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ON STOCHASTICALLY COMPLETE MANIFOLDS

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A Riemannian manifold is said to be *stochastically complete* if every two Wiener processes on it have the same transition function. For example, a Euclidean space is stochastically complete, but a proper open subset of $\mathbf{R}^n$ is not, because Wiener processes with different boundary conditions have different transition functions. We remark that there is always at least one Wiener process (i.e., a diffusion process generated by the Laplace operator) on an arbitrary smooth connected Riemannian manifold (see [1]).

It is known that a necessary condition for stochastic completeness of a manifold is its completeness as a metric space. However, not every metrically complete manifold is stochastically complete: a Wiener process can with positive probability leave a manifold in a finite time, and the subsequent motion of the Brownian particle is determined by the conditions on the boundary at infinity and is thus nonunique (see [2]). Yau [3] proved that if a complete manifold has Ricci curvature that is bounded below, then it is stochastically complete. We now formulate our main result.

THEOREM 1. *Let M be a complete Riemannian manifold, and let $V(r)$ be the volume of a geodesic ball of radius r with fixed center $O \in M$. If*

$$\int^{\infty} (r/\log V(r))\, dr = \infty, \tag{1}$$

then M is stochastically complete.

REMARKS. 1) If the Ricci curvature is bounded below, then $V(r) \le e^{Cr}$, and hence (1) holds. What is more, (1) also holds if

$$V(r) \le e^{Cr^2}$$

or

$$V(r) \le e^{Cr^2 \log r},$$

etc.

2) Theorem 1 is valid also for manifolds with a boundary if the reflection condition is assumed on the boundary, i.e., the one-sided Neumann condition.

3) Condition (1) is sharp in the following sense. If $\int^{\infty}(r/\log f(r))\, dr < \infty$ for a positive function $f(r)$ (regular in some sense), then there is a complete manifold M such that $V(r) \le Cf(r)$, and M is not stochastically complete.

PROOF OF THEOREM 1. To a Wiener process there corresponds a transition function $\mathbf{P}_t(x, N)$: the probability of hitting a Borel set $N \subset M$ from the point x in a time t. Further,

$$0 \le \mathbf{P}_t(x, N) \le 1, \tag{2}$$

and for every continuous bounded function $v(y)$ on M the function

$$u(t, x) = \int_M v(y) \mathbf{P}_t(x, dy) \tag{3}$$

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satisfies the inverse diffusion equation

$$\partial u/\partial t = \tfrac{1}{2}\Delta u \tag{4}$$

and the initial condition $u|_{t=0} = v$.

If there exist two transition functions $\mathbf{P}^{(1)}$ and $\mathbf{P}^{(2)}$, then we consider the difference $u = u^{(1)} - u^{(2)}$ between the functions $u^{(1)}$ and $u^{(2)}$ determined from (3). The function u satisfies equation (4) with the initial condition

$$u|_{t=0} = 0. \tag{5}$$

Moreover, it follows from (2) and (3) that $|u^{(i)}| \le \sup|v| < \infty$; hence u is bounded. It can be deduced from (4), (5) and the boundedness of u that $u \equiv 0$ (and so $\mathbf{P}^{(1)} = \mathbf{P}^{(2)}$). This follows from the next theorem, which is of independent interest.

THEOREM 2. *Suppose that M is a complete Riemannian manifold, and $u(t,x)$ is a solution of* (4) *with the initial condition* (5) *defined in the strip $M_T = M \times [0,T]$. Suppose that for any $R > 0$*

$$\int_0^T \int_{B_R} u^2(t,x)\,dx\,dt < e^{f(R)},$$

where B_R is a geodesic ball with fixed center $O \in M$, and $f(R)$ is a monotonically increasing function such that

$$\int^{\infty} (r/f(r))\,dr = \infty. \tag{7}$$

Then $u \equiv 0$ in M_T.

REMARKS. 1) Theorem 2 is valid also for manifolds with boundary ∂M under the condition that on the boundary $u(t,x)$ satisfies the Neumann condition (ν is the normal)

$$\partial u/\partial \nu\,|_{\partial M} = 0. \tag{8}$$

What is more, instead of (4) we can consider the more general parabolic equation

$$p(x)\partial u/\partial t = \operatorname{div}(a(t,x)\nabla u) + b(t,x)\nabla u + c(t,x)u, \tag{9}$$

where $a(t,x)$ is a positive selfadjoint operator $T_xM \to T_xM$ depending smoothly on t and x, $b(t,x)$ is a smooth vector field, and $p(x)$ and $c(t,x)$ are smooth functions, with $p(x) > 0$.

Suppose that all the expressions $\|a\|, \|a^{-1}\|, p(x), p(x)^{-1}, |b|$, and c, are uniformly bounded above. Then every solution of (9) with the conditions (5), (6), and (8) (where ν is the conormal corresponding to the operator a) is equal to zero in M_T.

2) An analogous theorem was proved in [4] in the case when M is a domain in $\mathbf{R}^n$ and the lower terms in (9) are absent. Here we give a simple proof for an arbitrary manifold. We emphasize that (1) is not assumed in the formulation of Theorem 2, but only metric completeness is required. The last requirement is essential.

The proof of Theorem 1 is completed as follows. Since u is bounded, for any $T > 0$ and $R > 0$

$$\int_0^T \int_{B_R} u^2\,dx\,dt \le CTV(R).$$

Let $f(R) = \log V(R)$; then (7) follows from (1), and $u \equiv 0$ by Theorem 2.

We proceed to a proof of Theorem 2. The main point in the proof is the use of a felicitously chosen test function. Let $\rho(x)$ be a Lipschitz function on M such that $|\nabla\rho| \le 1$. For example, this can be the function giving the distance to some set. We consider the function $g(t,x) = \rho(x)^2/2(t-s)$, defined for $t \neq s$ (s fixed). It follows from $|\nabla\rho| \leq 1$ that g satisfies

$$\partial g/\partial t + \tfrac{1}{2}|\nabla g|^2 \leq 0. \tag{10}$$

We also consider for each $R > 0$ a standard cutoff function $\eta(x)$ with compact support in the ball B_{2R} and equal to 1 in the ball $B_{(3/2)R}$. Let us multiply equation (4) by $e^g\eta^2 u$ and integrate over the cylinder $\mathrm{Cyl} = B_{2R} \times [t-\Delta t, t]$ for some t and Δt:

$$2\iint_{\mathrm{Cyl}} u_\tau u\eta^2 e^g\,dx\,d\tau = \iint_{\mathrm{Cyl}} \Delta u \cdot u\eta^2 e^g\,dx\,d\tau.$$

We next use integration by parts. The resulting expressions $(\nabla u, \nabla\eta)\eta u$ and $(\nabla u, \nabla g)u$ can be estimated from above in terms of $\frac{1}{4}|\nabla u|^2\eta^2 + |\nabla\eta|^2u^2$ and $\frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2|\nabla g|^2$ respectively. As a result, the three integrals containing $|\nabla u|^2\eta^2e^g$ are annihilated, and we get that

$$\begin{aligned}&\int_{B_{2R}} u^2\eta^2e^g|_{t-\Delta t}^{t} - \iint_{\mathrm{Cyl}} u^2\eta^2e^g\partial g/\partial\tau \\ &\quad\leq 2\iint_{\mathrm{Cyl}} |\nabla\eta|^2u^2e^g + \frac{1}{2}\iint_{\mathrm{Cyl}} u^2\eta^2e^g|\nabla g|^2.\end{aligned} \tag{11}$$

Using (10), we can throw away the second terms on both sides of (11); observing that $|\nabla\eta| \leq C/R$ (here and below the letter C denotes an absolute positive constant), we get that

$$\begin{aligned}\int_{B_R} u^2(t,x)e^g\,dx \leq \int_{B_R} &u^2(t-\Delta t,x)e^g\,dx \\ &+ \frac{C}{R^2}\int_{t-\Delta t}^{t} d\tau\int_{B_{2R}\setminus B_{(3/2)R}} u^2(\tau,x)e^g\,dx.\end{aligned} \tag{12}$$

We now make the form of g concrete. Let $\rho(x)$ be the function giving the distance to the ball B_R, i.e., if $x \in B_R$, then $\rho(x) = 0$, and if x is at a distance $r > R$ from the point O, then $\rho(x) = r - R$. Also, let $s = t + \Delta t$, i.e.,

$$g(\tau,x) = -\rho(x)^2/2(t+\Delta t-\tau) \leq 0,$$

and $g(\tau,x) = 0$ for $x \in B_R$. Therefore, in the first two integrals in (12) the factor e^g can be omitted. For $x \in B_{2R}\setminus B_{(3/2)R}$ we have that

$$\rho(x)^2/2(s-\tau) = (r-R)^2/2(s-\tau) \geq C^{-1}R^2/\Delta t.$$

Choose Δt so that $C^{-1}R^2/\Delta t \geq f(2R)$, i.e., $\Delta t \leq C^{-1}(2R)^2/f(2R)$. Then we get from (12) that

$$\int_{B_R} u^2(t,x)\,dx \leq \int_{B_{2R}} u^2(t-\Delta t,x)\,dx + \frac{C}{R^2}\iint_{\mathrm{Cyl}} u^2(\tau,x)e^{-f(2R)}\,dx\,d\tau,$$

or using (6),

$$\int_{B_R} u^2(t,x)dx \leq \int_{B_{2R}} u^2(t-\Delta t,x)\,dx + C/R^2. \tag{13}$$

We next take the sequence of radii $R_k = 2^kR$, $k = 0,1,2,\ldots$, and a sequence

$$\Delta t_k \leq C^{-1}R_{k+1}^2/f(R_{k+1}).$$

It can easily be deduced from (7) that $\sum_0^\infty R_k^2/f(R_k) = \infty$ (to do this, reduce (7) by a change of variable to the integral of a monotone function, and the rest is obvious). Therefore, the sequence Δt_k can be chosen so that $\Delta t_0 + \Delta t_1 + \cdots + \Delta t_m = t$ for some

m. If in (13) we now estimate the integral on the right-hand side again according to (13) and continue this up to the time zero, when $u = 0$, we get

$$\int_{B_R} u^2(t,x)\,dx \le C \sum_{k=0}^{m} \frac{1}{R_k^2} \le \frac{C}{R^2}.$$

Letting $R \to \infty$ and using the fact that $t < T$ is arbitrary, we get that $u \equiv 0$.

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