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STOCHASTICALLY COMPLETE MANIFOLDS AND SUMMABLE HARMONIC FUNCTIONS

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ABSTRACT. Main result: if on a geodesically complete Riemannian manifold M the volume V_R of a geodesic ball of radius R with fixed center satisfies the condition

$$\int^{\infty} \frac{R\,dR}{\ln V_R} = \infty,$$

then every nonnegative integrable superharmonic function on M is equal to a constant.
Bibliography: 18 titles.

Introduction

This article is devoted to two questions that appear at first glance to have little connection with each other. Let M be a connected smooth noncompact Riemannian manifold. We consider a minimal Wiener process on M, i.e., a diffusion process generated by the Laplace-Beltrami operator Δ with absorption condition at ∞. If the probability of absorption at ∞ in a finite amount of time is equal to zero, then M is said to be *stochastically complete*. For example, $\mathbf{R}^n$ is stochastically complete, but a bounded domain in $\mathbf{R}^n$ is not. It turns out that there are geodesically complete manifolds that are not stochastically complete. An example was considered in [1] (see also §3).

Yau [17] proved that a complete Riemannian manifold with Ricci curvature bounded below is stochastically complete. This theorem has been refined in a number of papers (see, for example, [8] and [14]): the Ricci curvature was allowed to decrease to $-\infty$ in a sufficient slow manner. In [5] the author proved a more general condition for stochastic completeness in terms of the growth of the volume of a geodesic ball (see §1 below). In §3 we present examples confirming the sharpness of this condition.

The second question considered here has to do with the Liouville problem. Yau [15] proved that on a complete Riemannian manifold every harmonic function (i.e., every solution of the Laplace-Beltrami equation $\Delta u = 0$) in the class $L^p(M)$, $1 < p < \infty$, is equal to a constant; in other words, the L^p-Liouville theorem holds. See [11] for some refinements, and see [9], [16], and [4] about the L^∞-Liouville theorem. Here we consider the case $p = 1$. For some time it was not known whether the L^1-Liouville theorem holds on any complete Riemannian manifold. In several papers reference was made to a preprint of Chung in which a complete two-dimensional manifold having a nontrivial integrable harmonic function was constructed for the

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first time. This example (or a closely related example) was published by Li and Schoen in [10]. The manifold in this example has finite volume, and is thereby stochastically complete, which refutes the conjecture that the L^1-Liouville theorem holds on a stochastically complete manifold. A sufficient condition is given in [10] for the L^1-Liouville theorem to hold in terms of the decrease of the Ricci curvature. Nevertheless, there is a connection between stochastic completeness and integrable harmonic functions. Namely, we prove in §2 that the L^1-Liouville theorem holds on a stochastically complete manifold for *nonnegative* harmonic and even superharmonic functions. In combination with a geometric condition for stochastic completeness [5], our main result can be formulated as follows: if on a geodesically complete Riemannian manifold the volume V_R of a geodesic ball of radius R with fixed center satisfies the inequality $V_R \leq e^{CR^2}$, then every nonnegative superharmonic function in $L^1(M)$ is a constant. Attention is drawn to the beautiful analogy with the Cheng-Yau theorem [2]: if $V_R \leq CR^2$ on a geodesically complete manifold, then every nonnegative superharmonic function on M is equal to a constant.

In §3 we present examples of complete manifolds of arbitrary dimension that admit nontrivial positive harmonic functions in $L^1(M)$, and we prove that the restrictions on the growth of V_R in the main theorem are sharp.

§1. Some facts about stochastically complete manifolds

For each precompact domain $\Omega \subset M$ with smooth boundary we define the Green's function $G_\Omega(x,y,t)$ of the heat equation, i.e., the function of $(x,t) \in \overline{\Omega} \times (0,+\infty)$ that satisfies for each $y \in \Omega$ the conditions

$$\partial G_\Omega/\partial t - \Delta G_\Omega = 0, \qquad G_\Omega|_{\partial\Omega} = 0, \qquad G_\Omega \to \delta_y(x) \quad \text{as } t \to 0.$$

It is well known that:

1) $G_\Omega(x,y,t)$, extended by zero for $t < 0$, is infinitely differentiable away from $(y,0)$;

2) $G_\Omega(x,y,t) = G_\Omega(y,x,t)$ for any $x, y \in \Omega$;

3) $G_\Omega \geq 0$;

4) $\int_\Omega G_\Omega(x,y,t)\,dx \leq 1$;

5) $G_\Omega(x,y,t+s) = \int_\Omega G_\Omega(x,z,t)G_\Omega(z,y,s)\,dz$.

We fix a point $y \in M$ and enlarge the domain Ω. It follows from the maximum principle that if $\Omega_1 \subset \Omega_2$, then $G_{\Omega_1} \leq G_{\Omega_2}$. By 4), the integrals of G_Ω on each compact set in $M \times (0,+\infty)$ are uniformly bounded; therefore, the limit $G(x,y,t) = \lim_{\Omega \to M} G_\Omega(x,y,t)$ exists, where $\Omega \to M$ means the exhaustion of M by precompact open domains. It is easy to verify that $\partial G/\partial t - \Delta G = 0$ in $M \times (0,+\infty)$, $G \to \delta_y(x)$ as $t \to 0$, and the analogues of properties 1)–5) hold. It follows from the construction that $G(x,y,t)$ is a minimal positive fundamental solution of the heat equation (see [7] for more details).

In view of property 5) the function $G(x,y,t)$ is the kernel of a semigroup G_t acting in $L^p(M)$, defined by

$$G_t f = \int_M G(x,y,t)f(y)\,dy \quad \text{where } f \in L^p(M),\ 1 \leq p \leq \infty.$$

It can be proved that the semigroup G_t is contractive (i.e., $\|G_t\|_{L^p} \leq 1$), positive (i.e., $G_t f \geq 0$ for $f \geq 0$), and L^1_{loc}-continuous with respect to t (i.e., $G_t f \to f$ as $t \to 0$ in the sense of $L^1_{\text{loc}}(M)$; and if $p < \infty$, then G_t is strongly continuous (see [13] for details). Moreover, it follows from the maximum principle and the construction of $G(x,y,t)$ that if $f \geq 0$ and $u(x,t)$ is a positive solution of the heat equation with initial condition $u(x,t) \to f(x)$ as $t \to 0$ in the sense of $L^1_{\mathrm{loc}}(M)$, then $u(x,t) \geq G_t f$, i.e., $G_t f$ is a minimal positive solution of the Cauchy problem with initial function f.

From the probabilistic point of view $G(x,y,2t)$ is the transition density of a minimal Wiener process on M. Stochastic completeness of M is equivalent to the condition that $\int_M G(x,y,t)\,dt = 1$ for any $y \subset M$ and $t > 0$ (i.e., the process continues arbitrarily long with probability 1).

THEOREM 1. *The following conditions are equivalent:*

a) *M is stochastically complete.*

b) *The solution of the Cauchy problem $u_t - \Delta u = 0$, $u|_{t=0} = 0$, is unique in the class of functions bounded on $M \times [0,T]$ (the initial condition is understood in the sense of $L^1_{\mathrm{loc}}(M)$).*

c) *Every positive solution of the equation $\Delta v - \lambda v = 0$ on M is bounded, where $\lambda = \mathrm{const} > 0$.*

These statements are encountered in various forms in various papers (see, for example, [3]). Nevertheless, for the convenience of the reader we present a proof, especially because it is very simple.

a) $\Rightarrow$ b). Let $u(x,t)$ be a bounded solution of the heat equation with zero initial condition. It can be assumed that $|u| < 1$. Let $w = 1 - u$. Since $w > 0$ and $w|_{t=0} = 1$, we have in view of the properties of G_t that $w \geq G_t 1 = \int_M G(x,y,t)\,dy = 1$, and hence $w \geq 1$ and $u \leq 0$. It can be proved in exactly the same way that $u \geq 0$, which implies that $u \equiv 0$.

b) $\Rightarrow$ c). If the bounded positive function $v(x)$ satisfies the equation $\Delta v - \lambda v = 0$, then the function $u(x,t) = v(x)e^{\lambda t}$ satisfies the heat equation with initial condition $u|_{t=0} = v$ and is bounded on $M \times [0,T]$ for each $T > 0$. Since $G_t v$ is also a bounded solution of the indicated Cauchy problem, we have that $G_t v = v e^{\lambda t}$ by the condition in b). However, this is impossible, since $\|G_t v\|_{L^\infty} \leq \|v\|_{L^\infty} < \|v e^{\lambda t}\|_{L^\infty}$ for $t > 0$.

c) $\Rightarrow$ a). Suppose that M is not stochastically complete, i.e., $G_{t_0} 1(x_0) < 1$ at some point (x_0, t_0). Since $G_t 1(x)$ is a solution of the heat equation and $\sup G_t 1 = 1$, we have from the strict maximum principle that $G_t 1 < 1$ for $t > t_0$. Let $w(x) = \int_0^\infty e^{-\lambda t} G_t 1\,dt$. It can be verified immediately that $\Delta w - \lambda w = -1$ and $0 < w < \lambda^{-1}$. Therefore, the function $v = 1 - \lambda w$ satisfies the equation $\Delta v - \lambda v = 0$ and the restrictions $0 < v < 1$.

THEOREM 2. *Suppose that the manifold M is geodesically complete, and*

$$\int^\infty \frac{R\,dR}{\ln V_R} = \infty, \tag{1}$$

where V_R is the volume of a geodesic ball of radius R with fixed center. Then M is stochastically complete.

It was proved in [5] that under condition (1) part b) of Theorem 1 holds. Part a) thereby also holds; that is, M is stochastically complete.

We prove in §3 that condition (1) is sharp. We remark that (1) holds, for example, if $V_{R_m} \leq e^{CR_m^2}$ for some sequence $R_m \to \infty$.

§2. Positive harmonic functions in $L^1(M)$

Our main result is the following theorem.

THEOREM 3. *If M is a stochastically complete manifold, then every positive superharmonic function $u \in L^1(M)$ is equal to a constant.*

PROOF. The Green's function of the Laplace equation can be constructed in a way analogous to the way the Green's function of the heat equation was constructed in §1. For every precompact domain $\Omega \subset M$ with smooth boundary there exists a Green's function $g_\Omega(x, y)$ satisfying for each fixed $y \in \Omega$ the equation $\Delta g_\Omega = -\delta_y(x)$ and the boundary condition $g_\Omega|_{\partial\Omega} = 0$. Further: 1) g_Ω is infinitely differentiable away from y; 2) $g_\Omega(x, y) = g_\Omega(y, x)$ for any $x, y \in \Omega$; and 3) $g_\Omega \geq 0$.

The functions $G_\Omega(x, y, t)$ and $g_\Omega(x, y)$ are connected by the well-known relation

$$g_\Omega(x, y) = \int_0^\infty G_W(x, y, t)\, dt. \tag{2}$$

As Ω increases in size the sequence g_Ω increases and has a limit

$$g(x, y) = \lim_{\Omega \to M} g_\Omega(x, y)$$

which, true, can turn out to be infinite (for example, for $M = \mathbf{R}^2$). If $g(x, y) < \infty$ for $x \neq y$, then $g(x, y)$ is the smallest positive fundamental solution of the operator $-\Delta$, but if $g \equiv \infty$, then there are no positive fundamental solutions (see [6] for details). Manifolds such that $g \equiv \infty$ are called *manifolds of parabolic type*. It is known that M has parabolic type if and only if every positive superharmonic function on M is equal to a constant [12]. See [6] and [14] about geometric conditions for parabolicity.

Now let M be a stochastically complete manifold and u a positive superharmonic function not equal to a constant. We prove that $\int_M u\, dx = \infty$. It follows from the existence of such a function u that M is not parabolic, and hence that the Green's function $g(x, y)$ exists. We verify that

$$\int_M g(x, y)\, dx = \infty. \tag{3}$$

Indeed, it follows from (2) and the stochastic completeness of M that

$$\int_M g(x, y)\, dx = \int_M \int_0^\infty G(x, y, t)\, dt\, dx = \int_0^\infty \int_M G(x, y, t)\, dx\, dt = \int_0^\infty dt = \infty.$$

From this we conclude also that $\int_M u\, dx = \infty$. Let ω be a precompact open subset of M, and let $y \in \omega$. We find a constant $C > 0$ such that $Cu(x) > g(x, y)$ on $\partial\omega$. In particular, for any domain $\Omega \supset \omega$ we get that $Cu > g_\Omega(x, y)$ on $\partial\omega$. Since $g_\Omega|_{\partial\Omega} = 0$, it is also true that $Cu > g_\Omega$ on $\partial\Omega$. The fact that Cu is a superharmonic function implies that $Cu > g_\Omega$ on $\Omega \backslash \omega$. Taking the limit as $\Omega \to M$ gives us that $Cu \geq g$ on $M \backslash \omega$, and so

$$C \int_M u(x)\, dx \geq \int_{M \backslash \omega} g(x, y)\, dx = \int_M g(x, y)\, dx - \int_\omega g(x, y)\, dx. \tag{4}$$

Note that $\int_\omega g(x, y)\, dx < \infty$. Indeed, $g(x, y)$ has the same singularity as in $\mathbf{R}^n$ as $x \to y$, i.e., r^{2-n} or $-\ln r$, where r is the geodesic distance between the points x and y, $r \to 0$ [18], and this singularity is clearly integrable. It thus follows from (3) and (4) that $\int_M u(x)\, dx = \infty$, which is what was required to prove.

REMARK 1. If u is a nonnegative superharmonic function, then in view of the maximum principle either $u \equiv 0$ or $u > 0$, so that Theorem 3 is also valid for such functions.

REMARK 2. If the volume of M is infinite, then under the conditions of Theorem 3 there are no positive superharmonic functions $u \in L^1(M)$.

COROLLARY 1. *If M is a complete Riemannian manifold and*

$$\int^{\infty} \frac{R\,dR}{\ln V_R} = \infty,$$

then every nonnegative superharmonic function in $L^1(M)$ is equal to a constant.

§3. Some examples

Here we present conditions for stochastic completeness and for the validity of the L^1-Liouville theorem for spherically symmetric manifolds.

Denote by M_h the manifold $\mathbf{R} \times S^n$ (where S^n is the *unit sphere* in $\mathbf{R}^n$), equipped with the Riemannian metric $ds^2 = dr^2 + h(r)^2\,d\theta^2$. Here $r \in \mathbf{R}$, $\theta \in S^n$, dr^2 and $d\theta^2$ are the standard metrics on $\mathbf{R}$ and S^n and $h(r)$ is a positive smooth function. For each $r \in \mathbf{R}$ let S_r denote the set of all points of the form (r, θ) with $\theta \in S^n$. Obviously, S_r is the orbit of the group $\mathrm{SO}(n)$ of isometries acting on M_h. Let $\sigma(r) = \mathrm{meas}_{n-1} S_r = \omega_n h(r)^{n-1}$, where ω_n is the $(n-1)$-dimensional unit sphere in $\mathbf{R}^n$. Let W_R be the volume of the shell $\{0 < r < R\}$, i.e., $W_R = \int_0^R \sigma(r)\,dr$. Let

$$I = \int_0^{\infty} \frac{W_R}{W_R'}\,dR.$$

PROPOSITION. *Suppose that the function $h(r)$ is even. Then the manifold M_h is stochastically complete if and only if $I = \infty$.*

COROLLARY 2. *If $f\colon [0, \infty) \to (1, \infty)$ is a smooth downward convex function with $f' > 0$ and*

$$\int^{\infty} \frac{R\,dR}{f(R)} < \infty, \tag{5}$$

then there exists a geodesically complete but not stochastically complete manifold M such that $V_R \le Ce^{f(R)}$, where V_R is the volume of a geodesic ball of radius R with some center $O \in M$.

Indeed, let $\sigma(r) = f'(r)e^{f(r)}$ for sufficiently large r, and let $h(r) = (\sigma(r)/\omega_n)^{1/(n-1)}$. Then the manifold M_h is not stochastically complete. Indeed, for large R

$$W_R = \int_0^R \sigma(r)\,dr = e^{f(R)} + \mathrm{const}, \qquad \frac{W_R}{W_R'} = \frac{e^{f(R)} + \mathrm{const}}{f'(R)e^{f(R)}} \sim \frac{1}{f'(R)} \le \frac{R}{f(R)}$$

because f is convex. Therefore, it follows from (5) that $I < \infty$, and M_h is not stochastically complete, by the proposition.

Moreover, if $O \in S_0$, then $V_R \le 2W_R = 2e^{f(R)} + \mathrm{const} \le Ce^{f(R)}$ for sufficiently large C.

COROLLARY 3. *There exist stochastically complete manifolds for which the volume V_R grows arbitrarily fast.*

Indeed, the equality

$$I = \int_0^{\infty} \frac{dR}{(\ln W_R)'} = \infty$$

is possible for any restriction of the form $\ln W_R \geq f(R)$, where $f(R)$ is an arbitrary monotonically increasing function.

PROOF OF THE PROPOSITION. We now construct on the domain $\{r > 0\}$ of M_h a positive function $v(X)$ satisfying the equation $\Delta v - \lambda v = 0$ for some $\lambda > 0$ and the conditions $v|_{S_0} = 1$ and $\partial v/\partial\nu|_{S_0} = 0$ where ν is the normal to S_0. Obviously, such a function v can be extended evenly to the whole manifold M_h. If v is bounded, then, by Theorem 1, M_h is not stochastically complete. But if $v(x) \to \infty$ as $x \to \infty$, then M_h is stochastically complete in view of Theorem 2.4 in [3]. The function v will depend only on r, so it will be written as $v(r)$. It is not hard to verify that upon multiplication by $\sigma(r)$ the equation $\Delta v - \lambda v = 0$ is reduced to the form

$$(\sigma v')' - \lambda\sigma v = 0. \tag{6}$$

Obviously, the solution of this ordinary differential equation with the initial conditions $v(0) = 1$ and $v'(0) = 0$ is monotonically increasing, and thus stochastic completeness of M_h is equivalent to the condition $v(r) \to \infty$. From (6) and the initial conditions we get the integral equation

$$v(r) = \lambda \int_0^r \frac{d\xi}{\sigma(\xi)} \int_0^\xi \sigma(\eta)v(\eta)\,d\eta + 1.$$

If $I = \infty$, then from $v(\eta) \geq 1$ it follows that

$$v(r) \geq \lambda \int_0^r \frac{W_\xi}{W'_\xi}\,d\xi + 1 \to \infty$$

as $r \to \infty$, i.e., M_h is stochastically complete. If $I < \infty$, then it follows from $v(\eta) \leq v(r)$ that

$$v(r) \leq \lambda v(r) \int_0^r \frac{W_\xi}{W'_\xi}\,d\xi + 1 \leq \lambda I v(r) + 1,$$

and for $\lambda < I^{-1}$ this implies that $v \leq (1 - \lambda I)^{-1}$, i.e., v is bounded, and M is not stochastically complete.

We now proceed to the construction of a counterexample to the L^1-Liouville theorem. It is very easy to find all harmonic functions on M_h depending only on r. Indeed, the equation $\Delta v(r) = 0$ can be rewritten in the form $(\sigma v')' = 0$, from which we find that

$$v(r) = c_1 \int_0^r \frac{d\xi}{\sigma(\xi)} + c_2.$$

It turns out that these solutions include integrable functions: for this it is necessary that $\int_{-\infty}^\infty |v(r)|\sigma(r)\,dr < \infty$.

We analyze the two possible cases.

1. Let

$$v(r) = \int_0^r \frac{d\xi}{\sigma(\xi)}.$$

Then we get the following restriction on σ:

$$\int_0^\infty \sigma(r) \int_0^r \frac{d\xi}{\sigma(\xi)}\,dr < \infty, \qquad r > 0,$$

and an analogous condition for $\sigma(-r)$. Changing the order of integration and introducing the notation

$$W(R) = \int_R^\infty \sigma(\xi)\,d\xi, \qquad W(-R) = \int_{-\infty}^{-R} \sigma(\xi)\,d\xi,$$

where $R > 0$, we get that

$$\int^{\infty} \frac{W(R)}{-W'(R)}\,dR < \infty \tag{7}$$

and an analogous condition for $W(-R)$. Condition (7) is satisfied, for example, by the function $W(R) = e^{-R^{2+\varepsilon}}$, $\varepsilon > 0$. The corresponding manifold M_h constricts very rapidly both as $r \to +\infty$ and as $r \to -\infty$. The function $v(r)$ tends to $\pm\infty$ as $r \to \pm\infty$.

2. Let

$$v(r) = \int_r^{\infty} \frac{d\xi}{\sigma(\xi)}.$$

If $v(r)$ is integrable for $r \to -\infty$, then the conditions of the preceding case hold as $r \to -\infty$. If $v(r)$ is integrable for $r \to +\infty$, then

$$\int^{\infty} \frac{W(R)}{W'(R)}\,dR < \infty, \quad \text{where } W(R) = \int_0^R \sigma(r)\,dr.$$

This condition holds, for example, for $W(R) = e^{R^{2+\varepsilon}}$, $\varepsilon > 0$. In this case the manifold M_h expands strongly as $r \to +\infty$, and $v(r) \to 0$. Since $v(r)$ is positive, we get in a way analogous to that for Corollary 2 that the condition on the growth of the volume V_R in Corollary 1 is sharp.

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