Heat kernels on fractals and walk dimension

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Angers, June 2023

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1 Analysis on metric spaces: integration

Let (M, d, μ) be a *metric measure space*, where d is a metric and μ is a Radon measure on M. Assume in what follows that μ is α*-regular*, that is, for any metric ball

$$B(x,r) := \{y \in M : d(x,y) < r\}$$

of any radius $r < r_0$, we have

$$\mu(B(x,r)) \simeq r^\alpha, \tag{1}$$

where $\alpha > 0$. The sign $\simeq$ means "comparable", that is, the ratio of the two sides is bounded from above and below by positive constants.

It follows from (1) that

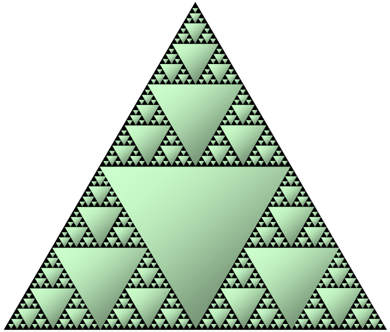
$$\dim_H M = \alpha \quad \text{and} \quad \mu \simeq \mathcal{H}_\alpha.$$

In some sense, α is a numerical characteristic of the integral calculus on M that is determined by integration against μ.

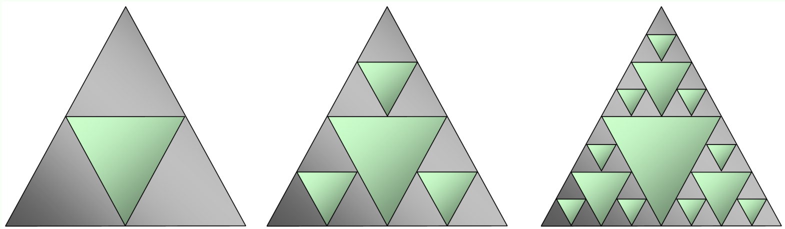
If α is fractional then α-regular spaces are frequently called *fractals*. They first appeared in mathematics as curious examples of sets (as the Cantor set).

However, at present, not only integral calculus is available on these spaces, but also, in a certain sense, *differential calculus*.

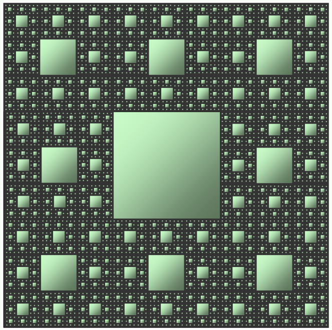
Here are some examples of fractals relevant to the topic of this talk:



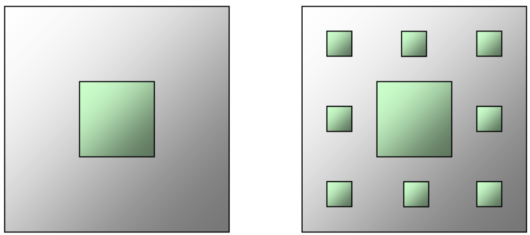
Sierpinski gasket (SG), $\alpha = \frac{\log 3}{\log 2} \approx 1.58$



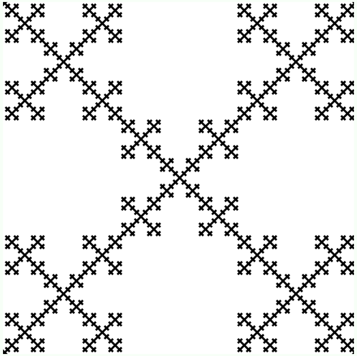
Three steps of construction of SG



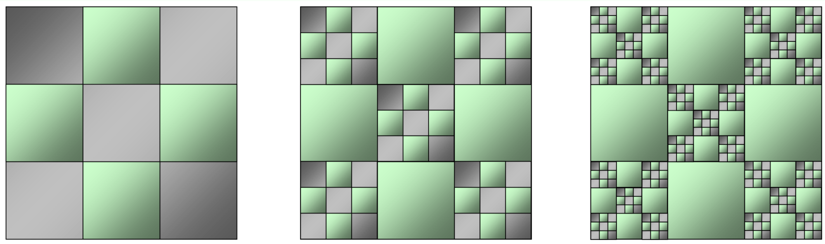
Sierpinski carpet (SC), $\alpha = \frac{\log 8}{\log 3} \approx 1.89$



Two steps of construction of SC



Vicsek snowflake (VS), $\alpha = \frac{\log 5}{\log 3} \approx 1.46$



Three steps of construction of VS

2 Analysis on metric spaces: differentiation

On many families of fractals, it is possible to construct a *Laplace-type* operator by means of the theory of Dirichlet forms of Fukushima.

A *Dirichlet form* on $L^2(M,\mu)$ is a pair $(\mathcal{E},\mathcal{F})$ where $\mathcal{F}$ is dense subspace of $L^2(M,\mu)$ and $\mathcal{E}$ is a bilinear form on $\mathcal{F}$ with the following properties:

1. It is *positive definite*, that is, $\mathcal{E}(f,f) \geq 0$ for all $f \in \mathcal{F}$.
2. It is *closed*, that is, $\mathcal{F}$ is complete with respect to the norm
$$\|f\|_{\mathcal{F}} := \left(\int_M f^2 d\mu + \mathcal{E}(f,f) \right)^{1/2}.$$
3. It is *Markovian*, that is, if $f \in \mathcal{F}$ then $\widetilde{f} := \min(f_+, 1) \in \mathcal{F}$ and $\mathcal{E}(\widetilde{f},\widetilde{f}) \leq \mathcal{E}(f,f)$.

Any Dirichlet form has the generator: a positive definite self-adjoint operator $\mathcal{L}$ in $L^2(M,\mu)$ with a dense domain $\operatorname{dom}(\mathcal{L}) \subset \mathcal{F}$ such that

$$(\mathcal{L}f,g)_{L^2} = \mathcal{E}(f,g) \text{ for all } f \in \operatorname{dom}(\mathcal{L}) \text{ and } g \in \mathcal{F}.$$

For example, the classical Dirichlet integral

$$\mathcal{E}(f,f)=\int_{\mathbb{R}^n}|\nabla f|^2\,dx \tag{2}$$

is a quadratic part of the following bilinear form

$$\mathcal{E}(f,g)=\int_{\mathbb{R}^n}\nabla f\cdot\nabla g\,dx,$$

which is a Dirichlet form with the domain $\mathcal{F}=W_2^1\left(\mathbb{R}^n\right)$.
The generator of this Dirichlet form is $\mathcal{L}=-\Delta$ with $\operatorname{dom}\left(\mathcal{L}\right)=W_2^2\left(\mathbb{R}^n\right)$.

Another example of a Dirichlet form in $\mathbb{R}^n$ is given by the quadratic form

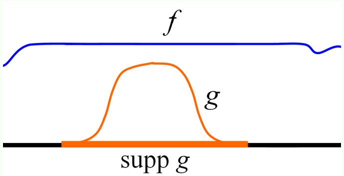
$$\mathcal{E}(f,f)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\left(f\left(x\right)-f\left(y\right)\right)^2}{\left|x-y\right|^{n+s}}dxdy, \tag{3}$$

where $s\in\left(0,2\right)$.
The domain of this Dirichlet form is $\mathcal{F}=B_{2,2}^{s/2}\left(\mathbb{R}^n\right)$, and the generator is $\mathcal{L}=\left(-\Delta\right)^{s/2}$.

A Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *strongly local* if $\mathcal{E}(f, g) = 0$ whenever

$$f = \text{const in a neighborhood of } \operatorname{supp} g.$$

For example, the Dirichlet form (2) is strongly local, while the Dirichlet form (3) is non-local.



The generator $\mathcal{L}$ of a Dirichlet form determines the *heat semigroup* $\{e^{-t\mathcal{L}}\}_{t\geq 0}$ in $L^2(M, \mu)$. In many cases, the operator $e^{-t\mathcal{L}}$ for $t > 0$ is an integral operator:

$$e^{-t\mathcal{L}} f\,(x) = \int_M p_t(x, y) f(y) d\mu(y) \text{ for all } f \in L^2,$$

where the integral kernel $p_t(x, y) \geq 0$ is called the *heat kernel* of $\mathcal{L}$ (or that of $(\mathcal{E}, \mathcal{F})$). For example, the local Dirichlet form (2) with the generator $\mathcal{L} = -\Delta$ has the heat kernel

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right). \quad (4)$$

This function is also known as the fundamental solution of the heat equation or as the Gauss-Weierstrass function or as the normal distribution with mean y and variance $2t$.

The non-local Dirichlet form (3) with the generator $\mathcal{L} = (-\Delta)^{s/2}$ has the heat kernel that admits the following estimate:

$$p_t(x,y) \simeq \frac{1}{t^{n/s}} \left(1 + \frac{|x-y|}{t^{1/s}} \right)^{-(n+s)} . \tag{5}$$

In the special case $s = 1$ the heat kernel of $(-\Delta)^{1/2}$ coincides with the Cauchy distribution with the scale parameter t:

$$p_t(x,y) = \frac{c_n t}{\left(t^2 + |x-y|^2\right)^{\frac{n+1}{2}}} = \frac{c_n}{t^n} \left(1 + \frac{|x-y|^2}{t^2} \right)^{-\frac{n+1}{2}} ,$$

where $c_n = \Gamma\left(\frac{n+1}{2}\right) / \pi^{(n+1)/2}$.

A Dirichlet form $(\mathcal{E}, \mathcal{F})$ is called *regular* if $\mathcal{F} \cap C_0(M)$ is dense both in $\mathcal{F}$ and $C_0(M)$. For example, the both Dirichlet forms (2) and (3) are regular.

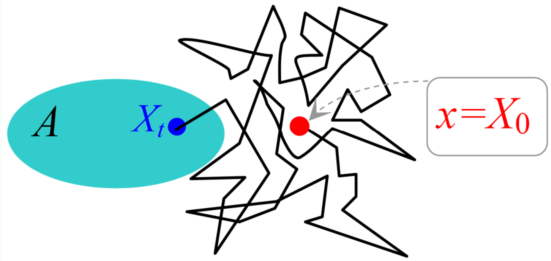
If a Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular then it determines a Markov processes $\{X_t\}_{t \geq 0}$ on M with the transition semigroup $e^{-t\mathcal{L}}$, which means that

$$\mathbb{E}_x f(X_t) = e^{-t\mathcal{L}} f(x) \quad \text{for all } f \in C_0(M) \text{ and } t \geq 0.$$

If the heat kernel of $(\mathcal{E}, \mathcal{F})$ exists then it is the transition density of this process:

$$\mathbb{P}_x(X_t \in A) = \int_A p_t(x, y)d\mu(y),$$

for any Borel set $A \subset M$ and $t > 0$.



If $(\mathcal{E}, \mathcal{F})$ is local then $\{X_t\}$ is a diffusion process (=with continuous trajectories), while otherwise the trajectories of the process $\{X_t\}$ contain jumps.

For example, the Dirichlet form (2) with the generator $\mathcal{L} = -\Delta$ determines Brownian motion in $\mathbb{R}^n$ with the transition density (4).

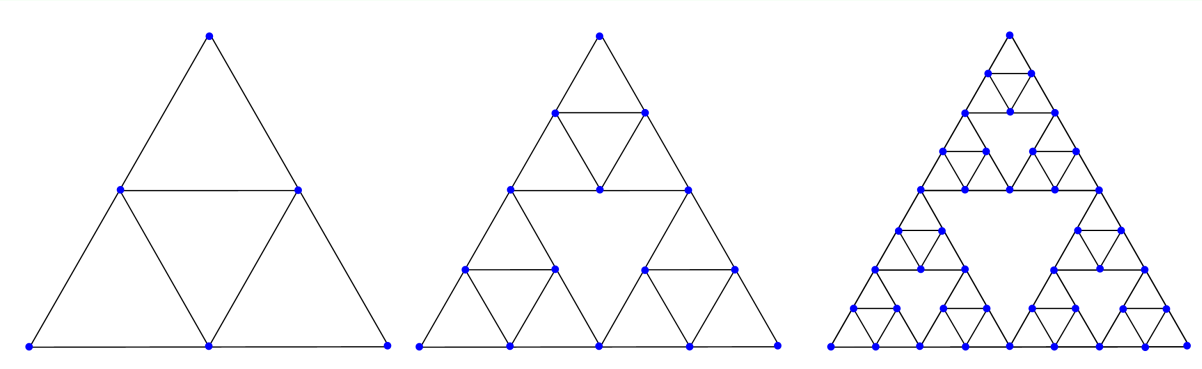
The Dirichlet form (3) with the generator $\mathcal{L} = (-\Delta)^{s/2}$ determines a symmetric stable Levy process in $\mathbb{R}^n$ of the index s with the transition density (5).

If a metric measure space (M, d, μ) possesses a strongly local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ then its generator $\mathcal{L}$ can be regarded as an analogue of the Laplace operator. In this sense $\mathcal{L}$ determines a differential calculus on M.

Large families of fractals admit non-trivial strongly local regular Dirichlet forms respecting their self-similarity and symmetry structures.

For example, such Dirichlet forms have been constructed on SG by Barlow–Perkins '88, Goldstein '87 and Kusuoka '87, on SC by Barlow–Bass '89 and Kusuoka–Zhou '92, on p.c.f. fractals (including VS) by Kigami '93.

An approach to construction of such a Dirichlet form is as follows. Each of the above fractals can be regarded as a limit of a sequence of graphs $\{\Gamma_n\}_{n=1}^{\infty}$.



Approximating graphs $\Gamma_1, \Gamma_2, \Gamma_3$ for SG

Define on each Γ_n a Dirichlet form $\mathcal{E}_n$ by

$$\mathcal{E}_n(f,f) = \sum_{x,y:\, x\sim y} (f(x) - f(y))^2$$

(where $x \sim y$ denotes neighboring vertices on Γ_n), and then consider a scaled limit

$$\mathcal{E}(f, f) = \lim_{n \to \infty} R_n \mathcal{E}_n(f, f) \tag{6}$$

with an appropriate renormalizing sequence $\{R_n\}$.

The main difficulty is to ensure the existence of $\{R_n\}$ such that this limit exists in $(0, \infty)$ for a dense in L^2 family of functions f.

For p.c.f. fractals one chooses $R_n = \rho^n$ where, for example, $\rho = \frac{5}{3}$ for SG and $\rho = 3$ for VS, and the limit in (6) exists due to monotonicity.

For SC the situation is much harder. Initially a strongly local Dirichlet form on SC was constructed by Barlow and Bass '89 in a different way by using a probabilistic approach. After a work of Barlow, Bass, Kumagai and Teplyaev '10 it became possible to claim that the limit (6) exists for a certain sequence $\{R_n\}$ such that $R_n \simeq \rho^n$, where the exact value of ρ is still unknown. Numerical computation indicates that $\rho \approx 1.25$.

Other methods of constructing a strongly local Dirichlet form on SC were proposed by Kusuoka and Zhou '92 and AG and M.Yang '19.

3 Walk dimension

In all the above examples of fractals, the strongly local Dirichlet form possesses the heat kernel that satisfies the following *sub-Gaussian* estimate:

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{7}$$

(where $C, c > 0$), for all $x, y \in M$ and $t \in (0, t_0)$ (Barlow–Perkins '88, Barlow–Bass '92).

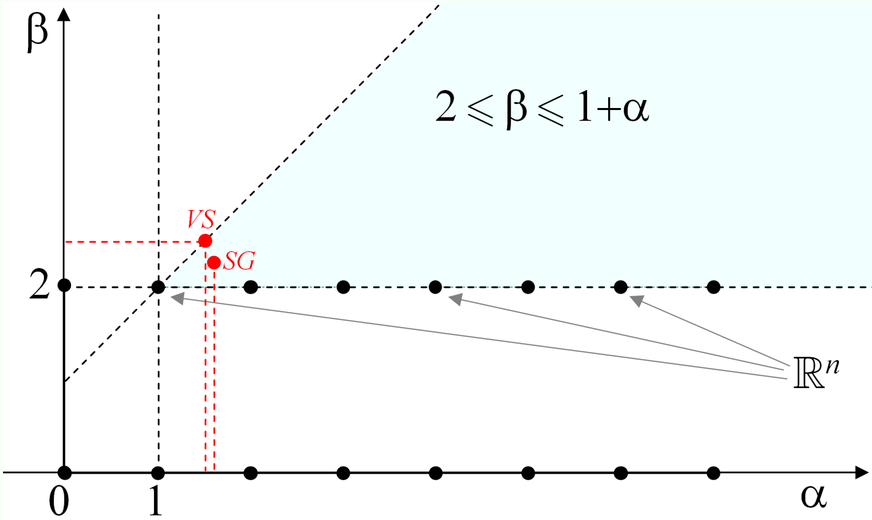
Here α is the Hausdorff dimension of the underlying metric space (M, d) while β is a new parameter that is called the *walk dimension*. It can be regarded as a numerical characteristic of the differential calculus on M that is determined by the generator $\mathcal{L}$.

It is known that always $\beta \geq 2$. Barlow '04 showed that if a pair (α, β) of reals satisfies

$$\alpha \geq 1 \text{ and } 2 \leq \beta \leq \alpha + 1,$$

then there exists a *geodesic* metric measure space with the heat kernel satisfying (7).

Hence, we obtain a large family of metric measure spaces that are characterized by a pair (α, β) where α is responsible for integration while β is responsible for differentiation.



The Euclidean space $\mathbb{R}^n$ belongs to this family with $\alpha = n$ and $\beta = 2$ (in the case $\beta = 2$ the estimate (7) becomes Gaussian).

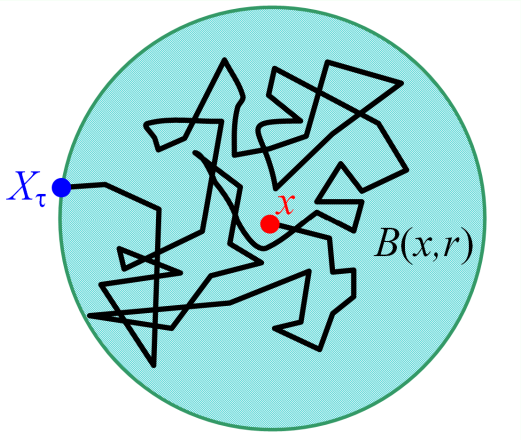
On fractals the values of β is determined by the scaling parameter ρ. It is known that:

- on SG: $\beta = \frac{\log 5}{\log 2} \approx 2.32$ (and $\alpha = \frac{\log 3}{\log 2} \approx 1.58$)
- on VS: $\beta = \frac{\log 15}{\log 3} \approx 2.46$ (and $\alpha = \frac{\log 5}{\log 3} \approx 1.46$)
- on SC: $\beta = \frac{\log(8\rho)}{\log 3} \approx 2.10$ (and $\alpha = \frac{\log 8}{\log 3} \approx 1.89$).

The walk dimension β has the following probabilistic meaning.

For any open set $\Omega \subset M$, denote by τ_Ω the first exit time of diffusion X_t from Ω:

$$\tau_\Omega = \inf \{t > 0 : X_t \notin \Omega\} .$$



It is known that if (7) holds then for any ball $B(x, r)$ with $r < r_0$,

$$\mathbb{E}_x \tau_{B(x,r)} \simeq r^\beta .$$

Hence, the parameter β can be regarded as a certain characteristic of the diffusion process, which is determined by the Dirichlet form $(\mathcal{E}, \mathcal{F})$.

However, as we will see below, β is in fact determined by the metric space (M, d) alone!

For that we need a different characterization of β that is provided by a family of *Besov spaces* on (M, d, μ).

4 Besov spaces and characterization of β

Given an α-regular metric measure space (M, d, μ), it is possible to define a family $B^{\sigma}_{p,q}$ of Besov spaces, where $p, q \in [1, \infty]$, $\sigma > 0$. Here we need only the following special cases: for any $\sigma > 0$ the space $B^{\sigma}_{2,2}$ consists of functions $f \in L^2(M, \mu)$ such that

$$\|f\|^2_{\dot{B}^{\sigma}_{2,2}} := \int\int_{M\times M} \frac{|f(x) - f(y)|^2}{d(x, y)^{\alpha+2\sigma}} d\mu(x)d\mu(y) < \infty,$$

and $B^{\sigma}_{2,\infty}$ consists of functions $f \in L^2(M, \mu)$ such that

$$\|f\|^2_{\dot{B}^{\sigma}_{2,\infty}} := \sup_{0<r<r_0} \frac{1}{r^{\alpha+2\sigma}} \int\int_{\{d(x,y)<r\}} |f(x) - f(y)|^2 d\mu(x)d\mu(y) < \infty.$$

It is easy to see that the space $B^{\sigma}_{2,2}$ shrinks as σ increases. Define the *critical Besov exponent* by

$$\boxed{\sigma^* = \sup\{\sigma > 0 : B^{\sigma}_{2,2} \text{ is dense in } L^2\}}. \tag{8}$$

If $\sigma < 1$ then $B^{\sigma}_{2,2}$ contains all Lipschitz functions with compact support. Hence, $\sigma^* \geq 1$. In $\mathbb{R}^n$, if $\sigma > 1$ then $B^{\sigma}_{2,2} = \{0\}$ so that $\sigma^* = 1$. On most fractal spaces $\sigma^* > 1$.

We say that a metric space (M, d) is regular if (M, d, μ) is α-regular for some measure μ and some $\alpha > 0$. The critical exponent σ^* is defined by (8) for any regular metric space, and the value of σ^* does not depend on the choice of α and μ because $\alpha = \dim_H M$ and $\mu \simeq \mathcal{H}_\alpha$. Hence, σ^* is an invariant of a regular metric space.

Theorem 1 (AG, Jiaxin Hu, K.-S. Lau '03) *Let $(\mathcal{E}, \mathcal{F})$ be a strongly local Dirichlet form on M such that its heat kernel exists and satisfies the sub-Gaussian estimate*

$$p_t(x, y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x, y)}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{9}$$

with some α and β. Then the following is true:

(*a*) *the space M is α-regular*;

(*b*) $\beta = 2\sigma^*$ (consequently, $\beta \geq 2$);

(*c*) $\mathcal{F} = B^{\sigma^*}_{2,\infty}$ *and* $\mathcal{E}(f, f) \simeq \|f\|^2_{\dot{B}^{\sigma^*}_{2,\infty}}$.

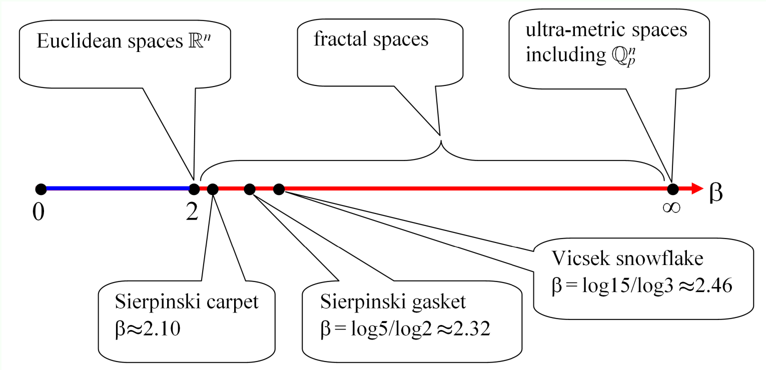
Similar results for SG – Jonsson '96, and for d-sets in $\mathbb{R}^n$ – K. Pietruska-Pałuba '00.

In the view of Theorem 1, we **redefine** now the notion of the walk dimension by setting

$$\boxed{\beta := 2\sigma^*}.$$

With this definition the walk dimension β becomes a second invariant of a regular metric space after the Hausdorff dimension α.

Here is a classification of regular metric spaces according to their walk dimension $\beta = 2\sigma^*$.



A metric space (M, d) is called *ultra-metric* if it satisfies a stronger triangle inequality

$$d(x, y) \leq \max\left(d\left(x, z\right), d\left(y, z\right)\right) \quad \text{for all } x, y, z \in M.$$

For example, the field $\mathbb{Q}_p$ of p-adic numbers with the p-adic distance $|x - y|_p$ is an ultra-metric space. All ultra-metric spaces are totally disconnected and, hence, cannot carry a non-trivial diffusion. On the other hand, on such spaces, for any $\sigma > 0$, the space $B^{\sigma}_{2,2}$ contains indicator functions $\mathbf{1}_B$ of all balls and, hence, is dense in L^2. Consequently, $\sigma^* = \infty$.

5 An approach to construction of local Dirichlet forms

An open question. *Let (M, d, μ) be an α-regular metric measure space (or even self-similar). Assume $\sigma^* < \infty$. Is there a strongly local regular Dirichlet form in M?*
Does its heat kernel satisfy the sub-Gaussian estimate (9) with $\beta = 2\sigma^*$?

Here is a possible approach to construction of such a Dirichlet form based on the family of Besov spaces. For any $\sigma < \sigma^*$ we need to define in $B^{\sigma}_{2,2}$ a quadratic form $\mathcal{E}_\sigma(f, f)$ with the following properties:

(i) $$\mathcal{E}_\sigma(f, f) \simeq \|f\|^2_{\dot{B}^{\sigma}_{2,2}} = \iint_{M\times M} \frac{|f(x) - f(y)|^2}{d(x, y)^{\alpha+2\sigma}} d\mu(x) d\mu(y);$$

(ii) the following limit should exist in some sense: $\lim_{\sigma\to\sigma^*} (\sigma^* - \sigma)\, \mathcal{E}_\sigma =: \mathcal{E}$

(iii) and the limit $\mathcal{E}$ should be a strongly local regular Dirichlet form on M.

In $\mathbb{R}^n$ this method works with $\mathcal{E}_\sigma(f, f) = \|f\|^2_{\dot{B}^{\sigma}_{2,2}}$. For SG and SC this method was realized by AG and M.Yang '18 and '19. However, in the general case there are too many difficulties. Perhaps, some additional conditions should be imposed.

6 Heat kernel estimates of self-similar type

Let (M,d) be metric space and μ be an α-regular measure on M.

Theorem 2 (AG, T.Kumagai '08) *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on M such that*

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}}\Phi\left(c\frac{d(x,y)}{t^{1/\beta}}\right),$$

where $\alpha,\beta>0$ and Φ is a positive function on $[0,\infty)$. Then the following dichotomy holds:
(i) either the Dirichlet form $\mathcal{E}$ is strongly local,

$$\Phi(s) \asymp C\exp\left(-cs^{\frac{\beta}{\beta-1}}\right)$$

and $\mathcal{F}=B^{\beta/2}_{2,\infty}$, $\mathcal{E}(f,f)\simeq\|f\|^2_{\dot{B}^{\beta/2}_{2,\infty}}$;
(ii) or the Dirichlet form $\mathcal{E}$ is non-local,

$$\Phi(s)\simeq(1+s)^{-(\alpha+\beta)}$$

and $\mathcal{F}=B^{\beta/2}_{2,2}$, $\mathcal{E}(f,f)\simeq\|f\|^2_{\dot{B}^{\beta/2}_{2,2}}$.

That is, in the first case we obtain the sub-Gaussian estimate

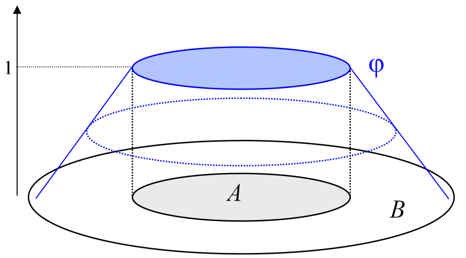
$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{sub-G}$$

while in the second case we obtain a *stable-like estimate*

$$\begin{aligned} p_t(x,y) &\simeq \frac{1}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \qquad \text{(stable)} \\ &\simeq \min\left(\frac{1}{t^{\alpha/\beta}}, \frac{t}{d(x,y)^{\alpha+\beta}}\right). \end{aligned}$$

Next, we discuss the condition on (M,d,μ) and $(\mathcal{E},\mathcal{F})$ that ensure the estimates (sub-G) or (stable). For that we need some additional notions.

7 Capacity and generalized capacity



a cutoff function φ of (A,B)

Let $A \Subset B$ be two precompact open subsets of M.

Define the capacity of (A, B) as follows:

$$\operatorname{cap}(A, B) = \inf \left\{ \mathcal{E}(\varphi, \varphi) : \varphi \in \mathcal{F},\ \varphi|_{\overline{A}} = 1,\ \varphi|_{B^c} = 0 \right\}.$$

Definition. We say that $(\mathcal{E}, \mathcal{F})$ satisfies the *capacity condition* with parameter $\beta > 0$ if there exists a constant $C > 0$ such that, for any two concentric balls $B_0 := B(x, R)$ and $B := B(x, R + r)$,

$$\operatorname{cap}(B_0, B) \leq C \frac{\mu(B)}{r^\beta}. \tag{cap}$$

The condition (cap) is equivalent to the existence of a cutoff function φ of (B_0, B) such that

$$\mathcal{E}(\varphi, \varphi) \leq C \frac{\mu(B)}{r^\beta}.$$

For any function $u \in L^\infty \cap \mathcal{F}$ and a real $\kappa \geq 1$ define the *generalized capacity* of A in B by

$$\mathrm{cap}_u^{(\kappa)}(A,B) = \inf\left\{\mathcal{E}\left(u^2\varphi,\varphi\right) : \varphi \in \mathcal{F},\ 0 \leq \varphi \leq \kappa,\ \ \varphi|_{\overline{A}} \geq 1,\ \varphi|_{B^c} = 0\right\}.$$

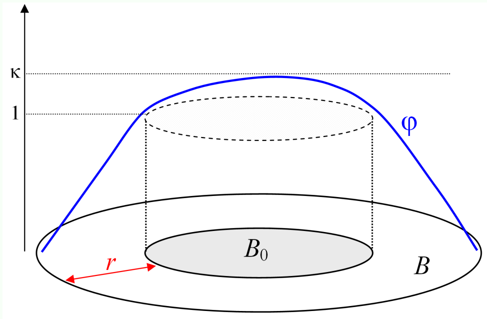
For example, if $u \equiv 1$ then $\mathrm{cap}_u^{(\kappa)}(A,B) = \mathrm{cap}(A,B)$.

Definition. $(\mathcal{E},\mathcal{F})$ satisfies the *generalized capacity condition* (Gcap) with parameter $\beta > 0$ if there exist $\kappa \geq 1$, $C > 0$ such that, for any $u \in \mathcal{F} \cap L^\infty$ and for any two balls $B_0 := B(x,R)$ and $B := B(x, R+r)$,

$$\mathrm{cap}_u^{(\kappa)}(B_0,B) \leq \frac{C}{r^\beta}\int_B u^2 d\mu. \qquad \text{(Gcap)}$$

Equivalently, for any $u \in \mathcal{F} \cap L^\infty$ there exists a cutoff function φ of pair (B_0,B) such that

$$\mathcal{E}\left(u^2\varphi,\varphi\right) \leq \frac{C}{r^\beta}\int_B u^2 d\mu.$$



a cutoff function φ of (B_0,B)

Clearly, (Gcap)$\Rightarrow$(cap).

8 Estimating heat kernels: strongly local case

Assume that all metric balls in (M,d) a precompact. In this section, we assume in addition that (M,d) satisfies the *chain condition*: if $\exists C$ such that for all $x,y\in M$ and for $n\in\mathbb{N}$ there exists a sequence $\{x_k\}_{k=0}^{n}$ of points in M such that $x_0=x$, $x_n=y$, and

$$d(x_{k-1},x_k)\le C\frac{d(x,y)}{n},\quad \text{for all } k=1,...,n.$$

Definition. We say that a strongly local Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(M,\mu)$ satisfies the *Poincaré inequality* with parameter $\beta>0$ if, for any ball $B=B(x,r)$ on M and for any function $f\in\mathcal{F}$,

$$\mathcal{E}_B(f,f):=\int_B d\Gamma(f,f)\ge\frac{c}{r^\beta}\int_{\varepsilon B}\left(f-\overline{f}\right)^2 d\mu, \qquad (PI)$$

where $\overline{f}=⨍_{\varepsilon B} f d\mu$, and c,ε are small positive constants independent of B and f.

For example, (PI) holds in $\mathbb{R}^n$ with $\beta=2$ and $\varepsilon=1$.

Theorem 3 (AG, Jiaxin Hu, K.S.Lau '15)

Let (M,d) satisfy the chain condition. Let μ be an α-regular measure on M and $(\mathcal{E},\mathcal{F})$ be a strongly local regular Dirichlet form on $L^2(M,\mu)$. Then

$$(PI) + (\mathrm{Gcap}) \Leftrightarrow \quad p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right).$$

.

Conjecture. *The condition* (Gcap) *here can be replaced by* (cap).

9 Estimating heat kernels: jump case

Let now $(\mathcal{E},\mathcal{F})$ be a jump type Dirichlet form given by

$$\mathcal{E}(f,f)=\iint_{M\times M}(f(x)-f(y))^2 J(x,y)d\mu(x)d\mu(y),$$

where J is a symmetric jump kernel. We use the following condition instead of (PI):

$$J(x,y)\simeq d(x,y)^{-(\alpha+\beta)}. \tag{J}$$

Theorem 4 (AG, Eryan Hu, Jiaxin Hu '18 and Z.Q.Chen, T.Kumagai, J.Wang '20)

$$(J)+(\mathrm{Gcap}) \;\Leftrightarrow\; p_t(x,y)\simeq\frac{1}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}.$$

In the case $\beta<2$ it is easy to show that $(J)\Rightarrow(\mathrm{Gcap})$ so that in this case (Gcap) can be dropped.

Conjecture. *The condition* (Gcap) *here can be replaced by* (cap).

10 Faber-Krahn inequality and upper bounds

We say that the *Faber-Krahn inequality* (FK) with parameter $\beta > 0$ holds if, for any precompact open set $\Omega \subset M$,

$$\lambda_1(\Omega) \geq c\mu(\Omega)^{-\beta/\alpha}, \tag{FK}$$

where $\lambda_1(\Omega) = \inf \operatorname{spec}\left(\mathcal{L}^{\Omega}\right)$. Or, equivalently, (FK) holds if

$$\inf_{\varphi \in \mathcal{F} \cap C_0(\Omega) \setminus \{0\}} \frac{\mathcal{E}(\varphi, \varphi)}{\|\varphi\|_{L^2}^2} \geq c\mu(\Omega)^{-\beta/\alpha}.$$

It is known that (FK) is equivalent to the *diagonal upper estimate* of the heat kernel

$$p_t(x, y) \leq Ct^{-\alpha/\beta}.$$

It is also known that

$$J(x, y) \geq \frac{c}{d(x, y)^{\alpha+\beta}} \Rightarrow (FK).$$

In some sense, (FK) can be regarded as an integral version of a pointwise lower bound of J.

Denote by (C) the hypothesis that $(\mathcal{E}, \mathcal{F})$ is *conservative*, that is, $P_t 1 = 1$.

Now let us state a result about the off-diagonal upper estimate

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} . \qquad (UE)$$

It is easy to prove that (UE) implies that

$$J(x,y) \le \frac{C}{d(x,y)^{\alpha+\beta}}. \qquad (J_\le)$$

Theorem 5 (AG, J.Hu, K.-S.Lau K.-S. '14 and Z.-Q.Chen, T.Kumagai, J.Wang '21)

$$(J_\le) + (FK) + (\mathrm{Gcap}) \Leftrightarrow (UE) + (C)\,.$$

11 Tail estimates

Fix some $\beta > 0$, $q \in [1, \infty]$ and consider the following hypothesis for the tail of J:

$$\boxed{\|J(x,\cdot)\|_{L^q(B^c(x,r))} \leq \frac{C}{r^{\alpha/q'+\beta}}}, \qquad (TJ_q)$$

for all $x \in M$ and $r > 0$, where $q' = \frac{q}{q-1}$ is the Hölder conjugate of q.

It is easy to see that (TJ_q) becomes stronger when q increases.

For example, if $q = 1$ then $q' = \infty$ and (TJ_q) becomes

$$\int_{B^c(x,r)} J(x,y)d\mu(y) \leq \frac{C}{r^\beta}. \qquad (TJ_1)$$

If $q = 2$ then $q' = 2$ and (TJ_q) becomes

$$\left(\int_{B^c(x,r)} J^2(x,y)d\mu(y)\right)^{1/2} \leq \frac{C}{r^{\alpha/2+\beta}}. \qquad (TJ_2)$$

If $q=\infty$ then $q'=1$ and (TJ_q) becomes

$$\operatorname*{essup}_{y\in B^c(x,r)} J(x,y) \le \frac{C}{r^{\alpha+\beta}}, \tag{TJ_∞}$$

which is equivalent to the upper bound in (J).

Consider the following hypotheses about the tail of the heat kernel $p_t(x,y)$:

$$\boxed{\|p_t(x,\cdot)\|_{L^q(B^c(x,r))} \le \frac{C}{t^{\alpha/(q'\beta)}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}} \simeq \frac{1}{t^{\alpha/(q'\beta)}} \wedge \frac{t}{r^{\alpha/q'+\beta}}, \tag{TP_q}$$

for all $x\in M$ and $r>0$. It is easy to prove that

$$(TP_q) \Rightarrow (TJ_q). \tag{10}$$

The condition (TP_q) gets stronger when q increases. For $q=1$, we have

$$\int_{B^c(x,r)} p_t(x,y)d\mu(y) \le C\frac{t}{r^\beta}, \tag{TP_1}$$

for $q=2$, we have

$$\int_{B^c(x,r)} p_t^2(x,y)d\mu(y) \le \frac{C}{t^{\alpha/(2\beta)}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha/2+\beta)}, \tag{TP_2}$$

for $q=\infty$, (TP_∞) becomes

$$\operatorname*{esssup}_{y\in B^c(x,r)} p_t(x,y) \le \frac{C}{t^{\alpha/\beta}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha+\beta)}, \qquad (TP_\infty)$$

which is equivalent to the upper bound in (stable).

Consider also the following family of off-diagonal *upper estimates* of the heat kernel:

$$\boxed{p_t(x,y) \le \frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}}, \qquad (UE_q)$$

for all $t>0$ and almost all $x,y\in M$. For example, for $q=\infty$ we have

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}, \qquad (UE_\infty)$$

which coincides with (TP_∞).

For $q=1$ we have a weaker estimate

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta}. \qquad (UE_1)$$

Now we can state our main result.

Theorem 6 (AG, E.Hu, J.Hu '23) *For any* $q \in [2, \infty]$

$$(TJ_q) + (FK) + (\mathrm{Gcap}) \Leftrightarrow (TP_q) + (C) \Rightarrow (UE_q).$$

Or, considering (FK), (Gcap), (C) *as standing assumptions, we have*

$$\boxed{(TJ_q) \Leftrightarrow (TP_q) \Rightarrow (UE_q)}.$$