Appendix of Theorems 2.3–2.5

The following tables give the behavior of the heat kernel $p(t,x,y)$ varying on the regimes of $x, y \in M$ and $t > t_0$. Let E_i and E_j be two different ends of M.

Table of (i): The case that all ends are subcritical. If the end is maximal, then the underwaved terms can be dominated by another term.

		$y \in E_i$, $\lvert y\rvert > \sqrt{t}$	$y \in E_i$, $\lvert y\rvert \approx \sqrt{t}$	$y \in E_i$, $\lvert y\rvert < \sqrt{t}$	$y \in K$	$y \in E_j$, $\lvert y\rvert < \sqrt{t}$	$y \in E_j$, $\lvert y\rvert \approx \sqrt{t}$	$y \in E_j$, $\lvert y\rvert > \sqrt{t}$
$x \in E_i$	$\lvert x\rvert > \sqrt{t}$	$\frac{C}{V_i(x,\sqrt{t})} e^{-b\frac{d^2}{t}}$	$\frac{C}{V_i(\sqrt{t})} e^{-b\frac{d^2}{t}}$	$C\left(\frac{D(y,t)}{\utilde{V_i(\sqrt{t})}} + \frac{1}{V_{\max}(\sqrt{t})}\right) e^{-b\frac{d^2}{t}}$	$\frac{C}{V_{\max}(\sqrt{t})} e^{-b\frac{d^2}{t}}$	$\frac{C}{V_{\max}(\sqrt{t})} e^{-b\frac{d^2}{t}}$	$\frac{C}{V_{\max}(\sqrt{t})} e^{-b\frac{d^2}{t}}$	$\frac{C}{V_{\max}(\sqrt{t})} e^{-b\frac{d^2}{t}}$
$x \in E_i$	$\lvert x\rvert \approx \sqrt{t}$	$\frac{C}{V_i(\sqrt{t})} e^{-b\frac{d^2}{t}}$	$\frac{C}{V_i(\sqrt{t})}$	$C\left(\frac{D(y,t)}{\utilde{V_i(\sqrt{t})}} + \frac{1}{V_{\max}(\sqrt{t})}\right)$	$\frac{C}{V_{\max}(\sqrt{t})}$	$\frac{C}{V_{\max}(\sqrt{t})}$	$\frac{C}{V_{\max}(\sqrt{t})}$	$\frac{C}{V_{\max}(\sqrt{t})} e^{-b\frac{d^2}{t}}$
$x \in E_i$	$\lvert x\rvert < \sqrt{t}$	$C\left(\frac{D(x,t)}{\utilde{V_i(\sqrt{t})}} + \frac{1}{V_{\max}(\sqrt{t})}\right) e^{-b\frac{d^2}{t}}$	$C\left(\frac{D(x,t)}{\utilde{V_i(\sqrt{t})}} + \frac{1}{V_{\max}(\sqrt{t})}\right)$	$C\left(\frac{D(x,t)D(y,t)}{\utilde{V_i(\sqrt{t})}} + \frac{1}{V_{\max}(\sqrt{t})}\right)$	$\frac{C}{V_{\max}(\sqrt{t})}$	$\frac{C}{V_{\max}(\sqrt{t})}$	$\frac{C}{V_{\max}(\sqrt{t})}$	$\frac{C}{V_{\max}(\sqrt{t})} e^{-b\frac{d^2}{t}}$
$x \in K$		$\frac{C}{V_{\max}(\sqrt{t})} e^{-b\frac{d^2}{t}}$	$\frac{C}{V_{\max}(\sqrt{t})}$	$\frac{C}{V_{\max}(\sqrt{t})}$	$\frac{C}{V_{\max}(\sqrt{t})}$	$\frac{C}{V_{\max}(\sqrt{t})}$	$\frac{C}{V_{\max}(\sqrt{t})}$	$\frac{C}{V_{\max}(\sqrt{t})} e^{-b\frac{d^2}{t}}$
$x \in E_j$	$\lvert x\rvert < \sqrt{t}$	$\frac{C}{V_{\max}(\sqrt{t})} e^{-b\frac{d^2}{t}}$	$\frac{C}{V_{\max}(\sqrt{t})}$	$\frac{C}{V_{\max}(\sqrt{t})}$	$\frac{C}{V_{\max}(\sqrt{t})}$	$C\left(\frac{D(x,t)D(y,t)}{\utilde{V_j(\sqrt{t})}} + \frac{1}{V_{\max}(\sqrt{t})}\right)$	$C\left(\frac{D(x,t)}{\utilde{V_j(\sqrt{t})}} + \frac{1}{V_{\max}(\sqrt{t})}\right)$	$C\left(\frac{D(x,t)}{\utilde{V_j(\sqrt{t})}} + \frac{1}{V_{\max}(\sqrt{t})}\right) e^{-b\frac{d^2}{t}}$
$x \in E_j$	$\lvert x\rvert \approx \sqrt{t}$	$\frac{C}{V_{\max}(\sqrt{t})} e^{-b\frac{d^2}{t}}$	$\frac{C}{V_{\max}(\sqrt{t})}$	$\frac{C}{V_{\max}(\sqrt{t})}$	$\frac{C}{V_{\max}(\sqrt{t})}$	$C\left(\frac{D(y,t)}{\utilde{V_j(\sqrt{t})}} + \frac{1}{V_{\max}(\sqrt{t})}\right)$	$\frac{C}{V_j(\sqrt{t})}$	$\frac{C}{V_j(\sqrt{t})} e^{-b\frac{d^2}{t}}$
$x \in E_j$	$\lvert x\rvert > \sqrt{t}$	$\frac{C}{V_{\max}(\sqrt{t})} e^{-b\frac{d^2}{t}}$	$\frac{C}{V_{\max}(\sqrt{t})} e^{-b\frac{d^2}{t}}$	$\frac{C}{V_{\max}(\sqrt{t})} e^{-b\frac{d^2}{t}}$	$\frac{C}{V_{\max}(\sqrt{t})} e^{-b\frac{d^2}{t}}$	$C\left(\frac{D(y,t)}{\utilde{V_j(\sqrt{t})}} + \frac{1}{V_{\max}(\sqrt{t})}\right) e^{-b\frac{d^2}{t}}$	$\frac{C}{V_j(\sqrt{t})} e^{-b\frac{d^2}{t}}$	$\frac{C}{V_j(x,\sqrt{t})} e^{-b\frac{d^2}{t}}$

(ii): Assume that there exists at least one critical end.

Table of $(ii)_1$: The case that M_i, M_j are subcritical.

<table>
<tr><th colspan="2" rowspan="2"></th><th colspan="3">$y \in E_i$</th><th rowspan="2">$y \in K$</th><th colspan="3">$y \in E_j$</th></tr>
<tr><th>$|y| > \sqrt{t}$</th><th>$|y| \approx \sqrt{t}$</th><th>$|y| < \sqrt{t}$</th><th>$|y| < \sqrt{t}$</th><th>$|y| \approx \sqrt{t}$</th><th>$|y| > \sqrt{t}$</th></tr>
<tr><td rowspan="3">$x \in E_i$</td><td>$|x| > \sqrt{t}$</td><td>$\frac{C}{V_i(x,\sqrt{t})} e^{-b\frac{d^2}{t}}$</td><td>$\frac{C}{V_i(\sqrt{t})} e^{-b\frac{d^2}{t}}$</td><td>$C\left(\frac{D(y,t)}{V_i(\sqrt{t})} + \frac{\log t}{t}\right) e^{-b\frac{d^2}{t}}$</td><td></td><td></td><td></td><td rowspan="4">$C\frac{\log t}{t} e^{-b\frac{d^2}{t}}$</td></tr>
<tr><td>$|x| \approx \sqrt{t}$</td><td>$\frac{C}{V_i(\sqrt{t})} e^{-b\frac{d^2}{t}}$</td><td>$\frac{C}{V_i(\sqrt{t})}$</td><td>$C\left(\frac{D(y,t)}{V_i(\sqrt{t})} + \frac{\log t}{t}\right)$</td><td></td><td></td><td rowspan="3">$C\frac{\log t}{t}$</td></tr>
<tr><td>$|x| < \sqrt{t}$</td><td>$C\left(\frac{D(x,t)}{V_i(\sqrt{t})} + \frac{\log t}{t}\right) e^{-b\frac{d^2}{t}}$</td><td>$C\left(\frac{D(x,t)}{V_i(\sqrt{t})} + \frac{\log t}{t}\right)$</td><td>$C\left(\frac{D(x,t)D(y,t)}{V_i(\sqrt{t})} + \frac{1+[D(x,t)+D(y,t)]\log t}{t}\right)$</td><td>$\frac{C}{t}\left(1 + D(x,t)\log t\right)$</td><td>$\frac{C}{t}\{1 + [D(x,t)+D(y,t)]\log t\}$</td></tr>
<tr><td colspan="2">$x \in K$</td><td rowspan="4">$C\frac{\log t}{t} e^{-b\frac{d^2}{t}}$</td><td rowspan="3">$C\frac{\log t}{t}$</td><td>$\frac{C}{t}\left(1 + D(y,t)\log t\right)$</td><td>$\frac{C}{t}$</td><td>$\frac{C}{t}\left(1 + D(y,t)\log t\right)$</td></tr>
<tr><td rowspan="3">$x \in E_j$</td><td>$|x| < \sqrt{t}$</td><td>$\frac{C}{t}\{1 + [D(x,t)+D(y,t)]\log t\}$</td><td>$\frac{C}{t}\left(1 + D(x,t)\log t\right)$</td><td>$C\left(\frac{D(x,t)D(y,t)}{V_j(\sqrt{t})} + \frac{1+[D(x,t)+D(y,t)]\log t}{t}\right)$</td><td>$C\left(\frac{D(x,t)}{V_j(\sqrt{t})} + \frac{\log t}{t}\right)$</td><td>$C\left(\frac{D(x,t)}{V_j(\sqrt{t})} + \frac{\log t}{t}\right) e^{-b\frac{d^2}{t}}$</td></tr>
<tr><td>$|x| \approx \sqrt{t}$</td><td></td><td></td><td>$C\left(\frac{D(y,t)}{V_j(\sqrt{t})} + \frac{\log t}{t}\right)$</td><td>$\frac{C}{V_j(\sqrt{t})}$</td><td>$\frac{C}{V_j(\sqrt{t})} e^{-b\frac{d^2}{t}}$</td></tr>
<tr><td>$|x| > \sqrt{t}$</td><td></td><td></td><td></td><td>$C\left(\frac{D(y,t)}{V_j(\sqrt{t})} + \frac{\log t}{t}\right) e^{-b\frac{d^2}{t}}$</td><td>$\frac{C}{V_j(\sqrt{t})} e^{-b\frac{d^2}{t}}$</td><td>$\frac{C}{V_j(x,\sqrt{t})} e^{-b\frac{d^2}{t}}$</td></tr>
</table>

Table of $(ii)_2$: The case that M_i, M_j are critical.

<table>
<tr><td colspan="2" rowspan="2"></td><td colspan="3">$y \in E_i$</td><td rowspan="2">$y \in K$</td><td colspan="3">$y \in E_j$</td></tr>
<tr><td>$|y| > \sqrt{t}$</td><td>$|y| \approx \sqrt{t}$</td><td>$|y| < \sqrt{t}$</td><td>$|y| < \sqrt{t}$</td><td>$|y| \approx \sqrt{t}$</td><td>$|y| > \sqrt{t}$</td></tr>
<tr><td rowspan="3">$x \in E_i$</td><td>$|x| > \sqrt{t}$</td><td>$\frac{C}{V_i(x,\sqrt{t})}e^{-b\frac{d^2}{t}}$</td><td colspan="3">$\frac{C}{t}e^{-b\frac{d^2}{t}}$</td><td>$C\frac{1+\left(\log\sqrt{t}-\log\|y\|\right)}{t\log t}e^{-b\frac{d^2}{t}}$</td><td>$\frac{C}{t\log t}e^{-b\frac{d^2}{t}}$</td><td>$\frac{C}{t}\left(\frac{1}{\log\|x\|}+\frac{1}{\log\|y\|}\right)e^{-b\frac{d^2}{t}}$</td></tr>
<tr><td>$|x| \approx \sqrt{t}$</td><td rowspan="3">$\frac{C}{t}e^{-b\frac{d^2}{t}}$</td><td colspan="3" rowspan="2"></td><td>$C\frac{1+\left(\log\sqrt{t}-\log\|y\|\right)}{t\log t}$</td><td>$\frac{C}{t\log t}$</td><td>$\frac{C}{t\log t}e^{-b\frac{d^2}{t}}$</td></tr>
<tr><td>$|x| < \sqrt{t}$</td><td>$C\frac{\log t+\log^2\sqrt{t}-\log\|x\|\log\|y\|}{t\log^2 t}$</td><td>$C\frac{1+\left(\log\sqrt{t}-\log\|x\|\right)}{t\log t}$</td><td>$C\frac{1+\left(\log\sqrt{t}-\log\|x\|\right)}{t\log t}e^{-b\frac{d^2}{t}}$</td></tr>
<tr><td colspan="2">$x \in K$</td><td colspan="2"></td><td colspan="3" rowspan="3">$\frac{C}{t}$</td><td rowspan="3">$\frac{C}{t}e^{-b\frac{d^2}{t}}$</td></tr>
<tr><td rowspan="3">$x \in E_j$</td><td>$|x| < \sqrt{t}$</td><td>$C\frac{1+\left(\log\sqrt{t}-\log\|x\|\right)}{t\log t}e^{-b\frac{d^2}{t}}$</td><td>$C\frac{1+\left(\log\sqrt{t}-\log\|x\|\right)}{t\log t}$</td><td>$C\frac{\log t+\log^2\sqrt{t}-\log\|x\|\log\|y\|}{t\log^2 t}$</td></tr>
<tr><td>$|x| \approx \sqrt{t}$</td><td>$\frac{C}{t\log t}e^{-b\frac{d^2}{t}}$</td><td>$\frac{C}{t\log t}$</td><td>$C\frac{1+\left(\log\sqrt{t}-\log\|y\|\right)}{t\log t}$</td></tr>
<tr><td>$|x| > \sqrt{t}$</td><td>$\frac{C}{t}\left(\frac{1}{\log\|x\|}+\frac{1}{\log\|y\|}\right)e^{-b\frac{d^2}{t}}$</td><td>$\frac{C}{t\log t}e^{-b\frac{d^2}{t}}$</td><td>$C\frac{1+\left(\log\sqrt{t}-\log\|y\|\right)}{t\log t}e^{-b\frac{d^2}{t}}$</td><td colspan="3">$\frac{C}{t}e^{-b\frac{d^2}{t}}$</td><td>$\frac{C}{V_j(x,\sqrt{t})}e^{-b\frac{d^2}{t}}$</td></tr>
</table>

Table $(ii)_3$: The case that M_i is subcritical and M_j is critical.

		$y \in E_i$			$y \in K$	$y \in E_j$		
		$\lvert y\rvert > \sqrt{t}$	$\lvert y\rvert \approx \sqrt{t}$	$\lvert y\rvert < \sqrt{t}$		$\lvert y\rvert < \sqrt{t}$	$\lvert y\rvert \approx \sqrt{t}$	$\lvert y\rvert > \sqrt{t}$
$x \in E_i$	$\lvert x\rvert > \sqrt{t}$	$\frac{C}{V_i(x,\sqrt{t})} e^{-b\frac{d^2}{t}}$	$\frac{C}{V_i(\sqrt{t})} e^{-b\frac{d^2}{t}}$	$C\left(\frac{D(y,t)}{V_i(\sqrt{t})} + \frac{\log t}{t}\right) e^{-b\frac{d^2}{t}}$	$C\frac{\log t}{t} e^{-b\frac{d^2}{t}}$	$\frac{C}{t}\left(1 + \log\frac{e\sqrt{t}}{\lvert y\rvert}\right) e^{-b\frac{d^2}{t}}$		
	$\lvert x\rvert \approx \sqrt{t}$	$\frac{C}{V_i(\sqrt{t})} e^{-b\frac{d^2}{t}}$	$\frac{C}{V_i(\sqrt{t})}$	$C\left(\frac{D(y,t)}{V_i(\sqrt{t})} + \frac{\log t}{t}\right)$	$C\frac{\log t}{t}$	$\frac{C}{t}\left(1 + \log\frac{e\sqrt{t}}{\lvert y\rvert}\right)$		
	$\lvert x\rvert < \sqrt{t}$	$C\left(\frac{D(x,t)}{V_i(\sqrt{t})} + \frac{\log t}{t}\right) e^{-b\frac{d^2}{t}}$	$C\left(\frac{D(x,t)}{V_i(\sqrt{t})} + \frac{\log t}{t}\right)$	$C\left(\frac{D(x,t)D(y,t)}{V_i(\sqrt{t})} + \frac{1+[D(x,t)+D(y,t)]\log t}{t}\right)$	$\frac{C}{t}\left(1 + D(x,t)\log t\right)$	$\frac{C}{t}\left(1 + D(x,t)\log\frac{e\sqrt{t}}{\lvert y\rvert}\right)$		$\frac{C}{t} e^{-b\frac{d^2}{t}}$
$x \in K$		$C\frac{\log t}{t} e^{-b\frac{d^2}{t}}$	$C\frac{\log t}{t}$	$\frac{C}{t}\left(1 + D(y,t)\log t\right)$				
$x \in E_j$	$\lvert x\rvert < \sqrt{t}$	$\frac{C}{t}\left(1 + \log\frac{e\sqrt{t}}{\lvert x\rvert}\right) e^{-b\frac{d^2}{t}}$	$\frac{C}{t}\left(1 + \log\frac{e\sqrt{t}}{\lvert x\rvert}\right)$	$\frac{C}{t}\left(1 + D(y,t)\log\frac{e\sqrt{t}}{\lvert x\rvert}\right)$		$\frac{C}{t}$		
	$\lvert x\rvert \approx \sqrt{t}$							
	$\lvert x\rvert > \sqrt{t}$			$\frac{C}{t} e^{-b\frac{d^2}{t}}$				$\frac{C}{V_j(x,\sqrt{t})} e^{-b\frac{d^2}{t}}$