INTEGRAL MAXIMUM PRINCIPLE AND ITS APPLICATIONS

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Abstract. The integral maximum principle for the heat equation on a Riemannian manifold is improved and applied to obtain estimates of double integrals of the heat kernel.

1. Introduction and main results

In the present paper we develop a general approach to some estimates of solutions to heat equation bases on so-called *integral maximum principle* . Suppose that M is a smooth connected complete non-compact Riemannian manifold and consider some precompact subregion $\Omega \subset M$. Suppose also that $u(x,t)$ is a (weak) solution to Dirichlet mixed boundary value problem in a cylinder $\Omega \times (0,T)$:

$$u_t - \Delta u = 0, \quad u|_{\partial\Omega\times(0,T)} = 0 \tag{1.1}$$

As it follows from the maximum principle, the function

$$\sup_{x\in\Omega} |u(x,t)|$$

is decreasing in t. Moreover, it is also well-known, the following integral

$$\int_\Omega u^2(x,t)dx$$

is a decreasing function of t too. This fact can be regarded as an integral version of the usual maximum principle.

There is a further development of this idea which has been applied in a series of works to obtain heat kernel estimates (see , for example, [1] , [3] , [7] , [5]) and consists of the fact that some weighted integral of u^2 decreases in t. Namely, this is applicable to the integral

$$I(t) = \int_\Omega u^2(x,t)e^{\xi}dx \tag{1.2}$$

provided the function $\xi(x,t)$ is locally Lipschitz and satisfies the relation

$$\xi_t + \frac{1}{2}|\nabla\xi|^2 \leq 0. \tag{1.3}$$

The simplest non-trivial examples of such functions ξ are as follows:

$$\xi = \frac{d(x)^2}{2t}$$

$d(x)$ being a locally Lipschitz function such that $|\nabla d(x)| \leq 1$ (for instance, a distance function from a set) and

$$\xi = \alpha d(x) - \frac{\alpha^2}{2}t$$

α being an arbitrary constant.

The following improvement of the maximum principle is proved in Section 2 below.

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Theorem 1 *(Integral maximum principle) Suppose that $u(x,t)$ is a (weak) solution to the mixed problem (1.1) and a locally Lipschitz function ξ satisfies the relation (1.3) in $\Omega \times (0,T)$, then the function*

$$I(t)\exp(2\lambda_1(\Omega)t)$$

is decreasing in $t \in (0,T)$ where $I(t)$ is defined by (1.2) and $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of Ω.

B.Davies [4] proved the following universal integral bound for the heat kernel $p(x,y,t)$ being the smallest positive fundamental solution to the heat equation (for details of the definition of the heat kernel see [2]). Let A and B be two Borel sets in M with finite volumes and let the distance $R = \operatorname{dist}(A,B)$ be positive, then

$$\int_A \int_B p(x,y,t)dxdy \le \sqrt{\mu A \mu B}\exp\left(-\frac{R^2}{4t}\right) \ . \tag{1.4}$$

This estimate is of much importance due to its generality: no a priori geometric assumption are needed for (1.4) to be valid. It turns out that Davies's estimate can be deduced with ease from the integral maximum principle. Moreover, Theorem 1 implies the improved version of (1.4) :

Theorem 2 *Let A,B be two Borel subsets in M of a finite volume and $R = \operatorname{dist}(A,B)$, then*

$$\int_A \int_B p(x,y,t)dxdy \le \sqrt{\mu A \mu B}\exp\left(-\frac{R^2}{4t} - \lambda_1(M)t\right) \ . \tag{1.5}$$

Here $\lambda_1(M)$ is by definition the bottom of the spectrum of the Laplacian in $L^2(M)$ that is called *the spectral radius* and coincides with $\inf \lambda_1(\Omega)$ over all precompact subregions Ω.

If the spectral radius of a manifold is positive then the estimate of theorem 2 gives the sharp speed of decay of heat kernel as $t \to \infty$ because as it is known

$$\lim_{t\to\infty} \frac{\log p(x,y,t)}{t} = -\lambda_1(M) \ .$$

Takeda [8] proved by a probabilistic method another kind of double integral estimate of heat kernel. Let A be an arbitrary compact of a positive volume on M and let us denote by A^R the open R-neighbourhood of A where $R > 0$. Let X_t be Brownian motion on manifold M governed by heat equation (1.1) . We shall consider the un-normalised law $\mathbf{P}_A$ of X_t under the condition that the initial point X_0 is uniformly distributed in A, where "un-normalised" means that the maximum value of $\mathbf{P}_A$ is equal to μA rather than to 1. Takeda's inequality for this setting estimates the probability $P(R,T)$ for X_t to exit A^R by a time T starting at a point of A that is the function

$$P(R,T) \equiv \mathbf{P}_A\left(\exists t \le T : X_t \notin A^R \middle| X_0 \in A\right) .$$

The following sharpened version of Takeda's inequality is due to T.Lyons [6]

$$P(R,T) \le 16\mu A^R \int_R^\infty \frac{1}{(4\pi T)^{\frac{1}{2}}} \exp\left(-\frac{\eta^2}{4T}\right)d\eta \tag{1.6}$$

In Section 3 we obtain by means of the integral maximum principle an analytic proof of a similar inequality which however doesn't cover (1.6) but sometimes is sharper.

Theorem 3 *Let $u(x,t)$ be a smooth subsolution to the heat equation in the cylinder $A^R \times [0,T]$ (where $A \subset M$ is a compact and R,T are arbitrary positive numbers) i.e.*

$$u_t - \Delta u \leq 0$$

and suppose that

$$0 \leq u(x,t) \leq 1 \quad and \quad u(x,0) = 0 \quad \forall x \in A^R, t \in [0,T], \tag{1.7}$$

then

$$\int_A u^2(x,T)dx \leq \mu\big(A^R \setminus A\big) \max(\frac{R^2}{2T}, \frac{2T}{R^2}) \exp(-\frac{R^2}{2T} + 1) \tag{1.8}$$

To explain connection of this theorem with inequality (1.6) we first mention that the following function

$$u(x,t) \equiv \mathbf{P}(\exists \tau \leq t : X_\tau \notin A^R | X_0 = x)$$

(where $\mathbf{P}$ denotes a probability measure) satisfies the heat equation in the cylinder in question and the conditions (1.7) . Thus, Theorem 3 is applicable to this function. Noting that the function $P(R,T)$ is equal to $\int_A u(x,T)dx$ and applying Cauchy-Schwarz inequality we get from (1.8)

$$P(R,T) \leq \sqrt{\mu(A)\mu(A^R \setminus A)} \max\left(\frac{R}{\sqrt{2T}}, \frac{\sqrt{2T}}{R} \right) \exp(-\frac{R^2}{4T} + \frac{1}{2}) \ . \tag{1.9}$$

Compare this inequality to that of (1.6) . It is easy to check that for all R,T the following estimate is valid

$$\int_R^\infty \exp(-\frac{\eta^2}{4T}) d\eta \leq \frac{2T}{R} \exp(-\frac{R^2}{4T})$$

and, moreover, the ratio of the left and the right sides here tends to 1 as $\frac{R^2}{T} \to \infty$.

Therefore, (1.6) implies that

$$P(R,T) \leq \frac{16}{\sqrt{\pi}} \mu(A^R) \frac{\sqrt{T}}{R} \exp(-\frac{R^2}{4T}) \tag{1.10}$$

and for large $\frac{R^2}{T}$ this inequality is only a bit weaker than (1.6) . On the other hand for $\frac{R^2}{2T} \geq 1$ (1.9) implies

$$P(R,T) \leq \sqrt{\frac{e}{2}} \sqrt{\mu(A)\mu(A^R \setminus A)} \frac{R}{\sqrt{T}} \exp(-\frac{R^2}{4T}) \tag{1.11}$$

or, applying $\sqrt{ab} \leq (a+b)/2$,

$$P(R,T) \leq \sqrt{\frac{e}{8}} \mu(A^R) \frac{R}{\sqrt{T}} \exp(-\frac{R^2}{4T}) \ . \tag{1.12}$$

Obviously, (1.10) is better for large $\frac{R^2}{T}$, but for intermediate values of $\frac{R^2}{T}$ (1.11) and (1.12) may give a sharper estimate, than (1.10) and (1.6) especially when the volume $\mu(A^R \setminus A)$ is small.

The inequalities (1.6) and (1.9) imply some estimates of heat kernel. It is obvious from a probabilistic point of view that $P(R,t)$ is an upper bound of the following integral of heat kernel

$$\int_A \int_{M\setminus A^R} p(x,y,t)dydx.$$

This can be explained from analytic point of view too. Indeed, applying theorem 3 to the function

$$v(x,t) = \int_{M\setminus A^R} p(x,y,t)dy$$

we obtain as above

$$\int_A \int_{M\setminus A^R} p(x,y,t)dydx = \int_A v(x,t)dx$$

$$\leq \sqrt{\mu(A)\mu(A^R\setminus A)}\max\left(\frac{R}{\sqrt{2T}},\frac{\sqrt{2T}}{R}\right)\exp(-\frac{R^2}{4T}+\frac{1}{2}).$$

2. Proof of theorems 1 and 2

To prove Theorem 1 consider a time derivative $I'(t)$ of the function

$$I(t) = \int_\Omega u^2(x,t)e^{\xi(x,t)}dx \tag{2.1}$$

Applying the equation (1.1) and the boundary condition (in a weak sense if the boundary of Ω is not smooth) we have

$$I'(t) = \int_\Omega \xi_t e^\xi u^2 + \int_\Omega 2ue^\xi u_t = \int_\Omega \xi_t e^\xi u^2 - \int_\Omega \left(\nabla\left(2ue^\xi\right),\nabla u\right)$$

(here we are applying inequality (1.3))

$$\leq -\frac{1}{2}\int_\Omega |\nabla\xi|^2 e^\xi u^2 - 2\int_\Omega(\nabla u,\nabla\xi)ue^\xi - 2\int_\Omega |\nabla u|^2 e^\xi$$

$$= -\frac{1}{2}\int_\Omega e^\xi\left(|\nabla\xi|^2 u^2 + 4(\nabla u,\nabla\xi)u + 4|\nabla u|^2\right) = -\frac{1}{2}\int_\Omega e^\xi\left(u\nabla\xi+2\nabla u\right)^2$$

On the other hand

$$\nabla(e^{\frac{\xi}{2}}u) = \frac{1}{2}e^{\frac{\xi}{2}}(u\nabla\xi+2\nabla u),$$

$$\left|\nabla(e^{\frac{\xi}{2}}u)\right|^2 = \frac{1}{4}e^\xi(u\nabla\xi+2\nabla u)^2$$

which implies the inequality

$$I'(t) \leq -2\int_\Omega \left|\nabla(e^{\frac{\xi}{2}}u)\right|^2 \tag{2.2}$$

We are left to notice that the function $v = e^{\xi/2}u$ as any other function vanishing on the boundary $\partial\Omega$ satisfies the relation

$$\int_\Omega |\nabla v|^2 \geq \lambda_1(\Omega)\int_\Omega v^2$$

Substituting into (2.2) we obtain a differential inequality

$$I'(t) \leq -2\lambda_1(\Omega)I(t)$$

whence the decreasing of $I(t)e^{2\lambda_1(\Omega)t}$ follows. □

Remark. One may replace u^2 in the statement of Theorem 1 by another power or function of the solution. Let $f(z)$ be a smooth function on $(0, +\infty)$ such that

$$f(z) > 0, \quad f(z)' > 0, \quad f(z)'' > 0$$

and

$$\kappa = \inf_{z>0} \frac{f''f}{f'^2} > 0$$

Suppose also that the function ξ satisfies the relation

$$\xi_t + \frac{|\nabla\xi|^2}{4\kappa} \leq 0$$

Then the interal

$$I_f(t) = \int_\Omega f(u(x,t))e^\xi dx$$

is a decreasing function of t.

For example, if $f(z) = z^p,\ p > 1$ then $\kappa = \frac{p-1}{p}$. Of course it would be interesting to specify a speed of decay of $I_f(t)$ as was done for $I(t)$ but the spectral radius seems not to suit this purpose.

Proof of theorem 2. It suffices to prove the theorem for the case when A and B are bounded sets. Indeed, if we have proved that, a general case will be reduced to that as follows. Consider a bounded region Ω then, by the hypothesis, we have the inequality

$$\int_{A\cap\Omega}\int_{B\cap\Omega} p(x,y,t)dxdy \leq \sqrt{\mu A \mu B}\exp\left(-\frac{R^2}{4t} - \lambda_1(M)t\right)$$

Letting $\Omega \to M$ we obtain (1.5) .

To prove theorem 2 for bounded sets A, B let us consider the distance function $d(x)$ from set A and put $\xi(x,t) = \alpha d(x) - \frac{\alpha^2}{2}t$ where constant $\alpha > 0$ is to be chosen below. Since $|\nabla d| \leq 1$ it follows that ξ satisfies the relation (1.3) . Let Ω be a large region containing both sets A, B and $p_\Omega(x,y,t)$ be a heat kernel in region Ω (with a vanishing Dirichlet boundary condition). Let us apply theorem 1 in region Ω to the function

$$u(x,t) = \int_A p_\Omega(x,y,t)dy$$

being a solution to Dirichlet mixed value problem in Ω with an initial value $u(x,0)=\mathbf{1}_A$. We have by theorem 1 that for any $t>0$ $I(t)\le \exp(-2\lambda_1(\Omega)t)I(0)$. Note that

$$I(0)=\int_\Omega \mathbf{1}_A^2 \exp(\alpha d(x))dx=\int_A \exp(\alpha d(x))dx=\mu A$$

for $d(x)|_A=0$. Therefore, we obtain

$$\int_\Omega u^2(x,t)exp(\alpha d(x)-\frac{\alpha^2}{2}t)dx\le \exp(-2\lambda_1(\Omega)t)\mu A$$

Reducing the domain of integration to B and taking into account that $d(x)|_B\ge \mathrm{dist}(A,B)=R$ we have

$$\int_B u^2(x,t)dx\le \exp\left(-\alpha R+\frac{\alpha^2}{2}t-2\lambda_1(\Omega)t\right)\mu A.$$

Finally, applying Cauchy-Schwarz inequality

$$\int_B\int_A p_\Omega(x,y,t)dydx=\int_B u(x,t)dx\le \left(\int_B u^2(x,t)dx\right)^{\frac{1}{2}}\sqrt{\mu B}$$

$$\le \exp\left(-\frac{\alpha}{2}R+\frac{\alpha^2}{4}t-\lambda_1(\Omega)t\right)\sqrt{\mu A\mu B}\ .$$

Taking here the optimal value $\alpha=\frac{R}{t}$ we obtain

$$\int_B\int_A p_\Omega(x,y,t)dydx\le \exp\left(-\frac{R^2}{4t}-\lambda_1(\Omega)t\right)\sqrt{\mu A\mu B}$$

We are left to let here $\Omega\to M$ and to mention that $\lambda_1(\Omega)\ge\lambda_1(M)$ and $p_\Omega\to p$ locally uniformly. $\square$

3. Proof of theorem 3

The proof of theorem 3 doesn't use theorem 1 directly. We shall apply the idea behind the proof of integral maximum principle in another situation. The proof will be split into three steps.

STEP 1. Let us denote by $d(x)$ the distance from x to the set A and consider a cut-off function $\varphi(x)$ such that

$$\varphi|_A=1,\quad \mathrm{supp}\varphi\subset A^R,$$

then the function $\eta\equiv(1-\delta)\xi$ satisfies the following inequality:

$$\int_{A^R} u^2e^{\eta(x,T)}\varphi^2(x)dx\le\frac{2}{\delta}\int_0^T\int_{A^R\setminus A}|\nabla\varphi|^2\,e^{\eta(x,t)}dx\,dt \tag{3.1}$$

where $\delta\in(0,1)$ is arbitrary.

Proof. We have

$$\frac{d}{dt}\int_{A^R} u^2 e^{\eta(x,t)}\varphi^2 = 2\int_{A^R} uu_t e^{\eta}\varphi^2 + \int_{A^R} u^2\eta_t e^{\eta}\varphi^2$$

$$\leq 2\int_{A^R} u\Delta u e^{\eta}\varphi^2 + \int_{A^R} u^2\eta_t e^{\eta}\varphi^2$$

$$= -2\int_{A^R} |\nabla u|^2 e^{\eta}\varphi^2 - 2\int_{A^R} u(\nabla u,\nabla\eta)e^{\eta}\varphi^2 - 4\int_{A^R} ue^{\eta}(\nabla u,\nabla\varphi)\varphi + \int_{A^R} u^2\eta_t e^{\eta}\varphi^2.$$

Applying the inequality

$$-2u(\nabla u,\nabla\varphi)\varphi \leq \delta^{-1}u^2|\nabla\varphi|^2 + \delta\varphi^2|\nabla u|^2$$

we get

$$\frac{d}{dt}\int_{A^R} u^2 e^{\eta}\varphi^2 \leq \frac{2}{\delta}\int_{A^R} u^2|\nabla\varphi|^2 e^{\eta}$$

$$-2\int_{A^R} e^{\eta}\varphi^2\Big((1-\delta)|\nabla u|^2 - u|\nabla u||\nabla\eta| - \frac{1}{2}\eta_t u^2\Big) \quad .$$

The expression in brackets on the right hand side of this inequality is equal to

$$(1-\delta)X^2 - |\nabla\eta| XY - \frac{1}{2}\eta_t Y^2$$

where $X = |\nabla u|$, $Y = u$. This quadratic polynomial of X, Y is non-negative if its discriminant is non-positive, i.e.

$$|\nabla\eta|^2 + 2(1-\delta)\eta_t \leq 0$$

which is true due to (1.3) . Recalling that $0 \leq u \leq 1$ we have

$$\frac{d}{dt}\int_{A^R} u^2 e^{\eta}\varphi^2 \leq \frac{2}{\delta}\int_{A^R} u^2|\nabla\varphi|^2 e^{\eta} \leq \frac{2}{\delta}\int_{A^R}|\nabla\varphi|^2 e^{\eta}.$$

Integrating this inequality with respect to t and taking into account that $|\nabla\varphi| = 0$ outside $A^R \setminus A$ we obtain (3.1) .

STEP 2. Let us prove the following estimate

$$\int_A u^2(x,T)dx \leq \frac{2\delta^{-1}\mu(A^R\setminus A)}{\left(\int_0^R\Big(\int_0^T \exp\Big((1-\delta)(\zeta(\rho,\tau)-\zeta(0,T))\Big)\,d\tau\Big)^{-\frac{1}{2}}d\rho\right)^2} \tag{3.2}$$

$\zeta(\rho,\tau)$ being a Lipschitz function in $[0,R]\times[0,T]$ satisfying in the following relation

$$\zeta_\tau + \frac{1}{2}\zeta_\rho^2 \leq 0. \tag{3.3}$$

Let us consider the function

$$\xi(x,t) = \zeta(d(x),t),$$

and apply (3.1) (note, that this function satisfies the condition (1.3)). Since for all $x \in A$ $\xi(x,T) = \zeta(0,T) \equiv$ const we can get from (3.1)

$$\int_A u^2(x,T)dx \le \frac{2}{\delta}\int_0^T \int_{A^R\setminus A} |\nabla\varphi|^2 e^{(1-\delta)(\xi(x,t)-\zeta(0,T))}dx\,dt. \tag{3.4}$$

Let us introduce a function

$$\omega(r) \equiv \int_0^T e^{(1-\delta)(\zeta(r,t)-\zeta(0,T))}dt$$

and suppose that φ depends on $d(x)$ only (i.e. we denote further by φ a function on $(0,R)$), then we can rewrite (3.4) as follows

$$\int_A u^2(x,T)dx \le \frac{2}{\delta}\int_0^R \varphi'(r)^2\omega(r)dV(r)$$

where

$$V(r) \equiv \mu(A^r \setminus A).$$

Optimizing the last integral over all Lipschitz functions φ on $(0,R)$ under conditions $\varphi(0) = 1,\ \varphi(R) = 0$ we obtain

$$\int_A u^2(x,T)dx \le \frac{2}{\delta}\left(\int_0^R \frac{dr}{V'(r)\omega(r)}\right)^{-1}. \tag{3.5}$$

Since

$$\int_0^R V'(r)dr\int_0^R \frac{dr}{V'(r)\omega(r)} \ge \left(\int_0^R \frac{dr}{\sqrt{\omega(r)}}\right)^2$$

we get from (3.5) an inequality

$$\int_A u^2(x,T)dx \le \frac{2}{\delta}V(R)\left(\int_0^R \frac{dr}{\sqrt{\omega(r)}}\right)^{-2},$$

which implies (3.2) .

STEP 3. Here we shall complete the proof choosing a suitable function ζ. If we take in (3.2) $\zeta \equiv 0$ and $\delta = 1$ we obtain a more or less trivial estimate

$$\int_A u^2(x,T)dx \le \mu(A^R \setminus A)\frac{2T}{R^2}. \tag{3.6}$$

Applying (3.2) to a function $\zeta(\rho,\tau) = -\alpha\rho - \frac{1}{2}\alpha^2\tau$ with $\alpha = \frac{R}{T}$ we get after integrating

$$\int_A u^2(x,T)dx \le \frac{1-\delta}{\delta}\frac{\mu(A^R\setminus A)}{e^{(1-\delta)\frac{R^2}{2T}} - 1}. \tag{3.7}$$

If $\frac{R^2}{2T} \geq 1$ then one can take $\delta = \frac{2T}{R^2}$ and (3.7) gives the following

$$\int_A u^2(x,T)dx \leq \left(\frac{R^2}{2T} - 1\right) \frac{\mu(A^R \setminus A)}{\exp\left(\frac{R^2}{2T} - 1\right) - 1}$$

or, applying the inequality

$$\frac{X-1}{e^{X-1}-1} \leq \frac{X}{e^{X-1}}$$

which is valid for all $X \geq 1$, we finally get that for $\frac{R^2}{2T} \geq 1$

$$\int_A u^2(x,T)dx \leq \mu(A^R \setminus A)\frac{R^2}{2T}\exp(-\frac{R^2}{2T}+1). \tag{3.8}$$

The desired estimate (1.8) follows from (3.8) and (3.6) immediately. □

Remark. One could expect that the spectral radius may be put into the estimate of theorem 3 like in theorems 1, 2 but this is not so. Indeed, the integral

$$\int_A u^2(x,T)dx$$

may tend to μA as $T \to \infty$ if u has a boundary value equal to 1 (on the contrary to the case of theorem 1 where a solution under consideration vanishes on a boundary). Hence, any upper bound of this integral cannot contain a term $\exp(-\lambda_1(M)T)$ vanishing as $T \to \infty$.

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REFERENCES

[1] **Aronson D.G.**, Non-negative solutions of linear parabolic equations, *Ann. Scuola Norm. Sup. Pisa. Cl. Sci. (4)* , **22** (1968) 607-694. Addendum **25** (1971) 221-228.

[2] **Chavel I.**, "Eigenvalues in Riemannian geometry" Academic Press, New York, 1984.

[3] **Cheng S.Y., Li P., Yau S.-T.**, On the upper estimate of the heat kernel of a complete Riemannian manifold, *Amer. J. Math.* , **103** (1981) no.5, 1021-1063.

[4] **Davies E.B.**, Heat kernel bounds, conservation of probability and the Feller property, *J. d'Analyse Math.* , **58** (1992) 99-119.

[5] **Grigor'yan A.A.**, On the fundamental solution of the heat equation on an arbitrary Riemannian manifol, (in Russian) *Mat. Zametki* , **41** (1987) no.3, 687-692. Engl. transl. *Math. Notes* , **41** (1987) no.5-6, 386-389.

[6] **Lyons T.**, Random thoughts on reversible potential theory, *in:* "Summer School in Potential Theory, Joensuu 1990" , edited by Ilpo Laine, University of Joensuu. Publications in Sciences 26, ISBN 951-696-837-6., 71-114.

[7] **Porper F.O., Eidel'man S.D.**, Two-side estimates of fundamental solutions of second-order parabolic equations and some applications, (in Russian) *Uspechi Mat. Nauk* , **39** (1984) no.3, 101-156. Engl. transl. *Russian Math. Surveys* , **39** (1984) no.3, 119-178.

[8] **Takeda M.**, On a martingale method for symmetric diffusion process and its applications, *Osaka J. Math* , **26** (1989) 605-623.