

Geometric analysis on manifolds with ends

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References

1. Introduction

In this survey article, we discuss some recent progress on geometric analysis on manifold with ends. In the final section, we construct manifolds with ends with oscillating volume functions which may turn out to have a different heat kernel estimates from those provided by known results.

Throughout the history of geometric analysis, manifolds with ends have appeared in several contexts. For example, Cai [2], Kasue [19] and Li-Tam [22] et. al. studied manifolds with non-negative Ricci (sectional) curvature outside a compact set where manifolds with ends play an important role. It should be pointed out that there are other recent works on manifolds with ends. See, for instance, Carron [3], Doan [5], Duong, Li and Sikora [6], Hassel, Nix and Sikora [17], Hassel and Sikora [18].

Because of the bottleneck structure inherent to most manifolds with ends, geometric and analytic properties of manifolds with ends are very different from a manifold such as \mathbb{R}^n . For example, in 1979, Kuz'menko and Molchanov [21] proved the following:

THEOREM 1.1. *On $M = \mathbb{R}^3 \# \mathbb{R}^3$, the connected sum of two copies of \mathbb{R}^3 , the weak Liouville property does not hold. Namely, there exists a non-trivial bounded harmonic function.*

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It is a well-known fact that the *parabolic Harnack inequality* ((PHI) in short) implies the weak Liouville property. See [14, Section 2.1] and [26, 5.4.5] for details. By a contraposition argument, the above theorem implies that (PHI) does not hold on $\mathbb{R}^3 \# \mathbb{R}^3$.

Denote by $p(t, x, y)$ the heat kernel of a non-compact weighted manifold (M, d, μ) , that is, the minimal positive fundamental solution of the heat equation $\partial_t u = \Delta u$, where Δ is the weighed Laplacian. In 1986, Li and Yau proved in [23] that

$$(LY) \quad p(t, x, y) \asymp \frac{c}{V(x, \sqrt{t})} e^{-bd^2(x, y)/t}$$

holds on non-compact manifold with non-negative Ricci curvature. Here $V(x, r) := \mu(B(x, r))$, the measure of the open geodesic ball $B(x, r) = \{y \in M : d(x, y) < r\}$ and the sign \asymp means that both \leq and \geq hold but with different values of the positive constants C and b . We call this estimate a Li-Yau type bound and write (LY) in short. The following theorem is a combined result of [7], [25] based on previous contributions of Moser [24], Kusuoka–Stroock [20] et al.

THEOREM 1.2. *On a geodesically complete, non-compact weighted manifold M , the following conditions are equivalent:*

- (1) *The Li-Yau type heat kernel estimates (LY).*
- (2) *The parabolic Harnack inequality (PHI).*
- (3) *The Poincaré inequality (PI): there exists $C, \kappa > 0$ such that for any $x \in M$, $r > 0$ and $f \in C^\infty(\overline{B(x, r)})$,*

$$(PI) \quad \int_{B(x, r)} |f - f_{B(x, r)}|^2 d\mu \leq Cr^2 \int_{B(x, r)} |\nabla f|^2 d\mu,$$

where $f_{B(x, r)} = \frac{1}{V(x, r)} \int_{B(x, r)} f d\mu$, and the volume doubling property (VD): there exists $C > 0$ such that for any $x \in M$, $r > 0$,

$$V(x, 2r) \leq CV(x, r).$$

Combining the above two theorems, the connected sum $M = M_1 \# M_2 = \mathbb{R}^3 \# \mathbb{R}^3$ satisfies neither (PI) nor (LY). Indeed, the function

$$f(x) = \begin{cases} 1 & x \in M_1, \\ -1 & x \in M_2 \end{cases}$$

implies that $f_{B(o, r)} = 0$ for a central reference point $o \in M$ and

$$\int_{B(o, r)} |f - f_{B(o, r)}|^2 d\mu \simeq r^3, \quad \int_{B(o, r)} |\nabla f|^2 d\mu \simeq \text{const},$$

which fails (PI). Here $f \simeq g$ means

$$cf \leq g \leq Cf$$

with some positive constants $0 < c \leq C$ on a suitable range of functions f, g . Moreover, Benjamini, Chavel and Feldman [1] obtained in 1996 the following heat kernel estimate.

THEOREM 1.3. *For $n \geq 3$, let $M = M_1 \# M_2 = \mathbb{R}^n \# \mathbb{R}^n$. There exists $\varepsilon > 0$ such that for $x \in M_1, y \in M_2$ with $|x| \simeq |y| \simeq \sqrt{t}$,*

$$p(t, x, y) \leq \frac{1}{t^{\frac{n+\varepsilon}{2}}} \ll \frac{1}{t^{n/2}}.$$

This theorem asserts that the heat kernel between two different ends is significantly smaller than that on one end because of a *bottleneck effect*.

In view of the above facts, it is natural to ask on a manifold with ends the behavior of the heat kernel $p(t, x, y)$ and the estimate of

$$\Lambda(B(x, r)) := \sup_{\substack{f \in C^1(\overline{B(x, r)}) \\ f \neq \text{const}}} \frac{\int_{B(x, r)} |f - f_{B(x, r)}|^2 d\mu}{\int_{B(x, r)} |\nabla f|^2 d\mu}, \quad (1.1)$$

which is called the *Poincaré constant*.

Notation. Throughout this article, the letters c, c', C, C', C'' denote positive constants whose values may be different at different instances. When the value of a constant is significant, it will be explicitly stated.

2. The state of the art

2.1. Setting. First of all, we begin with the definition of what we call a manifold with finitely many ends. For a fixed integer $k \geq 2$, let M_1, \dots, M_k be a sequence of geodesically complete, non-compact weighted manifolds of the same dimension.

DEFINITION 2.1. We say that a weighted manifold M is a manifold with k ends M_1, M_2, \dots, M_k and write

$$M = M_1 \# \dots \# M_k \quad (2.1)$$

if there is a compact set $K \subset M$ so that $M \setminus K$ consists of k connected components E_1, E_2, \dots, E_k such that each E_i is isometric (as a weighted manifold) to $M_i \setminus K_i$ for some compact set $K_i \subset M_i$ (see Fig. 1). Each E_i (or M_i) will be referred to as an *end* of M .

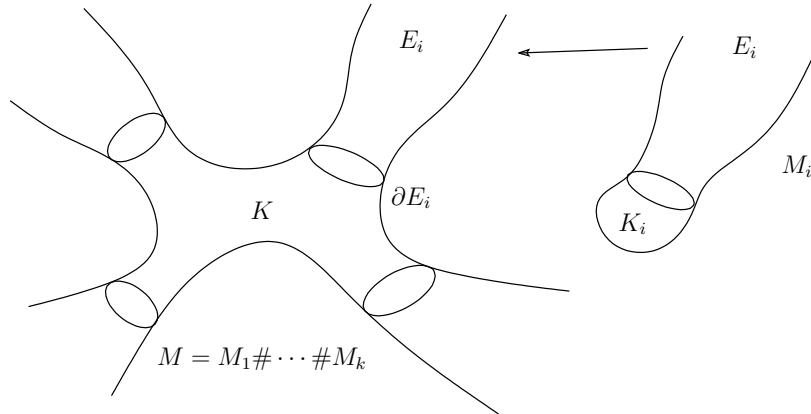


FIGURE 1. Manifold with ends

Here we remark that the definition of end given above is different from the usual notion defined as a connected component of the ideal boundary.

We say that a manifold M is *parabolic* if any positive superharmonic function on M is constant, and *non-parabolic* otherwise. See [8] for details.

Throughout this article, we always assume that each end M_i satisfies (VD) and (PI). Moreover, if the end M_i is parabolic, then we also assume that M_i satisfies the *relatively connected annuli* condition defined as follows.

DEFINITION 2.2 ((RCA)). A weighted manifold M satisfies *relatively connected annuli* condition ((RCA) in short) with respect to a reference point $o \in M$ if there exists a positive constant $A > 1$ such that for any $r > A^2$ and all $x, y \in M$ with $d(o, x) = d(o, y) = r$, there exists a continuous path from x to y staying in $B(o, Ar) \setminus B(o, A^{-1}r)$. See Fig. 2 and 3 for typical positive and negative examples.

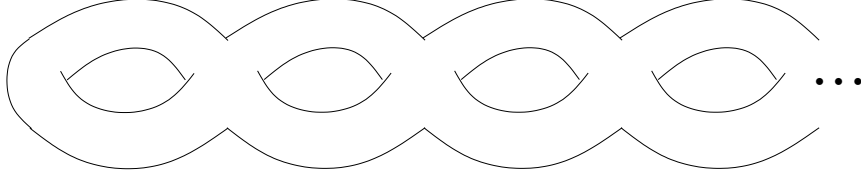


FIGURE 2. Manifold with (RCA)

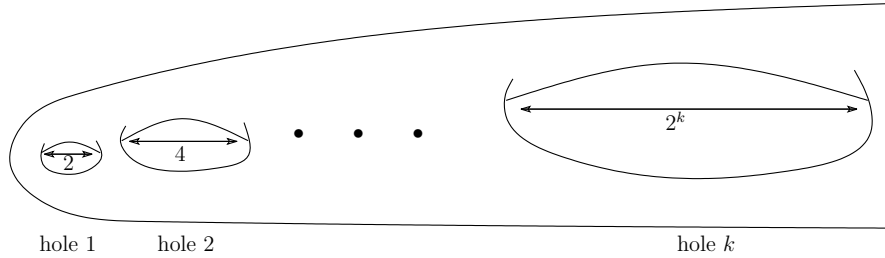


FIGURE 3. Manifold without (RCA)

The assumption (RCA) seems technical but it makes it possible to obtain an optimal estimates of the first exit (hitting) probability and the Dirichlet heat kernel in the exterior of a compact set of a parabolic manifold satisfying (LY). See [12] and [13] for details.

2.2. Heat kernel estimates.

2.2.1. *Off-diagonal estimates.* Let $M = M_1 \# \cdots \# M_k$ be a manifold with k ends. For $t > 0$, $x \in E_i$ and $y \in M$, let $p_{E_i}^D(t, x, y)$ be the extended Dirichlet heat kernel on an end E_i , that is, the Dirichlet heat kernel in $y \in E_i$ and extension to 0 if $y \notin E_i$. Let τ_{E_i} be the first exit time of the Brownian motion from E_i and then $\mathbb{P}_x(\tau_{E_i} < t)$ is the first exit probability starting from x by time t from E_i . We will use the following theorem to estimate the off-diagonal heat kernel estimates.

THEOREM 2.3 (Grigor'yan and Saloff-Coste [15, Theorem 3.5]). *Let $M = M_1 \# \cdots \# M_k$ be a manifold with k ends and fix a central reference point $o \in K$. For $x \in E_i$, $y \in E_j$ and $t > 1$,*

$$p(t, x, y) \simeq p_{E_i}^D(t, x, y) + p(t, o, o) \mathbb{P}_x(\tau_{E_i} < t) \mathbb{P}_y(\tau_{E_j} < t) + \int_0^t p(s, o, o) ds (\partial_t \mathbb{P}_x(\tau_{E_i} < t) \mathbb{P}_y(\tau_{E_j} < t) + \mathbb{P}_x(\tau_{E_i} < t) \partial_t \mathbb{P}_y(\tau_{E_j} < t)). \quad (2.2)$$

Under the assumption of (PI), (VD) on each end, and, in addition, (RCA) on each parabolic end, applying results in [12] and [13], the quantities $p_{E_i}^D(t, x, y)$, $\mathbb{P}_x(\tau_{E_i} < t)$, $\mathbb{P}_y(\tau_{E_j} < t)$, $\partial_t \mathbb{P}_x(\tau_{E_i} < t)$ and $\partial_t \mathbb{P}_y(\tau_{E_j} < t)$ can be estimated. Hence, estimating $p(t, o, o)$ becomes the key missing estimate to obtain off-diagonal bounds on manifolds with ends.

2.2.2. *Non-parabolic case.* First we consider heat kernel estimates on $M = M_1 \# \cdots \# M_k$, where M is non-parabolic, namely, at least one end M_i is non-parabolic.

For a fixed reference point $o_i \in K_i \subset M_i$, let

$$V_i(r) := \mu(B(o_i, r))$$

and

$$h_i(r) := 1 + \left(\int_1^r \frac{s ds}{V_i(s)} \right)_+.$$

Here we remark that, under the assumption (LY), M_i is parabolic if and only if

$$\lim_{r \rightarrow \infty} h_i(r) = \infty.$$

In 2009, Grigor'yan and Saloff-Coste [15] obtained the following (see also [16]).

THEOREM 2.4. *Let $M = M_1 \# \cdots \# M_k$ be a manifold with k ends. Assume that each end M_i satisfies (PI) and (VD) and that each parabolic end satisfies (RCA). Assume also that M is non-parabolic. Then for all $t > 0$,*

$$p(t, o, o) \simeq \frac{1}{\min_i V_i(\sqrt{t}) h_i^2(\sqrt{t})}.$$

If all ends M_1, \dots, M_k are non-parabolic, then all functions h_1, \dots, h_k are bounded. Hence, the above theorem implies that

$$p(t, o, o) \simeq \frac{1}{\min_i V_i(\sqrt{t})},$$

namely, the behavior of the heat kernel at the central reference point is determined by the **smallest end!**

As a typical example, let M be $M_1 \# M_2 = \mathbb{R}^n \# \mathbb{R}^n$, the connected sum of two copies of \mathbb{R}^n with $n \geq 3$. Then the above theorem implies that

$$p(t, o, o) \simeq \frac{1}{t^{n/2}}.$$

Substituting this estimates into Theorem 2.3, we obtain that for $x \in M_1, y \in M_2$,

$$p(t, x, y) \simeq \frac{1}{t^{n/2}} \left(\frac{1}{|x|^{n-2}} + \frac{1}{|y|^{n-2}} \right) e^{-bd^2(x,y)/t},$$

where $|x| = 1 + d(x, K)$.

2.2.3. *Parabolic case.* Next, we consider the case of manifolds with ends, $M = M_1 \# \cdots \# M_k$, which are parabolic, that is, for which all ends M_1, \dots, M_k are parabolic. To prove an optimal heat kernel estimates, we need the following assumptions on each end.

DEFINITION 2.5 (c.f. [10]). An end M_i is called *subcritical* if

$$h_i(r) \leq C \frac{r^2}{V_i(r)} \quad (\forall r > 1)$$

and *regular* if there exist $\gamma_1, \gamma_2 > 0$ satisfying $2\gamma_1 + \gamma_2 < 2$ such that

$$c \left(\frac{R}{r} \right)^{2-\gamma_2} \leq \frac{V_i(R)}{V_i(r)} \leq C \left(\frac{R}{r} \right)^{2+\gamma_1} \quad (\forall 1 < r \leq R). \quad (2.3)$$

For example, a manifold M with volume function $V(r) = r^\alpha (\log r)^\beta$ is parabolic if and only if either $\alpha < 2$ or $\alpha = 2$ and $\beta \leq 1$. Moreover, M is subcritical if $\alpha < 2$ and regular if $\alpha = 2$ and $\beta \leq 1$. We remark that if M_i satisfies (VD), then the *reverse doubling property* holds and that implies that for any subcritical end, there exists $\delta > 0$ such that

$$V_i(r) \leq Cr^{2-\delta} \quad (\forall r > 0). \quad (2.4)$$

For $r > 0$, let $m = m(r)$ be a number so that

$$V_m(r) = \max_i V_i(r). \quad (2.5)$$

We can now state the following result.

THEOREM 2.6 (Grigor'yan, Ishiwata, Saloff-Coste [10]). *Let $M = M_1 \# \cdots \# M_k$ be a manifold with k parabolic ends. Assume that each end M_i satisfies (PI), (VD), (RCA) and is either subcritical or regular. If there exist both of subcritical and regular ends, assume also that the constant δ in (2.4) satisfies $\delta > \gamma_2$, namely, for any subcritical volume function $V_i(r)$ and any regular volume function $V_j(r)$,*

$$V_i(r) \leq Cr^{2-\delta} \leq C'r^{2-\gamma_2} \leq C''V_j(r) \quad (\forall r > 0).$$

Moreover, assume that there exists an end M_m such that for all $i = 1, \dots, k$ and for all $r > 0$

$$V_m(r) \geq cV_i(r) \text{ and } V_m(r)h_m^2(r) \leq CV_i(r)h_i^2(r). \quad (2.6)$$

Then for $t > 0$

$$p(t, o, o) \simeq \frac{1}{V_m(\sqrt{t})}. \quad (2.7)$$

This means that the on-diagonal heat kernel estimates at the central reference point is determined by the **largest end!**

REMARK 2.7. In our approach, we require the existence of a fixed dominating end given by (2.6) for the optimal estimates in (2.7) to hold. Indeed, more generally, on a manifold with either regular or subcritical ends, we obtain for $t > 1$, (see [10] for the detail)

$$p(t, o, o) \leq C \frac{\min_i h_i^2(\sqrt{t})}{\min_i V_i(\sqrt{t})h_i^2(\sqrt{t})}. \quad (2.8)$$

The assumption in (2.6) implies that for all $r > 1$

$$\frac{\min_i h_i^2(r)}{\min_i V_i(r)h_i^2(r)} \leq \frac{C}{V_m(r)}, \quad (2.9)$$

which allows to apply [4, Theorem 7.2] for the matching lower bound. In Section 3, we construct manifolds with ends without a fixed dominating end and, in such cases, the estimates in (2.9) does not hold.

As illustrative examples, let $M = M_1 \# \cdots \# M_k$ be a manifold with parabolic ends, where each end M_i satisfies (PI), (VD), (RCA). Let α_i and β_i be sequences satisfying

$$(\alpha_1, \beta_1) \geq (\alpha_2, \beta_2) \geq \cdots \geq (\alpha_k, \beta_k) > (0, +\infty)$$

in the sense of lexicographical order, namely $(\alpha_i, \beta_i) > (\alpha_j, \beta_j)$ means that

$$\alpha_i > \alpha_j \text{ or } \alpha_i = \alpha_j \text{ and } \beta_i > \beta_j$$

and we assume that

$$V_i(r) \simeq r^{\alpha_i} (\log r)^{\beta_i}, \quad r > 2.$$

Here we need $(\alpha_1, \beta_1) \leq (2, 1)$ so that all ends M_1, \dots, M_k are parabolic. Then the above theorem implies that

$$p(t, o, o) \simeq \frac{1}{t^{\alpha_1/2} (\log t)^{\beta_1}}, \quad t > 2.$$

As an explicit example, suppose that $k = 2$ and $(\alpha_1, \beta_1) = (2, 0)$ and $(\alpha_2, \beta_2) = (1, 0)$. Substituting the above estimates into (2.2), we obtain for $x \in E_1, y \in E_2$ and $t > 1$

$$p(t, x, y) \simeq \begin{cases} \frac{1}{t} e^{-bd^2(x,y)/t} & \text{if } |x| > \sqrt{t} \\ \frac{1}{t} \left(1 + \frac{|y|}{\sqrt{t}} \log \frac{e\sqrt{t}}{|x|}\right) & \text{if } |x|, |y| \leq \sqrt{t} \\ \frac{1}{t} \left(\log \frac{e\sqrt{t}}{|x|}\right) e^{-bd^2(x,y)/t} & \text{if } |x| \leq \sqrt{t} < |y|. \end{cases}$$

REMARK 2.8. Assume that all ends of a manifold $M = M_1 \# \dots \# M_k$ are sub-critical. Then for $x \in E_i$ and $y \in E_j$ with $i \neq j$ and $t > 1$,

$$p(t, x, y) \asymp \frac{C}{V_m(\sqrt{t})} e^{-bd^2(x,y)/t}$$

(see [9, Theorem 2.3]).

2.3. Poincaré constant estimates. In this section, we consider the estimates of the Poincaré constant defined in (1.1). Recall that $M = M_1 \# \dots \# M_k$ is a manifold with ends M_1, \dots, M_k , where each end satisfies (VD) and (PI). Let $o \in K$ be a central reference point. Our main interest is to obtain the Poincaré constant $\Lambda(B(o, r))$ at the central point o . In fact, by the monotonicity of Λ together with a Whitney covering argument (see [10]), for $r > 2|x|$

$$\Lambda(B(x, r)) \simeq \Lambda(B(o, r)).$$

For $r > 0$, let $n = n(r)$ be the number so that

$$V_n(r) = \max_{i \neq m} V_i(r),$$

where $m = m(r)$ is the number of the largest end (see (2.5)). Then we obtain the following.

THEOREM 2.9 (Grigor'yan, Ishiwata, Saloff-Coste [10]). *Let $M = M_1 \# \dots \# M_k$ be a manifold with k non-parabolic ends. Assume that each end M_i satisfies (VD) and (PI). Then for sufficiently large $r > 1$*

$$\Lambda(B(o, r)) \leq CV_n(r).$$

Moreover, if for all $r > 0$

$$rV'(r) \leq CV(r),$$

then for sufficiently large $r > 1$

$$\Lambda(B(o, r)) \simeq V_n(r).$$

When M has at least one parabolic end, we assume the following additional condition (see [10] for details).

DEFINITION 2.10 ((COE)). We say that a manifold $M = \#_{i \in I} M_i$ has *critically ordered ends* and write (COE) in short if there exist $\varepsilon, \delta, \gamma_1, \gamma_2 > 0$ such that

$$\gamma_1 < \varepsilon, \quad \gamma_1 + \gamma_2 < \delta < 2, \quad 2\gamma_1 + \gamma_2 < 2,$$

and a decomposition

$$I = I_{super} \sqcup I_{middle} \sqcup I_{sub}$$

such that the following conditions are satisfied:

(a) For each $i \in I_{super}$ and all $r \geq 1$,

$$V_i(r) \geq cr^{2+\epsilon}.$$

(b) For each $i \in I_{sub}$, V_i is subcritical (see Definition 2.5) and

$$V_i(r) \leq Cr^{2-\delta}.$$

(c) For each $i \in I_{middle}$, V_i is regular (see (2.3)). Moreover, for any pair $i, j \in I_{middle}$ we have either $V_i \geq cV_j$ or $V_j \geq cV_i$ (i.e., the ends in I_{middle} can be ordered according to their volume growth uniformly over $r \in [1, \infty)$) and $V_i \geq cV_j$ implies that $V_i h_i \geq c'V_j h_j$. Besides, if M is parabolic (i.e., all ends are parabolic) then $V_i \geq CV_j$ also implies $V_i h_i^2 \leq C'V_j h_j^2$.

THEOREM 2.11 ([10]). *Let $M = M_1 \# \cdots \# M_k$ be a manifold with k ends, where each end satisfies (VD) and (PI). Suppose that there exists at least one parabolic end and each parabolic end satisfies (RCA). If M admits (COE), then for sufficiently large $r > 1$*

$$\Lambda(B(o, r)) \leq CV_n(r)h_n(r).$$

If, in addition, each V_i satisfies that

$$rV_i'(r) \leq CV_i(r) \quad (\forall r > 1),$$

then, for sufficiently large $r > 1$

$$\Lambda(B(o, r)) \simeq V_n(r)h_n(r).$$

These results say that the Poincaré constant $\Lambda(B(o, r))$ is determined by the **second largest end!**

As an explicit example, let $M = \mathbb{R}^n \# \mathbb{R}^n$ with $n \geq 2$. Then Theorems 2.9 and 2.11 imply that

$$\Lambda(B(o, r)) \simeq \begin{cases} r^n & n \geq 3, \\ r^2 \log r & n = 2. \end{cases}$$

Let $M = M_1 \# \cdots \# M_k$ be a manifold with ends. Assume that each end M_i satisfies (VD), (PI) and that each parabolic end satisfies (RCA). Suppose that for $i = 1, \dots, k$

$$V_i(r) \simeq r^{\alpha_i} (\log r)^{\beta_i},$$

where $(\alpha_1, \beta_1) \geq (\alpha_2, \beta_2) \geq \cdots \geq (\alpha_k, \beta_k)$ as the lexicographical order. Then

$$\begin{aligned} \Lambda(B(o, r)) &\simeq V_2(r)h_2(r) \\ &\simeq \begin{cases} r^{\alpha_2} (\log r)^{\beta_2} & \text{if } (\alpha_2, \beta_2) > (2, 1), \\ r^2 \log r (\log \log r)^2 & \text{if } (\alpha_2, \beta_2) = (2, 1), \\ r^2 \log r & \text{if } (2, -\infty) < (\alpha_2, \beta_2) < (2, 1), \\ r^2 & \text{if } (\alpha_2, \beta_2) < (2, -\infty). \end{cases} \end{aligned}$$

3. Manifold with ends with oscillating volume functions

3.1. Preliminaries. The purpose of this section is to construct manifolds with ends for which the estimate in (2.8) might not give an optimal bound. To obtain such a manifold, we need a manifold with (VD) and (PI) together with oscillating volume function. First, let us recall the following theorem.

THEOREM 3.1 (Grigor'yan and Saloff-Coste [14, Theorem 5.7]). *Let (M, μ) be a complete non-compact weighted manifold with (PHI) and (RCA) at a reference point $o \in M$. If a positive valued smooth function $W : [0, \infty) \rightarrow \mathbb{R}$ satisfies for all $r > 0$*

$$\sup_{[r, 2r]} W \leq C \inf_{[r, 2r]} W, \quad (3.1)$$

$$\int_0^r W^2(s) ds \leq CW^2(r)r^2, \quad (3.2)$$

then the weighted manifold $(M, W^2(d(o, \cdot))\mu)$ also satisfies (PHI).

Let (M_1, μ_1) be the 2-dimensional Euclidean space \mathbb{R}^2 with the Euclidean measure. We denote by (M_2, μ_2) a weighted manifold $(\mathbb{R}^2, W^2(d(o, \cdot))\mu_1)$, where the positive valued smooth function $W : [0, \infty) \rightarrow \mathbb{R}$ is defined as follows. For $\alpha > 2$, $0 < \beta < 2$ define a function W so that for all $k \in \mathbb{N}$

$$\int_0^r W^2(s) ds = \begin{cases} r^2 & 0 < r < a_1, a_k \leq r < b_k, \\ \left(\frac{r}{b_k}\right)^\alpha b_k^2 & b_k \leq r < c_k, \\ r^2 \log r & c_k \leq r < d_k, \\ \left(\frac{r}{d_k}\right)^\beta d_k^2 \log d_k & d_k \leq r < a_{k+1}, \end{cases} \quad (3.3)$$

where the sequences $a_k \leq b_k < c_k \leq d_k < a_{k+1}$ satisfy $a_1 > e$ and

$$b_k = \frac{c_k}{(\log c_k)^{\frac{1}{\alpha-2}}}, \quad (3.4)$$

$$a_{k+1} = d_k (\log d_k)^{\frac{1}{2-\beta}} \quad (3.5)$$

(see fig. 4). These sequences will be fixed later. Then the function W satisfies that

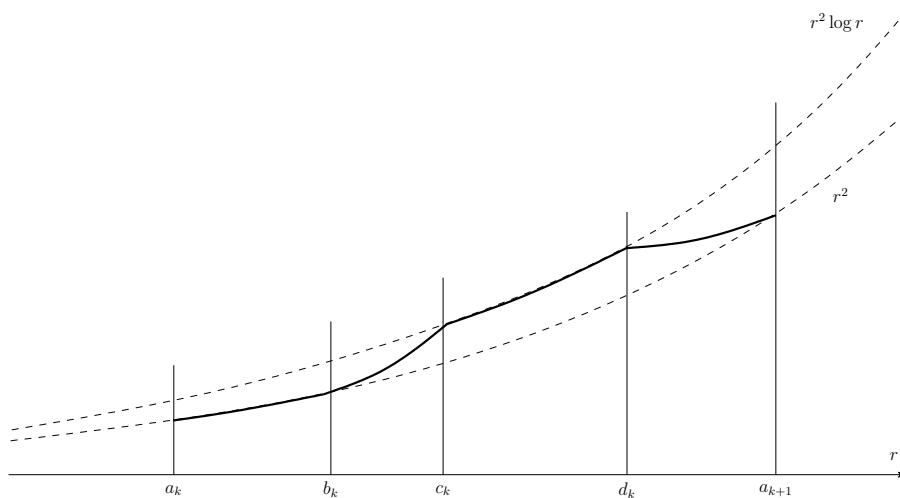


FIGURE 4. Oscillating volume function (thick line)

$$W(r) \simeq \begin{cases} 1 & 0 < r < a_1, a_k \leq r < b_k, \\ \left(\frac{r}{b_k}\right)^{\frac{\alpha-2}{2}} & b_k \leq r < c_k, \\ \sqrt{\log r} & c_k \leq r < d_k, \\ \sqrt{\log d_k} \left(\frac{r}{d_k}\right)^{-\frac{2-\beta}{2}} & d_k \leq r < a_{k+1}, \end{cases}$$

$$W'(r) \simeq \begin{cases} 0 & 0 < r < a_1, a_k < r < b_k, \\ r^{\frac{\alpha-4}{2}} b_k^{-\frac{\alpha-2}{2}} & b_k < r < c_k, \\ \frac{1}{\sqrt{\log r} r} & c_k < r < d_k, \\ -r^{-\frac{4-\beta}{2}} \sqrt{\log d_k} d_k^{\frac{2-\beta}{2}} & d_k < r < a_{k+1}. \end{cases}$$

Since

$$\frac{W'(r)}{W(r)} \simeq \begin{cases} 0 & 0 < r < a_1, a_k < r < b_k, \\ \frac{1}{r} & b_k < r < c_k, \\ \frac{1}{r \log r} & c_k < r < d_k, \\ -\frac{1}{r} & d_k < r < a_{k+1}, \end{cases} \quad (3.6)$$

the condition in (3.1) holds. Indeed, the estimates in (3.6) imply that for any $0 < r_1 < r_2$

$$-C \log \frac{r_2}{r_1} = - \int_{r_1}^{r_2} \frac{C ds}{s} \leq \int_{r_1}^{r_2} \frac{W'(s)}{W(s)} ds = \log \frac{W(r_2)}{W(r_1)} \leq \int_{r_1}^{r_2} \frac{C ds}{s} = C \log \frac{r_2}{r_1}.$$

Then we obtain for any $0 < r_1 \leq r_2 \leq 2r_1$

$$W(r_1) \simeq W(r_2).$$

Taking $r_1 \leq r_2, r_3 \leq 2r_1$ so that

$$W(r_2) = \sup_{[r_1, 2r_1]} W \text{ and } W(r_3) = \inf_{[r_1, 2r_1]} W,$$

we conclude the condition in (3.1).

Moreover, we see that

$$W^2(r)r^2 \simeq \begin{cases} r^2 & 0 < r < a_1, a_k \leq r < b_k, \\ r^\alpha b_k^{2-\alpha} & b_k \leq r < c_k, \\ r^2 \log r & c_k \leq r < d_k, \\ r^\beta d_k^{2-\beta} \log d_k & d_k \leq r < a_{k+1}, \end{cases}$$

which satisfies the condition in (3.2). Applying Theorem 3.1, the weighted manifold $(M_2, \mu_2) = (\mathbb{R}^2, W^2(d(o, \cdot))\mu_1)$ satisfies (PHI)¹.

Let $Z : [0, \infty) \rightarrow \mathbb{R}$ be a positive valued smooth function satisfying

$$Z(r) = \sqrt{1 + 2 \log r} \quad (r \geq 1).$$

Let (M_3, μ_3) be a weighted manifold $(\mathbb{R}^2, Z^2(d(o, \cdot))\mu_1)$. Then Theorem 3.1 implies also that (M_3, μ_3) satisfies (PHI).

¹We need a smooth modification of the function W satisfying (3.3) to apply Theorem 3.1. However, we omit the smoothing argument for simplicity.

For $i = 1, 2, 3$, we denote by $V_i(r)$ the volume function on (M_i, μ_i) at $o \in M_i = \mathbb{R}^2$. Then we obtain

$$\begin{aligned} V_1(r) &= \pi r^2, \\ V_2(r) &= 2\pi \int_0^r W^2(s) ds, \\ V_3(r) &= 2\pi \int_0^1 Z^2(s) ds + 2\pi r^2 \log r \quad (r \geq 1). \end{aligned}$$

It is easy to obtain for sufficiently large $r > 1$

$$h_1(r) \simeq \log r \quad \text{and} \quad h_3(r) \simeq \log \log r.$$

Now we estimate the function $h_2(r)$. Observe that

$$\begin{aligned} \int_{a_k}^{b_k} \frac{s ds}{V_2(s)} &= \log \frac{b_k}{a_k}, \\ \int_{b_k}^{c_k} \frac{s ds}{V_2(s)} &= \int_{b_k}^{c_k} b_k^{\alpha-2} s^{1-\alpha} ds = \frac{1}{\alpha-2} \left(1 - \frac{1}{\log c_k} \right) \simeq 1, \\ \int_{c_k}^{d_k} \frac{s ds}{V_2(s)} &= \int_{c_k}^{d_k} \frac{ds}{s \log s} = \log \left(\frac{\log d_k}{\log c_k} \right), \\ \int_{d_k}^{a_{k+1}} \frac{s ds}{V_2(s)} &= \int_{d_k}^{a_{k+1}} \frac{s^{1-\beta} ds}{d_k^{2-\beta} \log d_k} = \frac{1}{(2-\beta)} \left(1 - \frac{1}{\log d_k} \right) \simeq 1. \end{aligned}$$

Then we obtain

$$h_2(a_n) = 1 + \int_1^{a_1} \frac{s ds}{V_2(s)} + \sum_{k=1}^{n-1} \int_{a_k}^{a_{k+1}} \frac{s ds}{V_2(s)} \simeq \sum_{k=1}^n \left(\log \frac{b_k}{a_k} + \log \frac{\log d_k}{\log c_k} + 1 \right), \quad (3.7)$$

which shows that the behavior of $h_2(r)$ depends on the choice of sequences $a_k \leq b_k < c_k \leq d_k < a_{k+1}$ satisfying (3.4) and (3.5).

3.2. Example 1. For the first case, let us choose sequences $a_k \leq b_k < c_k \leq d_k < a_{k+1}$ so that for any $k \in \mathbb{N}$

$$a_k \simeq b_k, \quad c_k \simeq d_k. \quad (3.8)$$

In this case, we obtain the following.

LEMMA 3.2. *If a_1 is large enough and the sequences $a_k \leq b_k < c_k \leq d_k < a_{k+1}$ satisfy (3.4), (3.5) and (3.8), then for sufficiently large $r > 1$,*

$$h_2(r) \simeq \frac{\log r}{\log \log r}.$$

PROOF. By the estimate in (3.7), we obtain

$$h_2(a_n) \simeq n. \quad (3.9)$$

Let us consider the behavior a_n . By the assumption in (3.4), we always have

$$c_k \simeq b_k (\log b_k)^{\frac{1}{\alpha-2}}. \quad (3.10)$$

Assumptions in (3.4) and (3.5) imply that for any $k \in \mathbb{N}$

$$\begin{aligned} a_{k+1} &= d_k (\log d_k)^{\frac{1}{2-\beta}} \simeq c_k (\log c_k)^{\frac{1}{2-\beta}} \\ &\simeq b_k (\log b_k)^{\frac{1}{\alpha-2}} \left(\log \left(b_k (\log b_k)^{\frac{1}{\alpha-2}} \right) \right)^{\frac{1}{2-\beta}} \\ &= b_k (\log b_k)^{\frac{1}{\alpha-2}} \left(\log b_k + \frac{1}{\alpha-2} \log \log b_k \right)^{\frac{1}{2-\beta}} \\ &\simeq b_k (\log b_k)^{\frac{1}{\alpha-2} + \frac{1}{2-\beta}} = b_k (\log b_k)^\gamma \simeq a_k (\log a_k)^\gamma, \end{aligned}$$

where $\gamma = \frac{1}{\alpha-2} + \frac{1}{2-\beta}$. Taking a_1 large enough, there exist positive constants c, C and $\gamma' > \gamma$ such that for any $n \in \mathbb{N}$

$$cn^{\gamma n} \leq a_n \leq Cn^{\gamma' n}.$$

This implies that for sufficiently large $n \in \mathbb{N}$

$$\log a_n \simeq n \log n, \quad \log \log a_n \simeq \log n.$$

Then we obtain for sufficiently large $n \in \mathbb{N}$,

$$\frac{\log a_n}{\log \log a_n} \simeq n \simeq h_2(a_n),$$

which concludes the lemma. \square

Now we estimate the heat kernel on $M = M_1 \# M_2$. By the definition of $V_1(r)$, $V_2(r)$ and by Lemma 3.2, we obtain for sufficiently large $r > 1$

$$\begin{aligned} V_1(r) &= r^2, & V_2(r) &\simeq \begin{cases} r^2 & a_k \leq r < b_k \\ r^2 \log r & c_k \leq r < d_k, \end{cases} \\ h_1(r) &\simeq \log r, & h_2(r) &\simeq \frac{\log r}{\log \log r}, \\ V_1(r)h_1^2(r) &\simeq r^2(\log r)^2, & V_2(r)h_2^2(r) &\simeq \begin{cases} r^2 \frac{(\log r)^2}{(\log \log r)^2} & a_k \leq r < b_k \\ r^2 \frac{(\log r)^3}{(\log \log r)^2} & c_k \leq r < d_k. \end{cases} \end{aligned}$$

According to the heat kernel upper estimate in (2.8), we obtain for $c_k \leq \sqrt{t} < d_k$

$$p(t, o, o) \leq \frac{\min_i h_i^2(\sqrt{t})}{\min_i V_i(\sqrt{t})h_i^2(\sqrt{t})} \simeq \frac{\left(\frac{\log \sqrt{t}}{\log \log \sqrt{t}} \right)^2}{t (\log \sqrt{t})^2} \simeq \frac{1}{t (\log \log t)^2}.$$

Since $V(r) \simeq \max_i V_i(r) = r^2 \log r$ in this interval, the above upper estimate is much larger than

$$\frac{1}{V(\sqrt{t})} \simeq \frac{1}{t \log t},$$

which makes difficult to obtain matching lower bound by using [4, Theorem 7.2].

3.3. Example 2. Next, let us choose sequences $a_k \leq b_k < c_k \leq d_k < a_{k+1}$ so that for some $\delta > 1$ and for any $k \in \mathbb{N}$

$$a_k \simeq b_k, \quad c_k^\delta \simeq d_k. \quad (3.11)$$

In this case, we obtain the following.

LEMMA 3.3. *If a_1 is large enough and the sequences $a_k \leq b_k < c_k \leq d_k < a_{k+1}$ satisfy (3.4), (3.5) and (3.11), then for sufficiently large $r > 1$,*

$$h_2(r) \simeq \log \log r.$$

PROOF. The estimate in (3.7) yields that

$$h_2(a_n) \simeq n.$$

By the estimates in (3.10) and (3.11), we obtain

$$d_k \simeq c_k^\delta \simeq \left(b_k (\log b_k)^{\frac{1}{\alpha-2}} \right)^\delta \simeq a_k^\delta (\log a_k)^{\frac{\delta}{\alpha-2}}.$$

Assumptions in (3.4) and (3.5) imply that

$$\begin{aligned} a_{k+1} &= d_k (\log d_k)^{\frac{1}{2-\beta}} \\ &\simeq a_k^\delta (\log a_k)^{\frac{\delta}{\alpha-2}} \left(\delta \log a_k + \frac{\delta}{\alpha-2} \log \log a_k \right)^{\frac{1}{2-\beta}} \\ &\simeq a_k^\delta (\log a_k)^\theta, \end{aligned}$$

where $\theta = \frac{\delta}{\alpha-2} + \frac{1}{2-\beta}$. Hence, there exist positive constants c, C and $\eta > \delta$ such that for any $k \in \mathbb{N}$

$$ca_k^\delta \leq a_{k+1} \leq Ca_k^\eta.$$

This implies that for any $n \in \mathbb{N}$

$$c^{\frac{\delta(n-1)-1}{\delta-1}} a_1^{\delta(n-1)} \leq a_n \leq C^{\frac{\eta(n-1)-1}{\eta-1}} a_1^{\eta(n-1)}.$$

Then we obtain

$$\frac{\delta^{n-1}-1}{\delta-1} \log c + \delta^{(n-1)} \log a_1 \leq \log a_n \leq \frac{\eta^{(n-1)}-1}{\eta-1} \log C + \eta^{(n-1)} \log a_1.$$

Taking a_1 large enough, we obtain for sufficiently large $n \in \mathbb{N}$

$$\log \log a_n \simeq n \simeq h_2(a_n),$$

which concludes the lemma. \square

Now let us consider the estimate of the heat kernel on $M = M_2 \# M_3$. By the definition of $V_2(r)$, $V_3(r)$ and by Lemma 3.3, we obtain for sufficiently large $r > 1$

$$\begin{aligned} V_2(r) &\simeq \begin{cases} r^2 & a_k \leq r < b_k, \\ r^2 \log r & c_k \leq r < d_k, \end{cases} & V_3(r) &\simeq r^2 \log r, \\ h_2(r) &\simeq \log \log r, & h_3(r) &\simeq \log \log r, \\ V_2(r)h_2^2(r) &\simeq \begin{cases} r^2(\log \log r)^2 & a_k \leq r < b_k, \\ r^2 \log r (\log \log r)^2 & c_k \leq r < d_k. \end{cases} & V_3(r)h_3^2(r) &\simeq r^2 \log r (\log \log r)^2. \end{aligned}$$

Substituting above into (2.8), we obtain for sufficiently large $k \in \mathbb{N}$ and $a_k \leq \sqrt{t} < b_k$

$$p(t, o, o) \leq C \frac{(\log \log \sqrt{t})^2}{t(\log \log \sqrt{t})^2} \simeq \frac{1}{t}.$$

Since $V(r) \simeq \max_i V_i(r) \simeq r^2 \log r$ for all $r > 1$, the above upper estimate is much larger than

$$\frac{1}{V(\sqrt{t})} \simeq \frac{1}{t \log t},$$

which makes difficult to obtain matching lower bound by using [4, Theorem 7.2].

We hope to prove matching heat kernel lower bounds in forthcoming work.

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