s = 1$ the heat kernel of $(-\Delta)^{1/2}$ coincides with the Cauchy distribution with the scale parameter t:

$$p_t(x,y) = \frac{c_n t}{\left(t^2 + |x-y|^2\right)^{\frac{n+1}{2}}} = \frac{c_n}{t^n} \left(1 + \frac{|x-y|^2}{t^2}\right)^{-\frac{n+1}{2}},$$

where $c_n = \Gamma\left(\frac{n+1}{2}\right)/\pi^{(n+1)/2}$.

A Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *regular* if $\mathcal{F} \cap C_0(M)$ is dense both in $\mathcal{F}$ and $C_0(M)$. For example, the both Dirichlet forms (2) and (3) are regular.

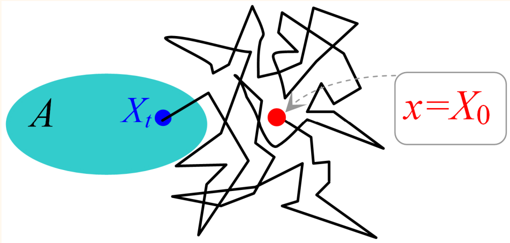
Any regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ determines a Markov processes $\{X_t\}_{t \geq 0}$ on M with the transition semigroup $e^{-t\mathcal{L}}$, which means that

$$\mathbb{E}_x f(X_t) = e^{-t\mathcal{L}} f(x) \quad \text{for all } f \in C_0(M) \text{ and } t \geq 0.$$

If the heat kernel of $(\mathcal{E}, \mathcal{F})$ exists then it serves as the transition density of $\{X_t\}$:

$$\mathbb{P}_x(X_t \in A) = \int_A p_t(x, y) d\mu(y),$$

for any Borel set $A \subset M$ and $t > 0$.



If $(\mathcal{E}, \mathcal{F})$ is local then $\{X_t\}$ is a diffusion process (=with continuous trajectories), while otherwise the trajectories of the process $\{X_t\}$ contain jumps.

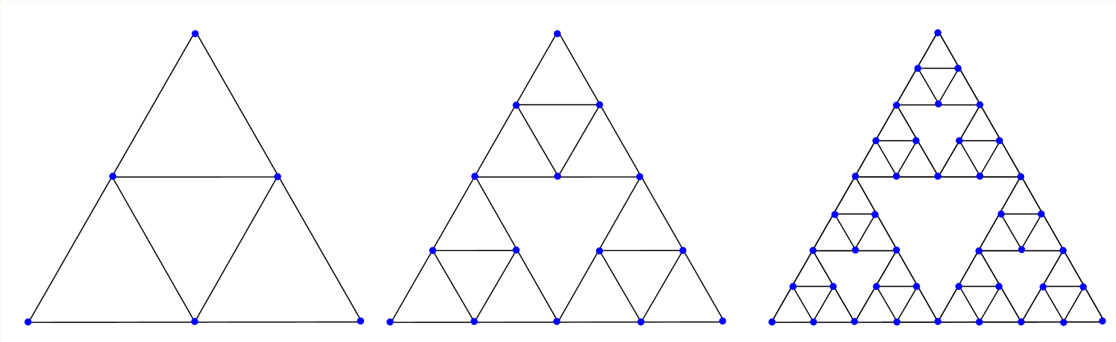
For example, the local Dirichlet form (2) with the generator $\mathcal{L} = -\Delta$ determines Brownian motion in $\mathbb{R}^n$ with the transition density (4).

The non-local Dirichlet form (3) with the generator $\mathcal{L} = (-\Delta)^{s/2}$ determines a symmetric stable Levy process in $\mathbb{R}^n$ of the index s with the transition density (5).

If a metric measure space M possesses a strongly local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ then its generator $\mathcal{L}$ can be regarded as an analogue of the Laplace operator; hence, it determines in some sense differential calculus on M.

Nontrivial strongly local regular Dirichlet forms have been successfully constructed on large families of fractals, in particular, on SG by Barlow–Perkins '88, Goldstein '87 and Kusuoka '87, on SC by Barlow–Bass '89 and Kusuoka–Zhou '92, on p.c.f. fractals (including VS) by Kigami '93.

Each of these fractals can be regarded as limit of a sequence of approximating graphs Γ_n.



Approximating graphs $\Gamma_1, \Gamma_2, \Gamma_3$ for SG

Define on each Γ_n a Dirichlet form $\mathcal{E}_n$ by

$$\mathcal{E}_n(f,f) = \sum_{x \sim y} (f(x) - f(y))^2$$

(where $x \sim y$ denotes neighboring vertices on Γ_n), and then consider a scaled limit

$$\mathcal{E}(f,f) = \lim_{n\to\infty} R_n \mathcal{E}_n(f,f) \tag{6}$$

with an appropriate renormalizing sequence $\{R_n\}$.
The main difficulty is to ensure the existence of $\{R_n\}$ such that this limit exists in $(0,\infty)$ for a dense in L^2 family of functions f.
For p.c.f. fractals one chooses $R_n = \rho^n$ where, for example, $\rho = \frac{5}{3}$ for SG and $\rho = 3$ for VS, and the limit in (6) exists due to monotonicity.

For SC the situation is much harder. Initially a strongly local Dirichlet form on SC was constructed by Barlow and Bass '89 in a different way by using a probabilistic approach. After a work of Barlow, Bass, Kumagai and Teplyaev '10 it became possible to claim that the limit (6) exists for a certain sequence $\{R_n\}$ such that $R_n \simeq \rho^n$, where the exact value of ρ is still unknown. Numerical computation indicates that $\rho \approx 1.25$.
Other methods of constructing a strongly local Dirichlet form on SC were proposed by Kusuoka and Zhou '92 and AG and M.Yang '19.

Walk dimension

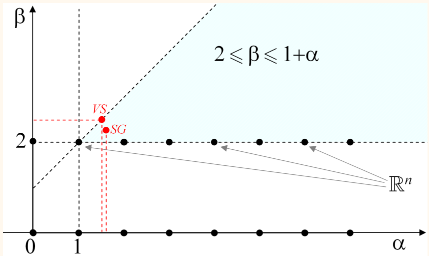
In all the above examples of fractals, the strongly local Dirichlet form possesses the heat kernel that satisfies the following *sub-Gaussian* estimate:

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{7}$$

(where $C, c > 0$), for all $x, y \in M$ and $t \in (0, t_0)$ (Barlow–Perkins '88, Barlow–Bass '92).

Here α is the Hausdorff dimension of the underlying metric space (M, d) while β is a new parameter that is called the *walk dimension*. It can be regarded as a numerical characteristic of the differential calculus on M that is determined by the generator $\mathcal{L}$.

It is known that always $\beta \geq 2$. Moreover, for any pair of reals $\alpha \geq 1$ and $\beta \in [2, \alpha + 1]$ there exists a *geodesic* metric measure space with the heat kernel satisfying (7) (Barlow '04).



Hence, we obtain a large family of regular metric measure spaces that are characterized by a pair (α, β), where α is responsible for integration while β is responsible for differentiation.

The Euclidean space $\mathbb{R}^n$ belongs to this family with $\alpha = n$ and $\beta = 2$ (in the case $\beta = 2$ the estimate (7) becomes Gaussian).

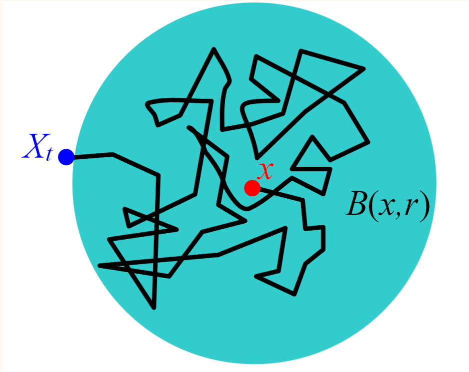
On fractals the values of β is determined by the scaling parameter ρ. It is known that:

- on SG: $\beta = \frac{\log 5}{\log 2} \approx 2.32$ (and $\alpha = \frac{\log 3}{\log 2} \approx 1.58$)
- on VS: $\beta = \frac{\log 15}{\log 3} \approx 2.46$ (and $\alpha = \frac{\log 5}{\log 3} \approx 1.46$)
- on SC: $\beta = \frac{\log(8\rho)}{\log 3} \approx 2.10$ (and $\alpha = \frac{\log 8}{\log 3} \approx 1.89$).

The walk dimension β has the following probabilistic meaning.

For any open set $\Omega \subset M$, denote by τ_Ω the first exit time of diffusion X_t from Ω:

$$\tau_\Omega = \inf\{t > 0 : X_t \notin \Omega\}.$$



It is known that if (7) holds, then for any ball $B(x, r)$ with $r < r_0$,

$$\mathbb{E}_x \tau_{B(x,r)} \simeq r^\beta.$$

That is, we have fractal scaling *time=distance*$^\beta$ that is different from Euclidean *time=distance*^{2}.

Besov spaces characterization of β

Given an α-regular metric measure space (M, d, μ), it is possible to define a family $B^{\sigma}_{p,q}$ of Besov spaces, where $p, q \in [1, \infty]$, $\sigma > 0$. Here we need only the following special cases: for any $\sigma > 0$ the space $B^{\sigma}_{2,2}$ consists of functions $f \in L^2(M, \mu)$ such that

$$\|f\|^2_{\dot{B}^{\sigma}_{2,2}} := \int\int_{M\times M} \frac{|f(x) - f(y)|^2}{d(x, y)^{\alpha+2\sigma}} d\mu(x)d\mu(y) < \infty,$$

and $B^{\sigma}_{2,\infty}$ consists of functions $f \in L^2(M, \mu)$ such that

$$\|f\|^2_{\dot{B}^{\sigma}_{2,\infty}} := \sup_{0<r<r_0} \frac{1}{r^{\alpha+2\sigma}} \int\int_{\{d(x,y)<r\}} |f(x) - f(y)|^2 d\mu(x)d\mu(y) < \infty.$$

It is easy to see that the space $B^{\sigma}_{2,2}$ shrinks as σ increases. Define

$$\boxed{\sigma^* = \sup\{\sigma > 0 : B^{\sigma}_{2,2} \text{ is dense in } L^2\}}. \tag{8}$$

If $\sigma < 1$ then $B^{\sigma}_{2,2}$ contains all Lipschitz functions with compact support. Hence, $\sigma^* \geq 1$. In $\mathbb{R}^n$, if $\sigma > 1$ then $B^{\sigma}_{2,2} = \{0\}$ so that $\sigma^* = 1$. On most fractal spaces $\sigma^* > 1$.

Theorem 1 (AG, Jiaxin Hu, Ka-Sing Lau) *Let $(\mathcal{E},\mathcal{F})$ be a strongly local Dirichlet form on (M,d,μ) such that its heat kernel exists and satisfies the sub-Gaussian estimate*

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{9}$$

with some α and β. Then the following is true:
(a) the space M is α-regular (consequently, $\alpha = \dim_H M$ and $\mu \simeq \mathcal{H}_a$);
(b) $\beta = 2\sigma^*$ (consequently, $\beta \geq 2$);
(c) $\mathcal{F} = B^{\sigma^*}_{2,\infty}$ *and* $\mathcal{E}(f,f) \simeq \|f\|^2_{\dot{B}^{\sigma^*}_{2,\infty}}$.

Corollary 2 *Both α and β in (9) are the invariants of the metric structure (M,d) alone.*

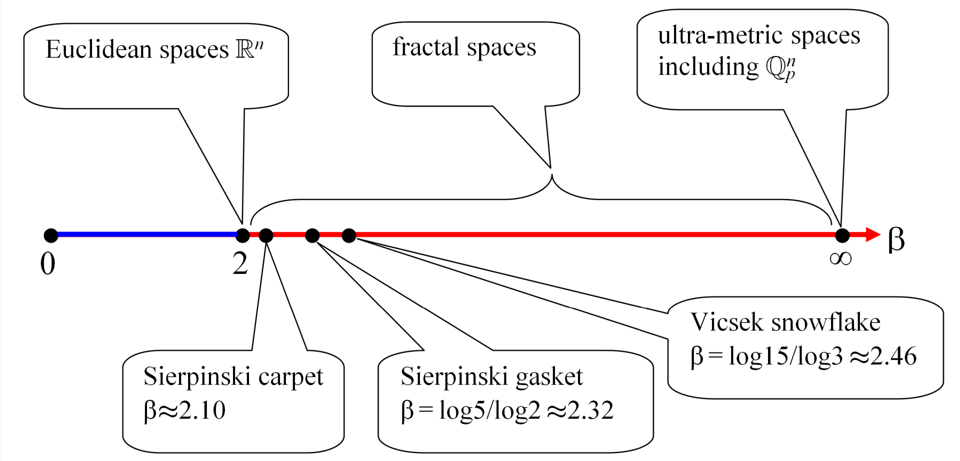
Indeed, σ^* is defined by using metric d and measure μ, while in this case $\mu \simeq \mathcal{H}_\alpha$ is also determined by d. Therefore, σ^* and β are also invariants of the metric space (M,d).

Note that σ^* is defined by (8) for any regular metric space. In the view of Theorem 1, we redefine now the notion of the walk dimension by setting

$$\boxed{\beta := 2\sigma^*}.$$

Hence, β is the second invariant of a regular metric space after the Hausdorff dimension α.

Here is a classification of regular metric spaces according to their walk dimension.



A metric space (M, d) is called *ultra-metric* if it satisfies a stronger triangle inequality

$$d(x,y) \leq \max\left(d\left(x,z\right), d\left(y,z\right)\right) \quad \text{for all } x, y, z \in M.$$

Examples of ultra-metric spaces: the field $\mathbb{Q}_p$ of p-adic numbers with the p-adic distance $|x-y|_p$ and Q_p^n with max-distance. All ultra-metric spaces are totally disconnected and, hence, cannot carry a non-trivial diffusion. On the other hand, on such spaces, for any $\sigma > 0$, the space $B_{2,2}^{\sigma}$ contains indicator functions $\mathbf{1}_B$ of all balls and, hence, is dense in L^2. Consequently, $\sigma^* = \infty$ (A.Bendikov, AG, Eryan Hu, Jiaxin Hu, '21).

An approach to construction of local Dirichlet forms

An open question. *Let (M, d, μ) be an α-regular metric measure space (or even self-similar). Assume $\sigma^* < \infty$. Does there exist a strongly local (regular) Dirichlet form in M? Does there exist a heat kernel satisfying the sub-Gaussian estimate* (9) *with $\beta = 2\sigma^*$? Which additional conditions may be required?*

Here is a possible approach to construction of such a Dirichlet form based on the family of Besov spaces. For any $\sigma < \sigma^*$ we need to define in $B^\sigma_{2,2}$ a quadratic form $\mathcal{E}_\sigma(f,f)$ with the following properties:

(i)
$$\mathcal{E}_\sigma(f,f) \simeq \|f\|^2_{\dot{B}^\sigma_{2,2}} = \int\int_{M\times M} \frac{|f(x) - f(y)|^2}{d(x,y)^{\alpha+2\sigma}} d\mu(x) d\mu(y),$$

(ii) there should exist in some sense the limit

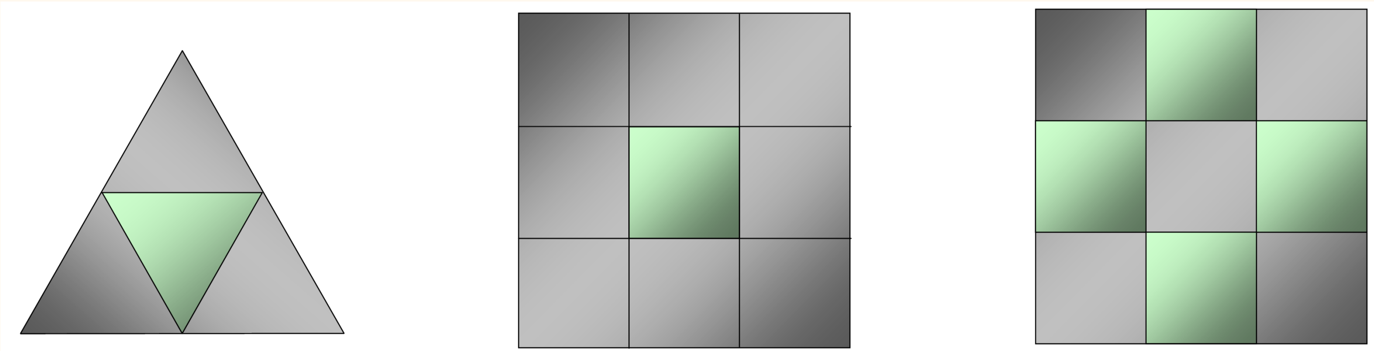
$$\lim_{\sigma\to\sigma^*} (\sigma^* - \sigma)\, \mathcal{E}_\sigma,$$

(iii) this limit should determine a strongly local regular Dirichlet form on M.

In $\mathbb{R}^n$ this method works with $\mathcal{E}_\sigma(f,f) = \|f\|^2_{\dot{B}^\sigma_{2,2}}$ and yields the Dirichlet integral. For SG and SC this method was realized by AG and Meng Yang '18 and '19. However, in the general case there are two many difficulties.

A related question: how to determine the walk dimension, even for self-similar sets?

Each self-similar set is determined by the first step in its construction:



SG SC VS

It is well known how to compute the Hausdorff dimension: $\alpha = \frac{\log A}{\log B}$ where A is the number of remaining cells after the first step, and B is the contraction ratio.

An open question. *How to compute the walk dimension β using the first step in the fractal construction? This must be some graph invariant.*

The exact value of β remains open for the Sierpinski carpet.

Self-similar heat kernels

Let (M, d) be metric space and μ be an α-regular measure on M.

Theorem 3 (AG–Takashi Kumagai) *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on M. Assume that its heat kernel satisfies the following estimate:*

$$p_t(x, y) \asymp \frac{C}{t^{\alpha/\beta}} \Phi\left(c \frac{d(x, y)}{t^{1/\beta}}\right),$$

where $\alpha, \beta > 0$ and Φ is a positive function on $[0, \infty)$. Then the following dichotomy holds :

- *either the Dirichlet form $\mathcal{E}$ is strongly local and $\Phi(s) \asymp C \exp(-cs^{\frac{\beta}{\beta-1}})$.*
- *or the Dirichlet form $\mathcal{E}$ is non-local and $\Phi(s) \simeq (1+s)^{-(\alpha+\beta)}$.*

That is, in the first case $p_t(x, y)$ satisfies the sub-Gaussian estimate

$$p_t(x, y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x, y)}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{10}$$

while in the second case we obtain a *stable-like estimate*

$$p_t(x, y) \simeq \frac{1}{t^{\alpha/\beta}}\left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{11}$$

Estimating heat kernels: strongly local case

Let M be a metric space with precompact balls, μ be an α-regular measure on M and $(\mathcal{E},\mathcal{F})$ be a strongly local regular Dirichlet form on M.

Definition. We say that (M,d) satisfies the *chain condition* (CC) if $\exists C$ such that for all $x,y\in M$ and for $n\in\mathbb{N}$ there exists a sequence $\{x_k\}_{k=0}^n$ of points in M such that $x_0=x$, $x_n=y$, and

$$d(x_{k-1},x_k)\le C\frac{d(x,y)}{n},\quad \text{for all } k=1,...,n.$$

Definition. We say that $(\mathcal{E},\mathcal{F})$ satisfies the Poincaré inequality with exponent β if, for any ball $B=B\left(x,r\right)$ on M and for any function $f\in\mathcal{F}$,

$$\mathcal{E}_B\left(f,f\right)\ge\frac{c}{r^\beta}\int_{\varepsilon B}\left(f-\overline{f}\right)^2d\mu, \tag{PI}$$

where $\overline{f}=⨍_{\varepsilon B}f d\mu$, and c,ε are small positive constant independent of B and f. For example, in $\mathbb{R}^n$ (PI) holds with $\beta=2$ and $\varepsilon=1$.
Let $A\Subset B$ be two open subset of M. Define the capacity of the capacitor (A,B) as follows:

$$\operatorname{cap}(A,B):=\inf\left\{\mathcal{E}\left(\varphi,\varphi\right):\varphi\in\mathcal{F},\ \varphi|_{\overline{A}}=1,\ \operatorname{supp}\varphi\Subset B\ \right\}.$$

Definition. We say that $(\mathcal{E},\mathcal{F})$ satisfies the *capacity condition* if, for any two concentric balls $B_0 := B(x,R)$ and $B := B(x,R+r)$,

$$\operatorname{cap}(B_0,B) \le C\frac{\mu(B)}{r^\beta}. \tag{cap}$$

Conjecture. $(CC) + (PI) + (\text{cap}) \Leftrightarrow (10)$

The implication $\Leftarrow$ is known to be true, so the main difficulty is in $\Rightarrow$.

Let $A \Subset B$ be two open subset of M. For any measurable function u on B, define the *generalized capacity* $\operatorname{cap}_u(A,B)$ by

$$\operatorname{cap}_u(A,B) = \inf\left\{\mathcal{E}\left(u^2\varphi,\varphi\right) : \varphi \in \mathcal{F},\ \varphi|_{\overline{A}} = 1,\ \operatorname{supp}\varphi \Subset B\right\}.$$

Definition. We say that the *generalized capacity condition* (Gcap) holds if, for any $u \in \mathcal{F}$ and for any two concentric balls $B_0 := B(x,R)$ and $B := B(x,R+r)$,

$$\operatorname{cap}_u(B_0,B) \le \frac{C}{r^\beta}\int_B u^2 d\mu. \tag{Gcap}$$

Theorem 4 (AG–J.Hu–K.S.Lau '15) $\quad (CC) + (PI) + (\text{Gcap}) \Leftrightarrow (10)$.

Estimating heat kernels: jump case

Let now $(\mathcal{E}, \mathcal{F})$ be a jump type Dirichlet form given by

$$\mathcal{E}(f, f) = \iint\limits_{M \times M} (f(x) - f(y))^2 J(x, y) d\mu(x) d\mu(y),$$

where J is a symmetric jump kernel. We use the following condition instead of the Poincaré inequality:

$$J(x, y) \simeq d(x, y)^{-(\alpha+\beta)}. \tag{J}$$

Theorem 5 (AG-E.Hu–J. Hu '16 and Z.Q.Chen-Kumagai-J.Wang '16)

$$(J) + (\mathrm{Gcap}) \Leftrightarrow (11).$$

In the case $\beta < 2$ it is easy to show that $(J) \Rightarrow$ (Gcap) so that in this case we obtain the equivalence

$$(J) \Leftrightarrow (11).$$

The latter was also shown by Chen and Kumagai '03, although under some additional assumptions about the metric structure of (M, d).

Conjecture. $(J) + (\mathrm{cap}) \Leftrightarrow (11)$.