Tail estimates of the heat kernel for jump processes

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August 2022

Contents

1 Stable-like estimates of the heat kernel

Let (M, d) be a locally compact separable metric space and μ be a Radon measure with full support on M. Let $(\mathcal{E}, \mathcal{F})$ be a regular jump type Dirichlet form with a symmetric jump kernel $J(x, y)$, that is,

$$\mathcal{E}(f, f) = \iint_{M \times M} (f(x) - f(y))^2 J(x, y) d\mu(x) d\mu(y).$$

Let $\{P_t\}$ denote the associated heat semigroup, that is, $P_t = e^{t\mathcal{L}}$, where $\mathcal{L}$ is the generator of $(\mathcal{E}, \mathcal{F})$, and $p_t(x, y)$ be the heat kernel, that is, the integral kernel of P_t, should it exist.

For example, if $M = \mathbb{R}^n$ and

$$J(x, y) = \frac{c}{|x - y|^{n+\beta}}$$

where $0 < \beta < 2$, then $\mathcal{L} = -(-\Delta)^{\beta/2}$ (that generates a symmetric stable process of index β), and

$$p_t(x, y) \simeq \frac{1}{t^{n/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(n+\beta)}. \tag{1}$$

We aim at similar estimates in a general metric measure space M.

Denote by $B(x,r)$ open metric balls in M. We assume always that μ is α-regular for some $\alpha > 0$, that is, for all $x \in M$ and $r > 0$,

$$\mu\left(B(x,r)\right) \simeq r^{\alpha}. \tag{V}$$

By a result of AG and T.Kumagai (2008), if the heat kernel satisfies a self-similar estimate

$$p_t(x,y) \simeq t^{-\gamma}\Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right)$$

for some $\beta, \gamma > 0$ and some function Φ then it is necessarily the following estimate:

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{2}$$

We refer to (2) as a *stable-like* estimate of the heat kernel because of its similarity to (1).

A natural question arises: what conditions on the jump kernel ensure (2)?

Z.-Q. Chen and T.Kumagai proved in *Stoch.Process.Appl.* **108** (2003) that if $\beta < 2$ then (2) is equivalent to the condition

$$J(x,y) \simeq d(x,y)^{-(\alpha+\beta)} \quad \text{for all } x,y \in M. \tag{J}$$

However, on most of fractal sets there exist regular Dirichlet forms with the jump kernel satisfying (J) with $\beta > 2$. In this case one needs one more condition: a *generalized capacity* condition denoted shortly by ($Gcap$) that will be explained below.

Condition ($Gcap$) is closely related to the *cutoff Sobolev inequality* introduced by M.Barlow and R.Bass in $Trans.AMS$ **356** (2004), and to the *energy inequality* of S.Andres and M.Barlow in $J.Reine\ Angew.Math.$ **699** (2015).

With help of this condition, the following result was proved in AG, E.Hu, J.Hu, $Adv.Math.$ **330** (2018) and in a more general setting – in Z.-Q.Chen, T.Kumagai, J.Wang, $Adv.Math.$ **374** (2020).

Theorem 1 *Under the standing assumption* (V) *we have, for any* $\beta > 0$,

$$(Gcap) + (J) \Leftrightarrow (2), \tag{3}$$

The main purpose of the present work is to obtain off-diagonal upper bounds of the heat kernel under *weaker* hypothesis about $J(x, y)$: using integral rather than pointwise estimates of $J(x, y)$.

2 Condition $(Gcap)$

Let us now state $(Gcap)$. Recall that the *capacity* associated with $(\mathcal{E}, \mathcal{F})$ is defined as follows: for any open set $U \subset M$ and a Borel set $A \subset U$ set

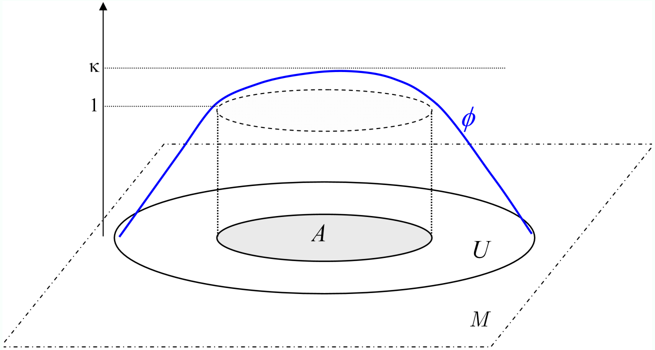
$$\operatorname{cap}(A, U) = \inf \left\{ \mathcal{E}(\phi, \phi) : \phi \in \mathcal{F},\ 0 \le \phi \le 1,\ \phi|_A = 1,\ \phi|_{U^c} = 0 \right\}.$$

Definition. For any bounded function $u \in \mathcal{F} + \mathrm{const}$ and a real $\kappa \ge 1$, define the *generalized capacity* of the pair (A, U) by

$$\operatorname{cap}_u^{(\kappa)}(A, U) = \inf_{\phi} \mathcal{E}\left(u^2\phi, \phi\right)$$

where inf is taken over all $\phi \in \mathcal{F}$ such that

$$0 \le \phi \le \kappa, \qquad \phi|_A \ge 1, \qquad \phi|_{U^c} = 0.$$



For example, if $\kappa = 1$ and $u \equiv 1$ then

$$\operatorname{cap}_u^{(\kappa)}(A, U) = \operatorname{cap}(A, U).$$

Definition. We say that the *generalized capacity* condition ($Gcap$) is satisfied if there exist two constants $\kappa \geq 1, C > 0$ such that, for any bounded function $u \in \mathcal{F} + \text{const}$ and for all concentric balls $B_0 := B(x, R)$, $B := B(x, R + r)$ with $x \in M$ and $R, r > 0$,

$$\operatorname{cap}_u^{(\kappa)}(B_0, B) \leq \frac{C}{r^\beta} \int_B u^2 d\mu. \tag{$Gcap$}$$

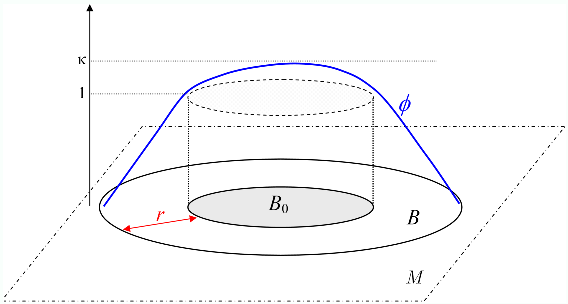
Equivalently, this condition means that, for any pair of concentric balls B_0, B as above

and for any bounded $u \in \mathcal{F} + \text{const}$,

there exists $\phi \in \mathcal{F}$ such that

$$0 \leq \phi \leq \kappa, \quad \phi|_{B_0} \geq 1, \quad \phi|_{B^c} = 0$$

and



$$\mathcal{E}\left(u^2\phi, \phi\right) \leq \frac{C}{r^\beta} \int_B u^2 d\mu. \tag{4}$$

Applying ($Gcap$) with $u \equiv 1$ and replacing ϕ with $\phi \wedge 1$, we obtain the *capacity* condition:

$$\operatorname{cap}(B_0, B) \leq \frac{C}{r^\beta} \mu(B). \tag{cap}$$

It would ideal if in all our results ($Gcap$) could be replaced by a much simpler condition (cap), but so far there is no technique for that.

Usually it is very difficult to verify ($Gcap$). However, there are two cases when ($Gcap$) is trivially satisfied. In the both cases, we assume that all balls are precompact.

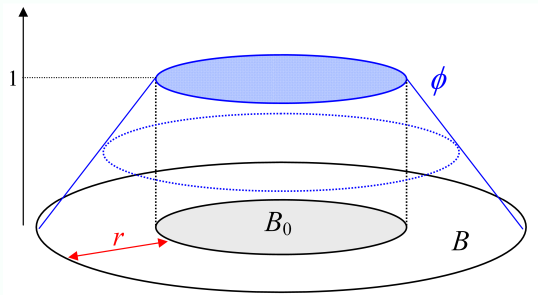
Case 1. If $\beta < 2$ and if the jump kernel satisfies the upper bound in (J), that is,

$$J(x,y) \le \frac{C}{d(x,y)^{\alpha+\beta}} \qquad (J_{\le})$$

then ($Gcap$) holds with $\kappa = 1$.

Indeed, in this case (4) is satisfied for a bump function ϕ of (B_0, B) which follows from

$$|\phi(x) - \phi(y)| \le \frac{d(x,y)}{r}.$$



However, on most fractal spaces there exist regular jump type Dirichlet forms satisfying (J) with $\beta > 2$. Besides, in our main results $J(x,y)$ does not have to satisfy ($J_{\le}$).

Case 2. Let (M, d) be an *ultra-metric* space, that is, d satisfies the ultra-metric triangle inequality

$$d(x, y) \leq \max\left(d(x, z), d(y, z)\right)$$

(for example, a field $\mathbb{Q}_p$ of p-adic numbers with p-adic distance is an ultra-metric space). Assume that the jump kernel satisfies a *tail estimate*

$$\int_{B^c(x,r)} J(x, y) d\mu(y) \leq \frac{C}{r^\beta}, \tag{TJ_1}$$

for all $x \in M$ and $r > 0$, where $\beta > 0$ is any. Then $(Gcap)$ is satisfied with $\kappa = 1$, because (4) is satisfied for $\phi = \mathbf{1}_B$.

In our main results, the jump kernels will always satisfy (TJ_1). Hence, if M is an ultra-metric space then the hypothesis $(Gcap)$ can be dropped.

For general metric spaces, it is an interesting open problem to understand which properties of d and J imply $(Gcap)$ for a given $\beta > 0$. This problem is open even for a simpler condition (cap).

3 Upper bounds of the heat kernel

Let us first ask when the following upper bound holds:

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}, \tag{UE}$$

for all $t > 0$ and almost all $x, y \in M$. If (UE) is satisfied then, using the identity,

$$J(x,y) = \lim_{t\to 0} \frac{p_t(x,y)}{2t}, \tag{5}$$

we obtain

$$J(x,y) \le \frac{C}{d(x,y)^{\alpha+\beta}}. \tag{$J_\le$}$$

For the opposite implication $(J_\le) \Rightarrow (UE)$ we need additional conditions.

Definition. We say that a *Faber-Krahn inequality* (FK) holds if, for any precompact open set $\Omega \subset M$,

$$\lambda_1(\Omega) \ge c\mu(\Omega)^{-\beta/\alpha}, \tag{FK}$$

where $\lambda_1(\Omega) = \inf \operatorname{spec}\left(-\mathcal{L}^\Omega\right)$.

Or, equivalently, (FK) holds if, for any $\varphi \in \mathcal{F} \cap C_0(\Omega) \setminus \{0\}$,

$$\frac{\mathcal{E}(\varphi,\varphi)}{\|\varphi\|_{L^2}^2} \geq c\mu(\Omega)^{-\beta/\alpha}.$$

It is known that (FK) is equivalent to the *diagonal upper estimate* of the heat kernel

$$\boxed{p_t(x,y) \leq Ct^{-\alpha/\beta}}. \tag{DUE}$$

It is also known that

$$J(x,y) \geq \frac{c}{d(x,y)^{\alpha+\beta}} \quad \Rightarrow \quad (FK). \tag{6}$$

Denote by (C) the hypothesis that $(\mathcal{E},\mathcal{F})$ is *conservative*, that is, $P_t 1 = 1$.

The following theorem can be extracted from the results of AG, J.Hu, K.-S.Lau K.-S. *Trans.AMS* **366** (2014) and Z.-Q.Chen, T.Kumagai, J.Wang, *Mem.AMS* **271** (2021).

Theorem 2 *Under the hypothesis (V) we have*

$$(FK) + (Gcap) + (J_{\leq}) \Leftrightarrow (UE) + (C).$$

Or, if we take (FK), $(Gcap)$ and (C) as standing assumptions, then

$$(J_{\leq}) \Leftrightarrow (UE).$$

4 Main results

Now we impose on $J(x,y)$ a weaker hypothesis than a pointwise upper bound ($J_{\leq}$). Fix some $\beta > 0$, $q \in [1,\infty]$ and consider the following hypothesis for the *tail* of for J:

$$\boxed{\|J(x,\cdot)\|_{L^q(B^c(x,r))} \leq \frac{C}{r^{\alpha/q'+\beta}}}, \tag{TJ_q}$$

for all $x \in M$ and $r > 0$, where $q' = \frac{q}{q-1}$ is the Hölder conjugate of q. The abbreviation TJ means "*Tail of J*". It is easy to see that (TJ_q) becomes stronger when q increases.

For example, if $q = 1$ then $q' = \infty$ and (TJ_q) becomes

$$\int_{B^c(x,r)} J(x,y)d\mu(y) \leq \frac{C}{r^{\beta}}. \tag{TJ_1}$$

In the case $q = 2$ (TJ_q) becomes

$$\left(\int_{B^c(x,r)} J^2(x,y)d\mu(y)\right)^{1/2} \leq \frac{C}{r^{\alpha/2+\beta}}, \tag{TJ_2}$$

and, in the case $q=\infty$, (TJ_q) becomes

$$\operatorname*{essup}_{y\in B^c(x,r)} J(x,y) \le \frac{C}{r^{\alpha+\beta}}, \tag{TJ_∞}$$

which is equivalent to $(J_\le)$.

Consider similar hypotheses about the tail of the heat kernel $p_t(x,y)$:

$$\boxed{\left\|p_t\left(x,\cdot\right)\right\|_{L^q(B^c(x,r))} \le \frac{C}{t^{\alpha/(q'\beta)}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}} \simeq \frac{1}{t^{\alpha/(q'\beta)}} \wedge \frac{t}{r^{\alpha/q'+\beta}}, \tag{TP_q}$$

for all $x\in M$ and $r>0$. The abbreviation TP means "*Tail of P*".

The condition (TP_q) gets stronger when q increases. By (5), we have

$$(TP_q) \Rightarrow (TJ_q). \tag{7}$$

For $q=1$, (TP_q) is equivalent to

$$\int_{B^c(x,r)} p_t(x,y)d\mu(y) \le C\frac{t}{r^\beta}, \tag{TP_1}$$

for $q = 2$, (TP_q) is equivalent to

$$\int_{B^c(x,r)} p_t^2(x,y)d\mu(y) \leq \frac{C}{t^{\alpha/(2\beta)}} \left(1 + \frac{r}{t^{1/\beta}}\right)^{-(\alpha/2+\beta)}, \tag{TP_2}$$

and, for $q = \infty$, (TP_q) is equivalent to

$$\operatorname*{esssup}_{y \in B^c(x,r)} p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{r}{t^{1/\beta}}\right)^{-(\alpha+\beta)}, \tag{TP_∞}$$

which coincides with (UE).

Finally, consider the following family of off-diagonal *Upper Estimates* of the heat kernel:

$$\boxed{p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}}, \tag{UE_q}$$

for all $t > 0$ and almost all $x, y \in M$. For example, for $q = \infty$ we have

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}, \tag{UE_∞}$$

which coincides with (UE) and (TP_∞).

For $q = 1$ we have a weaker estimate

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta}. \tag{UE_1}$$

Now we can state our main result.

Theorem 3 *Under the standing assumption* (V), *we have for any* $q \in [2, \infty]$ *the following equivalence/implication:*

$$(FK) + (Gcap) + (TJ_q) \Leftrightarrow (TP_q) + (C) \Rightarrow (UE_q).$$

Or, considering (FK), $(Gcap)$, (C) *as standing assumptions, we have*

$$\boxed{(TJ_q) \Leftrightarrow (TP_q) \Rightarrow (UE_q)}.$$

The case $q = \infty$ coincides with Theorem 2 while the case $q < \infty$ is completely new. The case $q \in [1, 2)$ is not covered but some of the implications are true for all $q \ge 1$.

If $q = 1$ and M is an ultra-metric space then (UE_1) can be obtained without using (TP_1):

$$(FK) + (TJ_1) \Rightarrow (UE_1),$$

which was proved in A.Bendikov, AG, E.Hu, J.Hu, *Ann. Scuola Norm. Sup. Pisa* **22** (2021) (($Gcap$) in this setting is true automatically). That paper contains also an example of a jump kernel satisfying

$$J(x,y) \geq \frac{c}{d(x,y)^{\alpha+\beta}} \tag{$J_\geq$}$$

(which implies (FK)) and

$$\int_{B^c(x,r)} J(x,y)d\mu(y) \leq \frac{C}{r^\beta},$$

(that is, (TJ_1)), and the upper bound (UE_1) is optimal in the sense that the exponent $-\beta$ cannot be replaced by $-(\beta+\varepsilon)$ for any $\varepsilon > 0$.

In the general case, replacing (FK) with a stronger condition ($J_\geq$) allows to obtain also a lower bound of the heat kernel.

Theorem 4 *Under the standing assumption (V), we have for any $q \in [2,\infty]$*

$$(J_\geq) + (Gcap) + (TJ_q) \Leftrightarrow (TP_q) + (LE) \Rightarrow (UE_q) + (LE),$$

where (LE) is the Lower Estimate

$$p_t(x,y) \geq \frac{c}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{LE}$$

5 Outline of the proof

We describe here the most important steps in the proof of the main implication

$$(FK) + (Gcap) + (TJ_q) \Rightarrow (TP_q).$$

Step 0. As it was already mentioned,

$$(FK) \Rightarrow (DUE).$$

However, in our proof we do not use this implication because we work in a more general setting of doubling spaces where this result is unavailable. We use an alternative proof of (DUE) with help of the mean value inequality.

Step 1. We prove that, for any $q \in [1, \infty]$,

$$(FK) + (Gcap) + (TJ_q) \Rightarrow (PMV_q)$$

where (PMV_q) stands for the *Parabolic Mean Value* inequality that means the following.

Fix in M a ball $B = B(x, R)$ and set $T = R^\beta$. Let u be a bounded non-negative function on $M \times (0, T]$ that is *subcaloric* in the cylinder $B \times (0, T]$:

that is, for any $t \in (0,T]$,

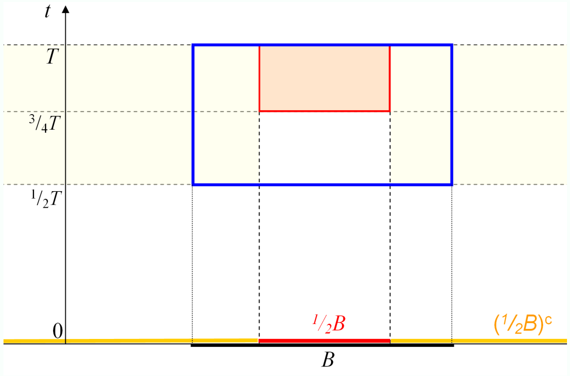
$$u(\cdot,t) \in \mathcal{F}_+ \cap L^\infty(M)$$

and u satisfies in $B \times (0,T]$

$$\partial_t u - \mathcal{L}u \le 0$$

in a certain weak sense.

Then, for any $\varepsilon \in (0,1]$,

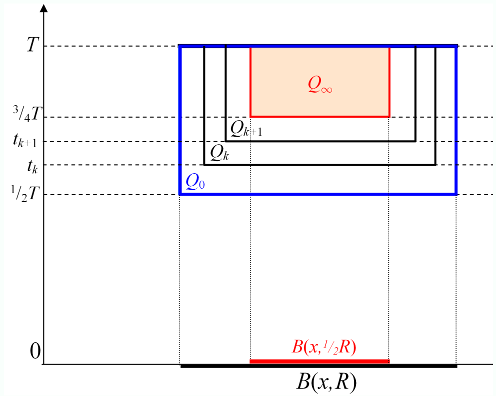


$$\sup_{t\in[\frac{3}{4}T,T]} \|u(\cdot,t)\|_{L^\infty(\frac{1}{2}B)} \le \left(\frac{C}{\varepsilon^{1/\beta}R}\right)^{\frac{\alpha+\beta}{2}} \|u\|_{L^2(B\times[\frac{1}{2}T,T])} + \frac{\varepsilon}{R^{\alpha/q'}} \sup_{t\in[\frac{1}{2}T,T]} \|u(\cdot,t)\|_{L^{q'}((\frac{1}{2}B)^c)}. \tag{PMV_q}$$

For that, we consider a shrinking sequence of cylinders $Q_k = B(x,r_k) \times [t_k,T]$, $k \ge 0$, an increasing sequence $b_k > 0$, a sequence

$$a_k := \int_{Q_k} (u-b_k)_+^2 \, d\mu dt$$

that clearly decreases, and prove that



$$a_{k+1} \leq \frac{C}{(b_{k+1}-b_k)^{2\frac{\beta}{\alpha}}} \left(\frac{r_k}{r_k - r_{k+1}} \right)^C \left(\frac{1}{(r_k - r_{k+1})^\beta} + \frac{1}{t_{k+1} - t_k} + \frac{s_k}{b_{k+1} - b_k} \right)^{1+\frac{\beta}{\alpha}} a_k^{1+\frac{\beta}{\alpha}},$$

where

$$s_k = \sup_{t \in [t_k, T]} \operatorname*{essup}_{z \in B(x, \frac{r_k + r_{k+1}}{2})} \int_{B^c(x, r_k)} u(y,t) J(z,y) d\mu(y).$$

The proof uses essentially (FK) and $(Gcap)$.

Choose

$$r_k = (\tfrac{1}{2} + 2^{-k-1})R \quad \text{and} \quad t_k = (\tfrac{3}{4} - 2^{-\beta k - 2})T,$$

so that

$$B \times [\tfrac{1}{2}T, T] = Q_0 \supset Q_k \supset Q_\infty = \tfrac{1}{2}B \times [\tfrac{3}{4}T, T].$$

Setting also $b_k = \left(1 - 2^{-k}\right) b$ for some $b > 0$, we obtain

$$a_{k+1} \leq C 2^{Ck} \left(1 + \frac{R^\beta s_k}{b} \right)^{1+\frac{\beta}{\alpha}} \frac{a_k^{1+\frac{\beta}{\alpha}}}{(R^{\alpha+\beta} b^2)^{\frac{\beta}{\alpha}}}. \tag{8}$$

Iterating (8), we show that, for a large enough b,

$$\lim_{k\to\infty} a_k = 0,$$

which implies that

$$u \leq b \ \text{ in } \ Q_\infty.$$

The choice of b depends on $\sup_k \frac{a_k}{R^{\alpha+\beta}} = \frac{a_o}{R^{\alpha+\beta}}$ and on an upper bound for $R^\beta s_k$. The value

$$\frac{a_0}{R^{\alpha+\beta}} = \frac{1}{R^{\alpha+\beta}} \left\| u \right\|^2_{L^2\left(B\times[\frac{1}{2}T,T]\right)}$$

yields the first term (PMV_q). Estimating s_k by means of the Hölder inequality and (TP_q) gives

$$\begin{aligned} R^\beta s_k &\leq R^\beta \sup_{t\in[\frac{1}{2}T,T]} \left\| u\left(\cdot,t\right) \right\|_{L^{q'}(\frac{1}{2}B)^c)} \frac{C}{\left(r_k - r_{k+1}\right)^{\alpha/q'+\beta}} \\ &= \frac{C2^{Ck}}{R^{\alpha/q'}} \sup_{t\in[\frac{1}{2}T,T]} \left\| u\left(\cdot,t\right) \right\|_{L^{q'}(\frac{1}{2}B)^c)} \end{aligned}$$

which yields the second term in (PMV_q).

Step 2. We prove that

$$(PMV_2) \Rightarrow (DUE).$$

For that apply (PMV_2) with $u(\cdot,t) = P_t f$ where $f \in C_0(M)$ and $f \geq 0$, and observe that the both terms in the right hand side of (PMV_q) are bounded by $\frac{C}{R^{a/2}} \|f\|_{L^2}$ which yields

$$\|P_T f\|_\infty \leq \frac{C}{T^{\alpha/(2\beta)}} \|f\|_2 ,$$

which then implies (DUE). Consequently, we obtain that, for any $q \in [2,\infty]$,

$$(FK) + (Gcap) + (TJ_q) \Rightarrow (DUE).$$

It follows from (DUE) that

$$\|p_t(x,\cdot)\|_{L^q(M)} \leq \frac{C}{t^{\alpha/(q'\beta)}}.$$

Hence, in order to prove (TP_q), it remains to prove

$$\boxed{\|p_t(x,\cdot)\|_{L^q(B^c(x,r))} \leq \frac{Ct}{r^{\alpha/q'+\beta}}} \tag{9}$$

assuming that $r^\beta \geq t$, which is done in the rest of the proof.

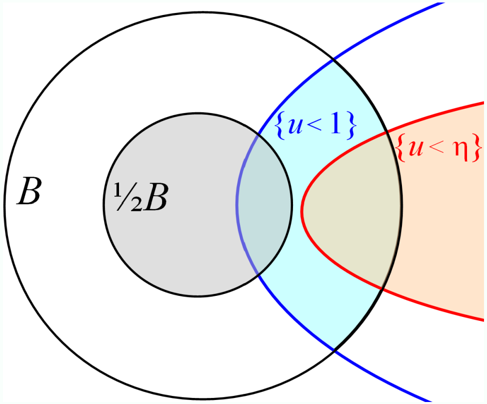
Step 3. We deduce from (PMV_1) a so called "Lemma of growth":

there exist some $\varepsilon, \eta \in (0,1)$ such that, for any ball $B \subset M$ and for any $u \in \mathcal{F}$ that is non-negative and bounded in M and superharmonic in B, if

$$\frac{\mu(B \cap \{u < 1\})}{\mu(B)} \leq \varepsilon,$$

then

$$\operatorname*{essinf}_{\frac{1}{2}B} u \geq \eta.$$



For that observe that $v = \frac{1}{u+a}$ is subharmonic for any $a > 0$. For subharmonic functions, we obtain from (PMV_1) the following multiplicative form of the mean value inequality (by choosing ε):

$$\|v\|_{L^\infty(\frac{1}{2}B)} \leq CA^\theta \max(A, T)^{1-\theta}, \tag{10}$$

where

$$A = \left(\fint_B v^2 d\mu\right)^{1/2}, \qquad T = \|v\|_{L^\infty((\frac{1}{2}B)^c)},$$

and $\theta = \theta(\alpha, \beta) \in (0,1)$.

Let us estimate A as follows:

$$A^2 = \frac{1}{\mu(B)} \left(\int_{B \cap u<1\}} + \int_{B \cap \{u \geq 1\}} \right) \frac{d\mu}{(u+a)^2}$$
$$\leq \frac{\mu(B \cap \{u<1\})}{\mu(B)} \frac{1}{a^2} + \frac{1}{(1+a)^2} \leq \frac{\varepsilon}{a^2} + \frac{1}{(1+a)^2} = \frac{2}{(1+a)^2},$$

for $a = \frac{1}{\varepsilon^{-1/2}-1}$. Estimating also trivially

$$\max(A, T) \leq \frac{1}{a},$$

we obtain from (10)

$$\operatorname*{essup}_{\frac{1}{2}B} \frac{1}{u+a} \leq C \left(\frac{2}{(1+a)^2} \right)^{\theta/2} \left(\frac{1}{a} \right)^{1-\theta} = \frac{C}{(1+a)^{\theta} a^{1-\theta}},$$

whence

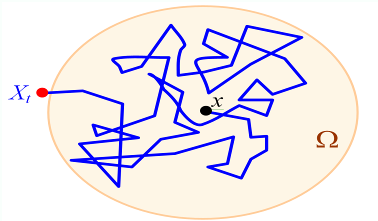
$$\operatorname*{essinf}_{\frac{1}{2}B} u \geq C^{-1} (1+a)^{\theta} a^{1-\theta} - a = a \left(C^{-1} \left(\tfrac{1}{a} + 1 \right)^{\theta} - 1 \right) =: \eta,$$

where $\eta > 0$ if a is small enough, that is, when ε is small enough.

Step 4. For any open set $\Omega \subset M$ and any $x \in \Omega$ set

$$E^{\Omega}(x) = \int_0^{\infty} P_t^{\Omega}\mathbf{1}(x)dt = \int_0^{\infty}\int_{\Omega} p_t^{\Omega}(x,y)d\mu(y)dt.$$

It has the probabilistic meaning of the *mean exit time* from Ω of the jump process X_t, associated with $(\mathcal{E},\mathcal{F})$, that starts at x: $E^{\Omega}(x) = \mathbb{E}_x(\tau^{\Omega})$, where τ^{Ω} is the first exit time from Ω.



In this step we prove that, under (FK), for any ball B of radius r,

$$\operatorname*{essup}_{B} E^{B} \le Cr^{\beta}. \tag{11}$$

Step 5. We prove the opposite inequality: the Lemma of growth and (cap) imply that

$$\operatorname*{essinf}_{\frac{1}{4}B} E^{B} \ge cr^{\beta}. \tag{12}$$

It is known that (11) and (12) imply (C).

Step 6. Using the upper and lower estimates of E^B, we deduce the *survival* inequality: there exist $\varepsilon > 0$ such that, for any ball B of radius r and for any $t > 0$,

$$P_t^B \mathbf{1}_B \geq \varepsilon - \frac{Ct}{r^\beta} \quad \text{in } \tfrac{1}{4}B. \tag{S}$$

In probabilistic terms,

$$P_t^B \mathbf{1}_B(x) = \mathbb{P}_x(\tau_B > t)$$

that is the probability of survival of the process in B up to time t assuming the killing condition in B^c.

Step 7. For any $\rho > 0$ consider a *truncated* Dirichlet form

$$\mathcal{E}^{(\rho)}(f,f) := \iint_{\{d(x,y)<\rho\}} (f(x) - f(y))^2 J(x,y) d(x) d\mu(y).$$

Denote by Q_t the heat semigroup of $\left(\mathcal{E}^{(\rho)}, \mathcal{F}\right)$ and by $q_t(x,y)$ its heat kernel. We prove that, under all the above hypotheses, the heat kernel of $\left(\mathcal{E}^{(\rho)}, \mathcal{F}\right)$ exists and satisfies the following diagonal upper bound

$$q_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \exp\left(\frac{Ct}{\rho^\beta}\right). \tag{13}$$

Step 8. We deduce from (S) a similar condition for the truncated semigroup Q_t:

$$Q_t^B \mathbf{1}_B \geq \varepsilon - Ct\left(r^{-\beta} + \rho^{-\beta}\right) \quad \text{in } \tfrac{1}{4}B$$

where $B = B(x,r)$. A certain iteration procedure allows to self-improve this estimate and to obtain that, for any $k \in \mathbb{N}$, if $r \geq 8k\rho$ then

$$Q_t^B \mathbf{1}_B \geq 1 - C(k)\left(\frac{t}{\rho^\beta}\right)^k,$$

which implies that

$$\int_{B^c(x,r)} q_t(x,y)d\mu(y) \leq C(k)\left(\frac{t}{\rho^\beta}\right)^k.$$

Combining this with (13), we obtain that, in the case $q < \infty$,

$$\|q_t(x,\cdot)\|_{L^q(B^c)} \leq \|q_t(x,\cdot)\|_{L^\infty(B^c)}^{1/q'} \|q_t(x,\cdot)\|_{L^1(B^c)}^{1/q} \leq \frac{C(k)}{t^{\alpha/(q'\beta)}} \exp\left(\frac{Ct}{\rho^\beta}\right)\left(\frac{t}{\rho^\beta}\right)^{\frac{k}{q}}. \quad (14)$$

In the case $q = \infty$ we improve (13) in a different way and obtain that if $r \geq 4k\rho$ then

$$\|q_t(x,\cdot)\|_{L^\infty(B^c)} \leq \frac{C(k)}{t^{\alpha/\beta}} \exp\left(\frac{Ct}{\rho^\beta}\right)\left(1 + \frac{\rho^\beta}{t}\right)^{\alpha/\beta}\left(\frac{t}{\rho^\beta}\right)^k. \quad (15)$$

Step 9. We prove that, under all the above conditions, including (TJ_q), we have, for any $t>0$ and for any ball $B=B(x,r)$,

$$\|p_t(x,\cdot)\|_{L^q(B^c)} \le \|q_t(x,\cdot)\|_{L^q(B^c)} + \frac{Ct}{\rho^{\alpha/q'+\beta}} \exp\left(\frac{Ct}{\rho^{\beta}}\right). \tag{16}$$

Step 10. In the case $q<\infty$, combining (14) and (16), we obtain that if $r \ge 8k\rho$ then

$$\|p_t(x,\cdot)\|_{L^q(B^c)} \le \frac{C(k)}{t^{\alpha/(q'\beta)}} \exp\left(\frac{Ct}{\rho^{\beta}}\right)\left(\frac{t}{\rho^{\beta}}\right)^{k/q} + \frac{Ct}{\rho^{\alpha/q'+\beta}} \exp\left(\frac{Ct}{\rho^{\beta}}\right).$$

Assuming that $r^{\beta} \ge t$ and setting $\rho = r/(8k)$, we obtain

$$\begin{aligned}\|p_t(x,\cdot)\|_{L^q(B^c)} &\le \frac{C(k)}{t^{\alpha/(q'\beta)}}\left(\frac{t}{r^{\beta}}\right)^{k/q} + \frac{C(k)t}{r^{\alpha/q'+\beta}} \\ &\le C\frac{t}{r^{\alpha/q'+\beta}},\end{aligned}$$

provided k is chosen so that

$$\left(\frac{t}{r^{\beta}}\right)^{k/q} \le \left(\frac{t}{r^{\beta}}\right)^{\frac{\alpha}{q'\beta}+1},$$

that is,

$$\frac{k}{q} \geq \frac{\alpha}{q'\beta} + 1.$$

This finishes the proof of (TP_q) if $q < \infty$.

In the case $q = \infty$ we obtain from (15) and (16), assuming that $r^\beta \geq t$ and setting $\rho = r/(4k)$ that

$$\begin{aligned} \|p_t(x,\cdot)\|_{L^q(B^c)} &\leq \frac{C(k)}{t^{\alpha/\beta}} \left(\frac{t}{r^\beta}\right)^{k-\frac{\alpha}{\beta}} + \frac{C(k)t}{r^{\alpha+\beta}} \\ &\leq C\frac{t}{r^{\alpha+\beta}}, \end{aligned}$$

provided k is chosen so that

$$\left(\frac{t}{r^\beta}\right)^{k-\frac{\alpha}{\beta}} \leq \left(\frac{t}{r^\beta}\right)^{\frac{\alpha}{\beta}+1}$$

that is,

$$k \geq 2\frac{\alpha}{\beta} + 1.$$

6 Consequences of (TP_q)

Let us first prove that if $q \in [2, \infty]$ then

$$\boxed{(TP_q) \Rightarrow (UE_q)}.$$

Setting $r = \frac{1}{2} d(x, y)$, we obtain by the semigroup property

$$\begin{aligned} p_{2t}(x, y) &= \int_M p_t(x, z)\, p_t(z, y)\, d\mu(z) \\ &\leq \left(\int_{B^c(x,r)} + \int_{B^c(y,r)} \right) p_t(x, z)\, p_t(z, y)\, d\mu(z). \end{aligned}$$

It suffices to estimate the first integral. By the Hölder inequality, we have

$$\int_{B^c(x,r)} p_t(x, z)\, p_t(z, y)\, d\mu(z) \leq \|p_t(x, \cdot)\|_{L^q(B^c(x,r))} \|p_t(\cdot, y)\|_{L^{q'}(M)}.$$

Since $q \geq 2$ and, hence, $q' \leq q$, we have not only (TP_q) but also $(TP_{q'})$. Hence,

$$\|p_t(x, \cdot)\|_{L^q(B^c(x,r))} \leq \frac{C}{t^{\alpha/(q'\beta)}} \left(1 + \frac{r}{t^{1/\beta}}\right)^{-(\alpha/q' + \beta)}$$

and

$$\left\| p_t\left(\cdot,y\right)\right\|_{L^{q'}(M)} \leq \frac{C}{t^{\alpha/(q\beta)}}.$$

Since $\frac{\alpha}{q'\beta}+\frac{\alpha}{q\beta}=\frac{\alpha}{\beta}$, we obtain

$$\int_{B^c(x,r)} p_t\left(x,z\right)p_t\left(z,y\right)d\mu\left(z\right) \leq \frac{C}{t^{\alpha/\beta}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}.$$

Estimating in the same manner the second integral, we obtain

$$p_{2t}\left(x,y\right) \leq \frac{C}{t^{\alpha/\beta}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)},$$

that is, (UE_q).

Since $(UE_q) \Rightarrow (DUE) \Rightarrow (FK)$, we obtain that

$$\boxed{(TP_q) \Rightarrow (FK)}.$$

The implication

$$\boxed{(TP_q) \Rightarrow (TJ_q)}$$

was already mentioned in (7).

Finally, the implication

$$\boxed{(TP_q) + (C) \Rightarrow (Gcap)}$$

is proved as follows. By (TP_q) we have also (TP_1), that is,

$$\int_{B^c(x,r)} p_t(x,y)d\mu(y) \le C\left(1 + \frac{r}{t^{1/\beta}}\right)^{-\beta} \le \frac{Ct}{r^\beta}.$$

This and (C) imply that

$$P_t^{B(x,r)}\mathbf{1}\,(x) \ge \varepsilon - \frac{Ct}{r^\beta}$$

that is, (S), and it is known that $(S) \Rightarrow (Gcap)$.

7 Appendix: $(Gcap)$ on ultra-metric spaces

Let us prove the following: if M is an ultra-metric space with compact balls and if J satisfies (TJ_1) that is,

$$\int_{B^c(x,\rho)} J(x,y)d\mu(y) \leq \frac{C}{\rho^\beta}$$

for some $\beta > 0$ and all $x \in M$ and $\rho > 0$, then $(Gcap)$ is satisfied. Indeed, given two concentric B_0 and B of radii R and $\rho = R + r$, it suffices to find a function $\phi \in \mathcal{F}$ such that

$$0 \leq \phi \leq 1, \quad \phi|_{B_0} = 1, \quad \phi|_{B^c} = 0$$

and

$$\mathcal{E}\left(u^2\phi, \phi\right) \leq \frac{C}{r^\beta} \int_B u^2 d\mu \tag{17}$$

for any $u \in \mathcal{F} + \text{const}$. A key point is that on ultra-metric space the indicator functions of balls belong to $\mathcal{F}$ so that we take

$$\phi = \mathbf{1}_B$$

(it is a consequence of the fact that in any ball B of radius r, the distance between any point inside B and any point outside B is at least r). With this ϕ we have

$$
\begin{aligned}
\mathcal{E}\left(u^2\phi,\phi\right) &= \iint_{M\times M}\left(u^2\varphi(x)-u^2\varphi(y)\right)(\varphi(x)-\varphi(y))\,J(x,y)d\mu(x)d\mu(y)\\
&= 2\int_{x\in B}\int_{y\in B^c}\left(u^2\varphi(x)-u^2\varphi(y)\right)(\varphi(x)-\varphi(y))\,J(x,y)d\mu(x)d\mu(y)\\
&= \int_{x\in B}\int_{y\in B^c} u^2(x)J(x,y)d\mu(x)d\mu(y)\\
&= \int_{x\in B} u^2(x)\left(\int_{B(x,\rho)^c} J(x,y)d\mu(y)\right)d\mu(x)\\
&\le \frac{C}{\rho^\beta}\int_B u^2 d\mu,
\end{aligned}
$$

whence (17) follows. We have used here that in an ultra-metric ball B of radius ρ any point is its center, that is, $B=B(x,\rho)$ for any $x\in B$.