# Tail estimates of heat kernels for jump processes

Alexander Grigor'yan
University of Bielefeld

Workshop on Heat Kernel and Related Topics in Hangzhou 2024
Zhejiang University of Technology

Based on a joint work with Eryan Hu and Jiaxin Hu

# Non-local heat kernels in $\mathbb{R}^n$

Consider in $\mathbb{R}^n$ a fractional Laplace operator $\mathcal{L} = (-\Delta)^{\beta/2}$ (that is a non-negative definite self-adjoint operator in $L^2(\mathbb{R}^n)$). If $\beta \in (0, 2)$ then this operator is the generator of the symmetric stable Levy process of index $\beta$. Denote by $p_t(x, y)$ the *heat kernel* of $\mathcal{L}$ that is the fundamental solution of the associated heat equation $\partial_t u = -\mathcal{L}u$ and, at the same time, the transition density of the Levy process. The heat kernel of $\mathcal{L}$ admits the estimate

$$p_t(x, y) \simeq \frac{1}{t^{n/\beta}} \left(1 + \frac{|x-y|}{t^{1/\beta}}\right)^{-(n+\beta)}, \tag{1}$$

where $A \simeq B$ means that $c_1 B \le A \le c_2 B$ for some positive constants $c_1, c_2$.

The operator $\mathcal{L} = (-\Delta)^{\beta/2}$ is the generator of the non-local Dirichlet form $(\mathcal{E}, \mathcal{F})$ where

$$\mathcal{E}(f, g) = C_n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))(g(x) - g(y))}{|x-y|^{n+\beta}} dxdy,$$

for all $f, g \in \mathcal{F} = B_{2,2}^{\beta/2}(\mathbb{R}^n)$. Recall that a Dirichlet form is related to its generator $\mathcal{L}$ by

$$\mathcal{E}(f, g) = (\mathcal{L}f, g)_{L^2} \text{ for all } f \in \operatorname{dom}(\mathcal{L}) \text{ and } g \in \mathcal{F}.$$

# Jump type Dirichlet forms on metric measure spaces

Let $(M, d)$ be a locally compact separable metric space and $\mu$ be a Radon measure with full support on $M$. Let $(\mathcal{E}, \mathcal{F})$ be a *regular jump type Dirichlet form*, where $\mathcal{F}$ is a dense subspace of $L^2(M, \mu)$ and $\mathcal{E}$ is a bilinear form on $\mathcal{F}$ given by

$$\mathcal{E}(f, g) = \iint_{M\times M} (f(x) - f(y))(g(x) - g(y)) J(x, y) d\mu(x) d\mu(y), \quad \forall f, g \in \mathcal{F}.$$

$J(x, y)$ is a *jump kernel*, that is, a non-negative measurable symmetric function on $M \times M$.

Let $\mathcal{L}$ be the (non-negative definite) generator of $(\mathcal{E}, \mathcal{F})$. For any $t \geq 0$, set $P_t = e^{-t\mathcal{L}}$ so that $\{P_t\}_{t\geq 0}$ is the *heat semigroup* of $(\mathcal{E}, \mathcal{F})$.

If, for any $t > 0$, the operator $P_t$ is an integral operator with the integral kernel $p_t(x, y)$,

$$P_t f(x) = \int_M p_t(x, y) f(y) d\mu(y) \ \text{ for all } f \in L^2(M, \mu),$$

then $p_t(x, y)$ is referred to as *the heat kernel* of $(\mathcal{E}, \mathcal{F})$.

Major problem: *obtaining upper and lower bounds of* $p_t(x, y)$ depending on the geometry of the underlying space and on $J$.

## Two-sides estimates of the heat kernel

Denote by $B(x,r)$ open metric balls in $M$. In this talk we always assume that $\mu$ is $\alpha$-regular for some $\alpha > 0$, that is, for all $x \in M$ and $r > 0$,

$$\mu\left(B(x,r)\right) \simeq r^{\alpha} \tag{V}$$

(although the main results are available also in the setting of a doubling measure).

By a result of AG and T.Kumagai (2008), if the heat kernel of a jump type Dirichlet form satisfies a self-similar estimate

$$p_t(x,y) \simeq \frac{1}{t^{\gamma}} \Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right)$$

for some $\beta, \gamma > 0$ and decreasing function $\Phi$, then it is necessarily the following estimate:

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{2}$$

We refer to (2) as a *stable-like* estimate of the heat kernel because it matches (1) with $\alpha = n$. The number $\beta$ is called the *index* of the corresponding Dirichlet form.

If (2) holds then, using the identity $J(x,y) = \lim_{t\to 0} \frac{1}{2t} p_t(x,y)$, we obtain that

$$J(x,y) \simeq d(x,y)^{-(\alpha+\beta)}. \tag{J}$$

Z.-Q. Chen and T.Kumagai (2003) proved that if $\beta < 2$ then, in fact,

$$(J) \Leftrightarrow (2).$$

However, on many fractal sets there exist regular Dirichlet forms with the heat kernel satisfying (2) with $\beta > 2$. Indeed, by works of M.Barlow, J.Kigami, et al., on large families of p.c.f. fractals and Sierpinski carpets, there are diffusion processes with heat kernels satisfying the *sub-Gaussian* estimate with the *walk dimension* $d_w > 2$. Using a subordination techniques, one obtains a jump process satisfying (2) for any index $\beta < d_w$. Clearly, $\beta$ can be $> 2$.

To obtain the implication $(J) \Rightarrow (2)$ in the case $\beta > 2$ in general, one has to assume one more hypothesis: a *generalized capacity* condition ($Gcap$) that will be explained below. This condition is closely related to the *cutoff Sobolev inequality* introduced by M.Barlow and R.Bass (2004), and to the *energy inequality* of S.Andres and M.Barlow (2015).

It was proved by AG, E.Hu, J.Hu (2018) and Chen, Kumagai, Wang (2020) that, for any $\beta > 0$,

$$(Gcap) + (J) \Leftrightarrow (2). \tag{3}$$

## Upper bounds of the heat kernel

The main question to be discussed here is how to obtain the estimates of the form

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-\gamma}, \tag{4}$$

with some $\beta, \gamma > 0$. Here necessarily $\gamma \le \alpha+\beta$ because otherwise (4) implies $J \equiv 0$. If $\gamma = \alpha+\beta$ then the necessary condition for (4) is

$$J(x,y) \le \frac{C}{d(x,y)^{\alpha+\beta}}. \tag{$J_\le$}$$

The case $\gamma < \alpha+\beta$ will be of main interest. In this case (4) does not imply any useful bound for $J$.

Alongside (4), consider for any $q \in [1,\infty]$ the following *tail estimate* of the heat kernel:

$$\boxed{\|p_t(x,\cdot)\|_{L^q(B^c(x,r))} \le \frac{C}{t^{\alpha/(q'\beta)}}\left(1+\frac{r}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}} \simeq \frac{1}{t^{\alpha/(q'\beta)}} \wedge \frac{t}{r^{\alpha/q'+\beta}}, \tag{$TP_q$}$$

for all $x \in M$ and $r > 0$, where $q' = \frac{q}{q-1}$. Condition ($TP_q$) gets stronger when $q$ increases.

If $q = 1$ then $q' = \infty$ and $(TP_q)$ is equivalent to

$$\int_{B^c(x,r)} p_t(x,y)d\mu(y) \le C\frac{t}{r^\beta}, \tag{$TP_1$}$$

if $q = 2$ then $q' = 2$ and $(TP_q)$ becomes

$$\left(\int_{B^c(x,r)} p_t^2(x,y)d\mu(y)\right)^{1/2} \le \frac{C}{t^{\alpha/(2\beta)}}\left(1 + \frac{r}{t^{1/\beta}}\right)^{-(\alpha/2+\beta)}, \tag{$TP_2$}$$

if $q = \infty$ then $q' = 1$ and $(TP_q)$ is becomes

$$\operatorname*{esssup}_{y\in B^c(x,r)} p_t(x,y) \le \frac{C}{t^{\alpha/\beta}}\left(1 + \frac{r}{t^{1/\beta}}\right)^{-(\alpha+\beta)}, \tag{$TP_\infty$}$$

which is equivalent to (4) with $\gamma = \alpha + \beta$.

**Lemma 1** *If $q \in [2, \infty]$ then $(TP_q)$ implies the following pointwise upper estimate*

$$\boxed{p_t(x,y) \le \frac{C}{t^{\alpha/\beta}}\left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}}, \tag{$UE_q$}$$

*for all $t > 0$ and $\mu$-almost all $x, y \in M$ (that is the estimate (4) with $\gamma = \alpha/q' + \beta$).*

Clearly, $(UE_q)$ gets stronger when $q$ increases. For example, if $q = 1$ then $(UE_q)$ becomes

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta}, \tag{$UE_1$}$$

if $q = 2$ then $(UE_q)$ becomes

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha/2+\beta)}, \tag{$UE_2$}$$

and if $q = \infty$ then $(UE_q)$ becomes

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}, \tag{$UE_\infty$}$$

which is equivalent to $(TP_\infty)$ and to (4) with $\gamma = \alpha + \beta$.

## Tail of jump kernel

In order to obtain the tail estimate $(TP_q)$ of the heat kernel, we will use the following hypothesis that is referred to as the *tail estimate of the jump kernel*:

$$\boxed{\|J(x,\cdot)\|_{L^q(B^c(x,r))} \le \frac{C}{r^{\alpha/q'+\beta}}}, \quad \forall x \in M,\ r > 0. \tag{$TJ_q$}$$

It is easy to verify that $(TJ_q)$ becomes stronger when $q$ increases.

For example, if $q = 1$ then $q' = \infty$ so that $(TJ_q)$ becomes

$$\int_{B^c(x,r)} J(x,y) d\mu(y) \le \frac{C}{r^\beta}, \tag{$TJ_1$}$$

If $q = 2$ then $q' = 2$ and $(TJ_q)$ becomes

$$\left( \int_{B^c(x,r)} J^2(x,y) d\mu(y) \right)^{1/2} \le \frac{C}{r^{\alpha/2+\beta}}. \tag{$TJ_2$}$$

If $q = \infty$ then $q' = 1$ and $(TJ_q)$ is equivalent to $(J_\le)$:

$$\operatorname*{essup}_{y \in B^c(x,r)} J(x,y) \le \frac{C}{r^{\alpha+\beta}}. \tag{$TJ_\infty$}$$

Since $J(x,y) = \lim_{t\to 0} \frac{1}{2t} p_t(x,y)$, we have the implication

$$(TP_q) \Rightarrow (TJ_q). \tag{5}$$

Our main result states that, under some additional hypotheses, also the converse implication holds, that is,

$$\text{(additional hypotheses)} + (TJ_q) \Rightarrow (TP_q).$$

## Additional hypothesis

We introduce here two hypotheses: the Faber-Krahn inequality and the generalized capacity inequality. They both can be stated for any regular Dirichlet form $(\mathcal{E}, \mathcal{F})$.

**Definition.** We say that the *Faber-Krahn inequality* $(FK)$ of index $\beta$ is satisfied for $(\mathcal{E}, \mathcal{F})$ if, for any precompact open set $\Omega \subset M$,

$$\lambda_1(\Omega) \geq c\mu(\Omega)^{-\beta/\alpha}, \tag{FK}$$

where $\lambda_1(\Omega) = \inf \operatorname{spec}(\mathcal{L}^\Omega)$ and $\mathcal{L}^\Omega$ is the generator of the restricted form $(\mathcal{E}, \mathcal{F}(\Omega))$.

Equivalently, $(FK)$ holds if, for any $\varphi \in \mathcal{F} \cap C_0(\Omega)$

$$\mathcal{E}(\varphi, \varphi) \geq c\mu(\Omega)^{-\beta/\alpha} \|\varphi\|_{L^2}^2.$$

It is known that $(FK)$ is equivalent to the *diagonal upper estimate* of the heat kernel

$$\boxed{p_t(x, y) \leq Ct^{-\alpha/\beta}}. \tag{DUE}$$

It is also known that

$$J(x, y) \geq \frac{c}{d(x, y)^{\alpha+\beta}} \quad \Rightarrow \quad (FK). \tag{6}$$

Hence, $(FK)$ can be regarded as an integral version of the lower bound of $J$.

Recall that the *capacity* associated with $(\mathcal{E}, \mathcal{F})$ is defined as follows: for any open set $U \subset M$ and a Borel set $A \subset U$ set

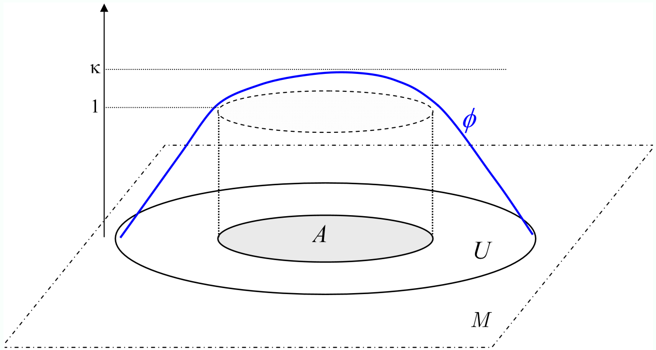
$$\operatorname{cap}(A, U) = \inf \left\{ \mathcal{E}(\phi, \phi) : \phi \in \mathcal{F},\ 0 \le \phi \le 1,\ \phi|_A = 1,\ \phi|_{U^c} = 0 \right\}.$$

**Definition.** For any bounded function $u \in \mathcal{F} + \text{const}$ and a real $\kappa \geq 1$, define the *generalized capacity* of the pair $(A, U)$ by

$$\operatorname{cap}_u^{(\kappa)}(A, U) = \inf_{\phi} \mathcal{E}(u^2 \phi, \phi),$$

where inf is taken over all $\phi \in \mathcal{F}$ such that

$$0 \le \phi \le \kappa, \quad \phi|_A \ge 1, \quad \phi|_{U^c} = 0.$$



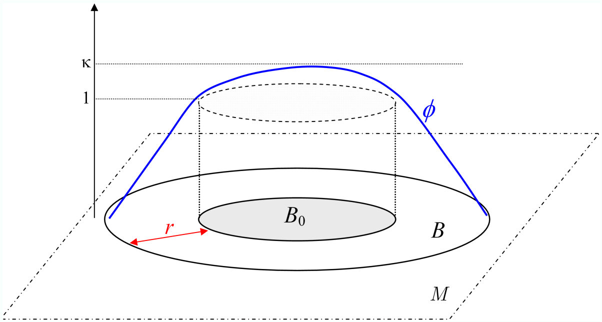

For example, if $\kappa = 1$ and $u \equiv 1$ then

$$\operatorname{cap}_u^{(\kappa)}(A, U) = \operatorname{cap}(A, U).$$

**Definition.** We say that the *generalized capacity condition* ($Gcap$) of index $\beta$ is satisfied for $(\mathcal{E},\mathcal{F})$ if there exist $\kappa\geq 1, C>0$ such that, for any bounded function $u\in\mathcal{F}+\mathrm{const}$ and for all concentric balls $B_0:=B(x,R)$, $B:=B(x,R+r)$ with $x\in M$ and $R,r>0$,

$$\mathrm{cap}_u^{(\kappa)}(B_0,B)\leq\frac{C}{r^\beta}\int_B u^2d\mu. \tag{$Gcap$}$$

Equivalently, this condition means that, for any pair of concentric balls $B_0, B$ as above and for any bounded $u\in\mathcal{F}+\mathrm{const}$, there exists $\phi\in\mathcal{F}$ such that




$$0\leq\phi\leq\kappa,\quad \phi|_{B_0}\geq 1,\quad \phi|_{B^c}=0$$

and the following inequality is true:

$$\mathcal{E}(u^2\phi,\phi)\leq\frac{C}{r^\beta}\int_B u^2d\mu. \tag{7}$$

Setting $u\equiv 1$ in (7) and replacing $\phi$ with $\phi\wedge 1$, we obtain the *capacity* condition:

$$\mathrm{cap}(B_0,B)\leq\frac{C}{r^\beta}\mu\left(B\right). \tag{$cap$}$$

Usually it is very difficult to verify ($Gcap$) (apart from some specific cases), and it is an open problem to develop methods for verification of ($Gcap$). In contrast to that, the capacity condition ($cap$) can be proved in many examples of interest.

**Conjecture.** *If in all our results ($Gcap$) can be replaced by ($cap$).*

# Main result

Denote by $(C)$ the hypothesis that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is conservative, that is, $P_t 1 \equiv 1$ for all $t > 0$.

**Theorem 2** (AG, E.Hu, J.Hu, 2024) *Let* $(V)$ *be satisfied. Then, for any* $q \in [1, \infty]$,

$$(FK) + (Gcap) + (TJ_q) \Leftrightarrow (TP_q) + (UE_q) + (C). \tag{8}$$

Recall for comparison that if $(\mathcal{E}, \mathcal{F})$ is a strongly local Dirichlet form then

$$(FK) + (Gcap) \Leftrightarrow \left\{ p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \exp\left( -c \left( \frac{d^\beta(x, y)}{t} \right)^{\frac{1}{\beta - 1}} \right) \right\} + (C),$$

by S.Andres and M.Barlow (2015) and AG, J.Hu, K.-S. Lau (2015).

In the present case of jump-type Dirichlet form, we add one more condition $(TJ_q)$ about the tail of $J$, and obtain both tail and pointwise estimates of the heat kernel.

If $q \geq 2$ then, by Lemma 1, $(TP_q) \Rightarrow (UE_q)$ so that Theorem 2 can be restated as follows:

$$(FK) + (Gcap) + (TJ_q) \Leftrightarrow (TP_q) + (C). \tag{9}$$

In fact, we have proved (9) for $q \in [2, \infty]$ in a more general setting when measure $\mu$ is doubling, and the scaling function $r^\beta$ is replaced by a general scaling function $W(x, r)$.

Recall some previously known related results. In the case $q = \infty$ we obtain from (8) that

$$(FK) + (Gcap) + (J_\leq) \Leftrightarrow (UE_\infty) + (C),$$

which is contained in the results of AG, J.Hu, K.-S.Lau (2014) and Z.-Q.Chen, T.Kumagai, J.Wang (2021).

Let $M$ be an ultra-metric space, that is, $d$ satisfies the ultra-metric triangle inequality

$$d(x, y) \leq \max(d(x, z), d(y, z)) \quad \forall x, y, z \in M.$$

For example, the field $\mathbb{Q}_p$ of $p$-adic numbers is an ultra-metric space with respect to the $p$-adic norm. Also, $\mathbb{Q}_p^n$ is an ultra-metric space with respect to max-product distance.

Let $q = 1$. It is known that in ultrametric spaces $(TJ_1) \Rightarrow (Gcap)$, and we obtain from (8) that

$$(FK) + (TJ_1) \Rightarrow (UE_1) \quad \text{that is, } p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-\beta},$$

which was previously proved by A.Bendikov, AG, E.Hu, J.Hu (2021).

# Approach to the proof of Theorem 2

## Strong generalized capacity condition and elliptic mean value inequality

In the definition of $(Gcap)$, in the inequality (7), that is, in

$$\mathcal{E}\left(u^2\phi, \phi\right) \leq \frac{C}{r^\beta}\int_B u^2 d\mu,$$

the cutoff function $\phi$ may depend on the weight $u$. Denote by $(Gcap')$ a stronger version of $(Gcap)$ when function $\phi$ depends only on the pair $B_0, B$ of the balls and serves all functions $u$ simultaneously.

The next theorem is the first one in a sequence of results leading to heat kernel upper bounds. We use again the following hypothesis about the tail of the jump kernel:

$$\boxed{\|J(x,\cdot)\|_{L^1(B^c(x,r))} \leq \frac{C}{r^\beta}}, \quad \forall x \in M, \, r > 0. \tag{TJ}$$

**Theorem 3** (AG, E.Hu, J.Hu, 2023) *Under the hypothesis $(V)$, we have the implication*

$$(FK) + (Gcap) + (TJ) \Rightarrow (Gcap').$$

The condition $(Gcap')$ will be used below for obtaining the *parabolic mean value inequality* that in turn is needed for heat kernel upper bounds. The proof of Theorem 3 uses the *elliptic mean value inequality* $(EMV)$.

**Definition.** We say that $(EMV)$ holds if, for any function $u \in \mathcal{F} \cap L^\infty$ that is non-negative and subharmonic in a ball $B = B(x_0, R)$, and for any $\varepsilon > 0$,

$$\operatorname*{esup}_{\frac{1}{2}B} u \leq C_\varepsilon \left( \fint_B u^2 \right)^{1/2} + \varepsilon \left\| u_+ \right\|_{L^\infty\left(\left(\frac{1}{2}B\right)^c\right)} .$$

The proof of Theorem 3 goes through the following implications (under the standing assumptions $(V)$, $(FK)$, $(TJ)$):

$$(Gcap) \Rightarrow (EMV) + (cap) \Rightarrow (Gcap') .$$

## Parabolic mean value inequality

**Theorem 4** (AG, E.Hu, J.Hu, 2023) *For any* $q\in[1,\infty]$, *we have*

$$(FK)+(Gcap')+(TJ_q)\Rightarrow(PMV_q),$$

*where* $(PMV_q)$ *stands for the* Parabolic Mean Value *inequality that means the following.* Fix an arbitrary ball $B=B(x,R)$ in $M$ and set $T=R^\beta$. Let $u$ be a bounded non-negative function on $M\times(0,T]$ that is *subcaloric* in the cylinder $B\times(0,T]$:

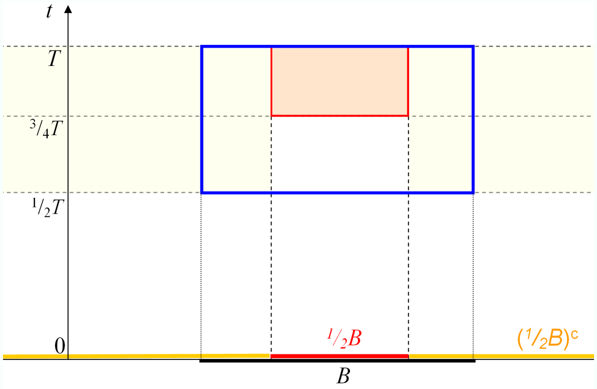
that is, for any $t\in(0,T]$,

$$u(\cdot,t)\in\mathcal{F}_+\cap L^\infty(M)$$

and $u$ satisfies in $B\times(0,T]$

$$\partial_t u+\mathcal{L}u\leq 0$$

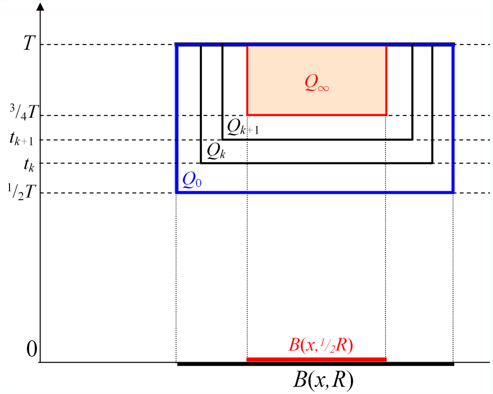
in a certain weak sense.




Then, for any $\varepsilon\in(0,1]$,

$$\sup_{t\in[\frac34 T,T]}\|u(\cdot,t)\|_{L^\infty(\frac12 B)}\leq C_\varepsilon\left(⨍_{B\times[\frac12 T,T]}u^2\right)^{1/2}+\frac{\varepsilon}{R^{\alpha/q'}}\sup_{t\in[\frac12 T,T]}\|u(\cdot,t)\|_{L^{q'}((\frac12 B)^c)}.\quad(PMV_q)$$

For the proof, consider a shrinking sequence of cylinders $Q_k = B(x, r_k) \times [t_k, T]$, $k \geq 0$, and an increasing sequence $b_k > 0$. Set




$$a_k := \int_{Q_k} (u - b_k)_+^2 \, d\mu dt$$

so that $a_k$ clearly decreases, and prove that

$$a_{k+1} \leq \frac{C}{(b_{k+1} - b_k)^{2\frac{\beta}{\alpha}}} \left( \frac{r_k}{r_k - r_{k+1}} \right)^C \left( \frac{1}{(r_k - r_{k+1})^\beta} + \frac{1}{t_{k+1} - t_k} + \frac{s_k}{b_{k+1} - b_k} \right)^{1+\frac{\beta}{\alpha}} a_k^{1+\frac{\beta}{\alpha}},$$

where

$$s_k = \sup_{t \in [t_k, T]} \operatorname*{essup}_{z \in B(x, \frac{r_k + r_{k+1}}{2})} \int_{B^c(x, r_k)} u(y,t) J(z,y) d\mu(y).$$

The proof of the relation between $a_k$ and $a_{k+1}$ uses essentially $(FK)$ and $(Gcap')$.

Choose

$$r_k = (\tfrac{1}{2} + 2^{-k-1})R \quad \text{and} \quad t_k = (\tfrac{3}{4} - 2^{-\beta k - 2})T,$$

so that

$$B\times[\tfrac{1}{2}T,T]=Q_0\supset Q_k\supset Q_\infty=\tfrac{1}{2}B\times[\tfrac{3}{4}T,T].$$

Setting also $b_k=\left(1-2^{-k}\right)b$ for some $b>0$, we obtain

$$a_{k+1}\le C2^{Ck}\left(1+\frac{R^\beta s_k}{b}\right)^{1+\frac{\beta}{\alpha}}\frac{a_k^{1+\frac{\beta}{\alpha}}}{\left(R^{\alpha+\beta}b^2\right)^{\frac{\beta}{\alpha}}}. \tag{10}$$

Iterating (10), we show that if $b$ is large enough then $\lim_{k\to\infty}a_k=0$,which implies that $u\le b$ in $Q_\infty$.The choice of $b$ depends on $\sup_k\frac{a_k}{R^{\alpha+\beta}}=\frac{a_o}{R^{\alpha+\beta}}$ and on an upper bound for $R^\beta s_k$. The value

$$\frac{a_0}{R^{\alpha+\beta}}\le\text{const}\,⨍_{B\times[\frac{1}{2}T,T]}u^2$$

yields the first term $(PMV_q)$. Estimating $s_k$ by means of the Hölder inequality and $(TP_q)$ gives

$$\begin{aligned}R^\beta s_k&\le R^\beta\sup_{t\in[\frac{1}{2}T,T]}\|u(\cdot,t)\|_{L^{q'}(\frac{1}{2}B)^c)}\frac{C}{\left(r_k-r_{k+1}\right)^{\alpha/q'+\beta}}\\&=\frac{C2^{Ck}}{R^{\alpha/q'}}\sup_{t\in[\frac{1}{2}T,T]}\|u(\cdot,t)\|_{L^{q'}(\frac{1}{2}B)^c)}\end{aligned}$$

which yields the second term in $(PMV_q)$.

## Outline of the proof of Theorem 2

Most of the proof is devoted to the implication

$$(FK) + (Gcap) + (TJ_q) \Rightarrow (TP_q)$$

**Step 0.** As it was already mentioned above,

$$(FK) \Rightarrow (DUE).$$

However, this implication does not work in a more general setting of doubling spaces, where we use an alternative proof of $(DUE)$ with help of the mean value inequality of Theorem 4.

**Step 1.** By Theorem 3, we have

$$(FK) + (Gcap) + (TJ) \Rightarrow (Gcap'),$$

and, by Theorem 4,

$$(FK) + (Gcap') + (TJ_q) \Rightarrow (PMV_q).$$

**Step 2**. We prove that

$$(PMV_2) \Rightarrow (DUE)\,.$$

For that apply $(PMV_2)$ with $u(\cdot,t) = P_t f$ where $f \in C_0(M)$ and $f \geq 0$, and observe that the both terms in the right hand side of $(PMV_q)$ are bounded by $\frac{C}{R^{a/2}}\|f\|_{L^2}$ which yields

$$\|P_T f\|_\infty \leq \frac{C}{T^{\alpha/(2\beta)}}\|f\|_2\,,$$

which then implies $(DUE)$. Consequently, we obtain that, for any $q \in [2,\infty]$,

$$(FK) + (Gcap) + (TJ_q) \Rightarrow (DUE)\,.$$

It follows from $(DUE)$ that

$$\|p_t(x,\cdot)\|_{L^q(M)} \leq \frac{C}{t^{\alpha/(q'\beta)}}.$$

Hence, in order to prove $(TP_q)$, it remains to prove

$$\boxed{\|p_t(x,\cdot)\|_{L^q(B^c(x,r))} \leq \frac{Ct}{r^{\alpha/q'+\beta}}} \tag{11}$$

assuming that $r^\beta \geq t$, which is done in the rest of the proof.

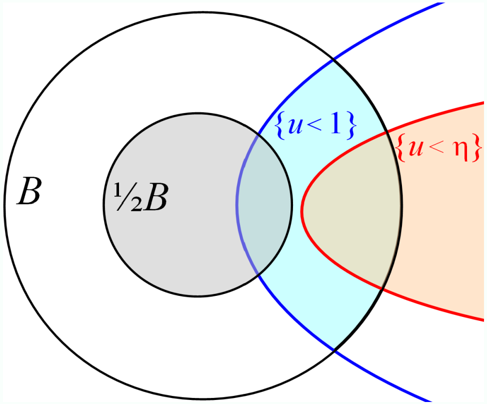
**Step 3**. We deduce from $(PMV_1)$ a so called "Lemma of growth":

there exist some $\varepsilon, \eta \in (0,1)$ such that, for any ball $B \subset M$ and for any $u \in \mathcal{F}$ that is non-negative and bounded in $M$ and superharmonic in $B$, if

$$\frac{\mu(B \cap \{u < 1\})}{\mu(B)} \leq \varepsilon,$$

then

$$\operatorname*{essinf}_{\frac{1}{2}B} u \geq \eta.$$




For that observe that $v = \frac{1}{u+a}$ is subharmonic for any $a > 0$. For subharmonic functions, we obtain from $(PMV_1)$ the following multiplicative form of the mean value inequality (by choosing $\varepsilon$):

$$\|v\|_{L^\infty(\frac{1}{2}B)} \leq CA^\theta \max(A,T)^{1-\theta}, \tag{12}$$

where

$$A = \left(⨍_B v^2 d\mu\right)^{1/2}, \qquad T = \|v\|_{L^\infty((\frac{1}{2}B)^c)},$$

and $\theta = \theta(\alpha, \beta) \in (0,1)$.

Let us estimate $A$ as follows:

$$A^2 = \frac{1}{\mu(B)}\left(\int_{B\cap u<1} + \int_{B\cap\{u\geq 1\}}\right)\frac{d\mu}{(u+a)^2}$$
$$\leq \frac{\mu(B\cap\{u<1\})}{\mu(B)}\frac{1}{a^2} + \frac{1}{(1+a)^2} \leq \frac{\varepsilon}{a^2} + \frac{1}{(1+a)^2} = \frac{2}{(1+a)^2},$$

for $a = \frac{1}{\varepsilon^{-1/2}-1}$. Estimating also trivially

$$\max(A, T) \leq \frac{1}{a},$$

we obtain from (12)

$$\operatorname*{essup}_{\frac{1}{2}B}\frac{1}{u+a} \leq C\left(\frac{2}{(1+a)^2}\right)^{\theta/2}\left(\frac{1}{a}\right)^{1-\theta} = \frac{C}{(1+a)^{\theta}a^{1-\theta}},$$

whence

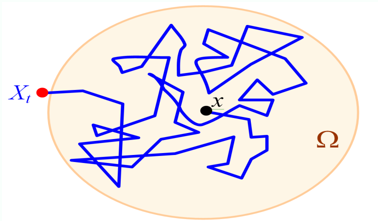
$$\operatorname*{essinf}_{\frac{1}{2}B} u \geq C^{-1}(1+a)^{\theta}a^{1-\theta} - a = a\left(C^{-1}\left(\tfrac{1}{a}+1\right)^{\theta} - 1\right) =: \eta,$$

where $\eta > 0$ if $a$ is small enough, that is, when $\varepsilon$ is small enough.

**Step 4.** For any open set $\Omega \subset M$ and any $x \in \Omega$ set

$$E^{\Omega}(x) = \int_0^{\infty} P_t^{\Omega}\mathbf{1}(x)dt = \int_0^{\infty}\int_{\Omega} p_t^{\Omega}(x,y)d\mu(y)dt.$$

It has the probabilistic meaning of the *mean exit time* from $\Omega$ of the jump process $X_t$, associated with $(\mathcal{E},\mathcal{F})$, that starts at $x$: $E^{\Omega}(x) = \mathbb{E}_x(\tau^{\Omega})$, where $\tau^{\Omega}$ is the first exit time from $\Omega$.




In this step we prove that, under $(FK)$, for any ball $B$ of radius $r$,

$$\operatorname*{essup}_{B} E^{B} \le C r^{\beta}. \tag{13}$$

**Step 5**. We prove the opposite inequality: the Lemma of growth and $(cap)$ imply that

$$\operatorname*{essinf}_{\frac{1}{4}B} E^{B} \ge c r^{\beta}. \tag{14}$$

It is known that (13) and (14) imply $(C)$.

**Step 6**. Using the upper and lower estimates of $E^B$, we deduce the *survival* inequality: there exist $\varepsilon > 0$ such that, for any ball $B$ of radius $r$ and for any $t > 0$,

$$P_t^B \mathbf{1}_B \geq \varepsilon - \frac{Ct}{r^\beta} \text{ in } \tfrac{1}{4}B. \tag{$S$}$$

In probabilistic terms,

$$P_t^B \mathbf{1}_B(x) = \mathbb{P}_x\left(\tau_B > t\right)$$

that is the probability of survival of the process in $B$ up to time $t$ assuming the killing condition in $B^c$.

**Step 7**. For any $\rho > 0$ consider a *truncated* Dirichlet form

$$\mathcal{E}^{(\rho)}(f,f) := \iint_{\{d(x,y)<\rho\}} (f(x) - f(y))^2 J(x,y) d(x) d\mu(y).$$

Denote by $Q_t$ the heat semigroup of $\left(\mathcal{E}^{(\rho)}, \mathcal{F}\right)$ and by $q_t(x,y)$ its heat kernel. We prove that, under all the above hypotheses, the heat kernel of $\left(\mathcal{E}^{(\rho)}, \mathcal{F}\right)$ exists and satisfies the following diagonal upper bound

$$q_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \exp\left(\frac{Ct}{\rho^\beta}\right). \tag{15}$$

**Step 8**. We deduce from $(S)$ a similar condition for the truncated semigroup $Q_t$:

$$Q_t^B \mathbf{1}_B \geq \varepsilon - Ct\left(r^{-\beta} + \rho^{-\beta}\right) \quad \text{in } \tfrac{1}{4}B$$

where $B = B(x,r)$. A certain iteration procedure allows to self-improve this estimate and to obtain that, for any $k \in \mathbb{N}$, if $r \geq 8k\rho$ then

$$Q_t^B \mathbf{1}_B \geq 1 - C(k)\left(\frac{t}{\rho^\beta}\right)^k,$$

which implies that

$$\int_{B^c(x,r)} q_t(x,y)d\mu(y) \leq C(k)\left(\frac{t}{\rho^\beta}\right)^k.$$

Combining this with (15), we obtain that, in the case $q < \infty$,

$$\|q_t(x,\cdot)\|_{L^q(B^c)} \leq \|q_t(x,\cdot)\|_{L^\infty(B^c)}^{1/q'} \|q_t(x,\cdot)\|_{L^1(B^c)}^{1/q} \leq \frac{C(k)}{t^{\alpha/(q'\beta)}} \exp\left(\frac{Ct}{\rho^\beta}\right)\left(\frac{t}{\rho^\beta}\right)^{\frac{k}{q}}. \qquad (16)$$

In the case $q = \infty$ we improve (15) in a different way and obtain that if $r \geq 4k\rho$ then

$$\|q_t(x,\cdot)\|_{L^\infty(B^c)} \leq \frac{C(k)}{t^{\alpha/\beta}} \exp\left(\frac{Ct}{\rho^\beta}\right)\left(1 + \frac{\rho^\beta}{t}\right)^{\alpha/\beta}\left(\frac{t}{\rho^\beta}\right)^k. \qquad (17)$$

**Step 9**. We prove that, under all the above conditions, including $(TJ_q)$, we have, for any $t>0$ and for any ball $B=B(x,r)$,

$$\|p_t(x,\cdot)\|_{L^q(B^c)} \leq \|q_t(x,\cdot)\|_{L^q(B^c)} + \frac{Ct}{\rho^{\alpha/q'+\beta}} \exp\left(\frac{Ct}{\rho^\beta}\right). \tag{18}$$

**Step 10**. In the case $q<\infty$, combining (16) and (18), we obtain that if $r \geq 8k\rho$ then

$$\|p_t(x,\cdot)\|_{L^q(B^c)} \leq \frac{C(k)}{t^{\alpha/(q'\beta)}} \exp\left(\frac{Ct}{\rho^\beta}\right)\left(\frac{t}{\rho^\beta}\right)^{k/q} + \frac{Ct}{\rho^{\alpha/q'+\beta}} \exp\left(\frac{Ct}{\rho^\beta}\right).$$

Assuming that $r^\beta \geq t$ and setting $\rho = r/(8k)$, we obtain

$$\begin{aligned}\|p_t(x,\cdot)\|_{L^q(B^c)} &\leq \frac{C(k)}{t^{\alpha/(q'\beta)}}\left(\frac{t}{r^\beta}\right)^{k/q} + \frac{C(k)t}{r^{\alpha/q'+\beta}} \\ &\leq C\frac{t}{r^{\alpha/q'+\beta}},\end{aligned}$$

provided $k$ is chosen so that

$$\left(\frac{t}{r^\beta}\right)^{k/q} \leq \left(\frac{t}{r^\beta}\right)^{\frac{\alpha}{q'\beta}+1},$$

that is,

$$\frac{k}{q} \geq \frac{\alpha}{q'\beta} + 1.$$

This finishes the proof of $(TP_q)$ if $q < \infty$.

In the case $q = \infty$ we obtain from (17) and (18), assuming that $r^\beta \geq t$ and setting $\rho = r/(4k)$ that

$$\begin{aligned}\|p_t(x,\cdot)\|_{L^q(B^c)} &\leq \frac{C(k)}{t^{\alpha/\beta}} \left(\frac{t}{r^\beta}\right)^{k-\frac{\alpha}{\beta}} + \frac{C(k)t}{r^{\alpha+\beta}} \\ &\leq C\frac{t}{r^{\alpha+\beta}},\end{aligned}$$

provided $k$ is chosen so that

$$\left(\frac{t}{r^\beta}\right)^{k-\frac{\alpha}{\beta}} \leq \left(\frac{t}{r^\beta}\right)^{\frac{\alpha}{\beta}+1}$$

that is,

$$k \geq 2\frac{\alpha}{\beta} + 1.$$

**Step 11**. We prove now consequences of $(TP_q)$. Let us first prove Lemma 1, that is, if $q \in [2, \infty]$ then

$$\boxed{(TP_q) \Rightarrow (UE_q)}.$$

Setting $r = \frac{1}{2} d(x, y)$, we obtain by the semigroup property

$$\begin{aligned} p_{2t}(x, y) &= \int_M p_t(x, z) \, p_t(z, y) \, d\mu(z) \\ &\leq \left( \int_{B^c(x,r)} + \int_{B^c(y,r)} \right) p_t(x, z) \, p_t(z, y) \, d\mu(z) \, . \end{aligned}$$

It suffices to estimate the first integral. By the Hölder inequality, we have

$$\int_{B^c(x,r)} p_t(x, z) \, p_t(z, y) \, d\mu(z) \leq \| p_t(x, \cdot) \|_{L^q(B^c(x,r))} \, \| p_t(\cdot, y) \|_{L^{q'}(M)} \, .$$

Since $q \geq 2$ and, hence, $q' \leq q$, we have not only $(TP_q)$ but also $(TP_{q'})$. Hence,

$$\| p_t(x, \cdot) \|_{L^q(B^c(x,r))} \leq \frac{C}{t^{\alpha/(q'\beta)}} \left( 1 + \frac{r}{t^{1/\beta}} \right)^{-(\alpha/q' + \beta)}$$

and

$$\| p_t(\cdot, y) \|_{L^{q'}(M)} \leq \frac{C}{t^{\alpha/(q\beta)}}.$$

Since $\frac{\alpha}{q'\beta} + \frac{\alpha}{q\beta} = \frac{\alpha}{\beta}$, we obtain

$$\int_{B^c(x,r)} p_t(x,z)\, p_t(z,y)\, d\mu(z) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{r}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)}.$$

Estimating in the same manner the second integral, we obtain

$$p_{2t}(x,y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{r}{t^{1/\beta}}\right)^{-(\alpha/q'+\beta)},$$

that is, $(UE_q)$.

**Step 12**. Since $(UE_q) \Rightarrow (DUE) \Rightarrow (FK)$, we obtain that

$$\boxed{(TP_q) \Rightarrow (FK)}.$$

The implication

$$\boxed{(TP_q) \Rightarrow (TJ_q)}$$

was already mentioned in (5).

**Step 13**. Finally, the implication

$$\boxed{(TP_q) + (C) \Rightarrow (Gcap)}$$

is proved as follows. By $(TP_q)$ we have also $(TP_1)$, that is,

$$\int_{B^c(x,r)} p_t(x,y)d\mu(y) \le C\left(1 + \frac{r}{t^{1/\beta}}\right)^{-\beta} \le \frac{Ct}{r^\beta}.$$

This and $(C)$ imply that

$$P_t^{B(x,r)}\mathbf{1}(x) \ge \varepsilon - \frac{Ct}{r^\beta}$$

that is, $(S)$, and it is known that $(S) \Rightarrow (Gcap)$.