UPPER BOUNDS OF HEAT KERNELS ON DOUBLING SPACES

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Abstract. In this paper we give various equivalent characterizations of *upper* estimates of heat kernels of regular, conservative and local Dirichlet forms on *doubling* spaces, from both the analytic and probabilistic points of view. The first part of this paper uses purely analytic arguemtn, while the second part focuses on the probabilistic aspects where the exit time plays an important role.

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1. Introduction

Let (M, d) be a locally compact separable metric space and μ be a Radon measure on M with full support. Let $(\mathcal{E}, \mathcal{F})$ be a local regular Dirichlet form in $L^2(M, \mu)$, Δ be its generator and $P_t = e^{t\Delta}$, $t \geq 0$, be the associated heat semigroup. A family $\{p_t\}_{t>0}$ of non-negative $\mu \times \mu$-measurable functions on $M \times M$ is called the *heat kernel* of the form $(\mathcal{E}, \mathcal{F})$ if p_t is the integral kernel of the operator P_t, that is, for any $t > 0$ and for any $f \in L^2(M, \mu)$,

$$P_t f(x) = \int_M p_t(x, y) f(y) \, d\mu(y) \tag{1.1}$$

for μ-almost all $x \in M$.

The purpose of this paper is to prove the existence of the heat kernel and to obtain certain upper estimates for it under appropriate assumptions. For any $x \in M$ and $r > 0$, set

$$B(x, r) = \{y \in M : d(x, y) < r\} \quad \text{and} \quad V(x, r) = \mu(B(x, r)).$$

We assume throughout that $0 < V(x, r) < \infty$. Fix a parameter $\beta > 1$ and consider the following condition, which in general may be true or not.

(UE_β) : The *upper estimate*: the heat kernel exists and satisfies the inequality

$$p_t(x, y) \leq \frac{C}{V(x, t^{1/\beta})} \exp\left(-c\left(\frac{d^\beta(x, y)}{t}\right)^{\frac{1}{\beta-1}}\right),$$

for some constants $C, c > 0$, for all $t > 0$ and μ-almost all $x, y \in M$.

This form of (UE_β) is motivated by the following two classes of examples.

1. Let M be a Riemannian manifold, d be the geodesic distance, μ be the Riemannian volume, and $\mathcal{E}$ be the canonical energy form given by

$$\mathcal{E}(f) = \int_M |\nabla f|^2 \, d\mu,$$

and $\mathcal{F} = W_0^{1,2}(M, \mu)$ (that is, $\mathcal{F}$ is the closure of $C_0^\infty(M)$ in $W^{1,2}(M, \mu)$). In this setting, the heat kernel $p_t(x, y)$ always exists and is a smooth function in (t, x, y). There is a vast literature devoted to upper and lower bounds of the heat kernel in connection with the geometry of M. See, for example, [8], [11], [18], [19], [20], [32], [34], [35], [36]. If $M = \mathbb{R}^n$ with the standard Euclidean structure then $V(x, r) = c_n r^n$ and

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{d(x, y)^2}{4t}\right),$$

so that (UE_2) obviously holds. Furthermore, if M is geodesically complete and the Ricci curvature of M is non-negative then the heat kernel also satisfies (UE_2) (see [30], [16], [33]).

2. Let M be one of the fractal spaces described, for example, in [1]. Typically, d is an extrinsic distance from the ambient Euclidean space, and μ is a Hausdorff

measure. On most of the basic fractal spaces, one has $V(x,r) \simeq r^{\alpha}$ where $\alpha = \dim_H M$. The energy form $(\mathcal{E}, \mathcal{F})$ is constructed in a certain (highly non-trivial) way using the self-similarity of the fractal. On large classes of fractals, it was proved that the heat kernel exists and is a continuous function of (t,x,y). Furthermore, on such fractals the heat kernel admits the upper estimate (UE_β) with some $\beta > 2$ (as well as a matching lower estimate). See, for example, [1], [2], [3], [5], [25], [27], [29].

In the both cases, the Dirichlet form gives rise to the associated diffusion process $\{X_t\}_{t\geq 0}$ on M, whose transition density with respect to measure μ is exactly the heat kernel $p_t(x,y)$. With some restrictions, such process exists also in the general case and can be used to set up reasonable conditions for heat kernel estimates.

The main purpose of this paper is to prove new equivalent conditions for the estimate (UE_β) (including the existence of the heat kernel) in various terms, both analytic and probabilistic, which will be explained in details in the next Section.

In the case of a Riemannian manifold, the necessary and sufficient condition for (UE_2) in terms of a certain *Faber-Krahn inequality* was proved in [17]. In the case of a general underlying space, Kigami [28] proved the necessary and sufficient conditions for (UE_β) in terms of a *Nash type inequality* and a *mean exit time estimate* (which involves the associated diffusion $\{X_t\}$), although under the additional a priori assumptions that the heat kernel exists, is a continuous function of (t,x,y), satisfies the estimate $\sup_x p_t(x,x) < \infty$ for all $t > 0$, and $\inf_x V(x,r) > 0$ for some $r > 0$.

Let us briefly list our new results.

1. We prove that, under mild general assumptions, the upper bound (UE_β) is equivalent to the following estimate:

$$p_t(x,y) \leq \frac{1}{V(x,t^{1/\beta})} \Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right), \tag{1.2}$$

where $\Phi(s)$ is a monotone decreasing function, that decays fast enough as $s \to \infty$. The fact that (1.2) implies (UE_β) can be regarded as a self-improvement phenomenon.

2. We prove that (UE_β) is equivalent to the conjunction of the Faber-Krahn inequality and some tail estimate of the heat kernel or that of the exit time.

3. We prove that (UE_β) is equivalent to a certain isoperimetric inequality for the mean exit time and the fact that this inequality is optimal for balls up to a constant factor.

4. We develop new techniques for comparison heat semigroups and heat kernels in different domains (cf. Theorem 4.6) that are used for obtaining heat kernel upper bounds.

The analytic conditions for (UE_β) are proved in Theorem 2.1, the probabilistic conditions – in Theorem 2.2. Let us emphasize that the proofs are completely analytic except for the cases where the probabilistic assumptions enter explicitly the statement.

An important feature of this paper is the level of generality, which distinguishes it from the previous ones and which is reflected in the following:

1. We make no a priori assumptions about the existence or regularity of the heat kernel.

2. We make no specific assumption on the distance function $d(x,y)$ (as being geodesic or satisfying the chain condition).

3. We do not assume that the metric balls are relatively compact (but do assume the local compactness of (M, d) and finiteness of the volumes of balls).

We hope that this level of generality will facilitate the applications of the above mentioned results[1]. At the same time, this setting poses certain technical challenges and makes the proofs noticeably longer and more elaborate than one would desire.

The structure of the paper is as follows. In Section 2 we introduce the necessary background material and state the main Theorems 2.1 and 2.2, providing also further comments of technical and historical nature.

In Section 3 we state and prove some basic properties of the heat semigroup.

In Section 4 we introduce one of the main tools of this paper - comparison estimates of heat semigroups in different domains (Theorem 4.6). The proofs here are based on the weak parabolic maximum principle of [21].

In Section 5 we prove Theorem 2.1. The major ingredients of the proof are Lemmas 5.5, 5.6 (based on Theorem 4.6) and Theorems 5.7, 5.8. In Section 6 we prove Theorem 2.2.

We should mention that a first version of this paper under the title "Heat kernel upper bounds on fractal spaces" was circulated in a preprint form by the first-named author since 2003. The preprint contained part of the present results in a more restricted setting. All references to that preprint should be replaced by references to the present paper.

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2. Statement of the main results

2.1. **Metric measure space.** Unless otherwise stated, here and in the rest of this paper (M, d) is a locally compact separable metric space and μ is a Radon measure on M with full support. As usual, the norm in the real Banach space $L^p := L^p(M, \mu)$ is defined by

$$\|f\|_p := \left(\int_M |f(x)|^p \, d\mu(x) \right)^{1/p}, \quad 1 \leq p < \infty,$$

and $\|f\|_\infty := \operatorname{esup}_{x \in M} |f(x)|$, where esup is the essential supremum. The inner product of functions $f, g \in L^2$ is denoted by (f, g).

As above, let $B(x, r)$ denote the metric ball in (M, d) and set $V(x, r) = \mu(B(x, r))$. The fact that μ has a full support is equivalent to having $V(x, r) > 0$ for all $x \in M$ and $r > 0$. We assume in addition that $V(x, r) < \infty$. If a ball $B(x, r)$ is precompact then the finiteness of $V(x, r)$ follows from the hypothesis that μ is Radon. However, in general we do not assume that all balls are precompact, but instead we take a much milder hypothesis of the finiteness of volumes of balls.

Consider the following conditions that in general may be true or not.

(VD): *Volume doubling property.* there is a constant $C_D \geq 1$ such that

$$V(x, 2r) \leq C_D V(x, r). \tag{2.1}$$

[1]For example, without the requirement of the relative compactness of the balls, one can remove from the space M a closed set of measure 0 without violating all other hypotheses.

for all $x \in M$ and $r > 0$.

It is known that (VD) implies the following: there exists $\alpha > 0$ such that

$$\frac{V(x,R)}{V(y,r)} \leq C_D \left(\frac{d(x,y)+R}{r}\right)^{\alpha} \tag{2.2}$$

for all $x, y \in M$ and $0 < r \leq R$ (see Proposition 5.1 below).

(RVD) : *Reverse volume doubling property*: there exist positive constants α' and c such that

$$\frac{V(x,R)}{V(x,r)} \geq c \left(\frac{R}{r}\right)^{\alpha'} \quad \text{for all } x \in M \text{ and } 0 < r \leq R.$$

It is known that (VD) implies (RVD) provided M is connected and unbounded (see Corollary 5.3 below).

2.2. **The Dirichlet forms.** A *Dirichlet form* $(\mathcal{E}, \mathcal{F})$ in $L^2(M,\mu)$ is a bilinear form $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ defined on a dense subspace $\mathcal{F}$ of $L^2(M,\mu)$, which satisfies in addition the following properties:

(1) Positivity: $\mathcal{E}(f) := \mathcal{E}(f,f) \geq 0$.
(2) Closedness: $\mathcal{F}$ is a Hilbert space with respect to the following inner product:

$$\mathcal{E}_1(f,g) := \mathcal{E}(f,g) + (f,g).$$

(3) The Markov property: if $f \in \mathcal{F}$ then also $\widetilde{f} := (f \wedge 1)_+$ belongs to $\mathcal{F}$ and $\mathcal{E}(\widetilde{f}) \leq \mathcal{E}(f)$.

Here we use the notation $a_+ := \max\{a, 0\}$.

Recall some further definitions and results from the theory of Dirichlet forms (cf. [15]). Any Dirichlet form has a *generator* Δ, which is a non-positive definite self-adjoint operator on $L^2(M,\mu)$ with domain $\mathcal{D} \subset \mathcal{F}$ such that

$$\mathcal{E}(f,g) = (-\Delta f, g)$$

for all $f \in \mathcal{D}$ and $g \in \mathcal{F}$. The generator determines the *heat semigroup* $\{P_t\}_{t\geq 0}$ defined by $P_t = e^{t\Delta}$.

It is known that the heat semigroup satisfies the following properties:

- $\{P_t\}_{t\geq 0}$ is *contractive* in L^2, that is $\|P_t f\|_2 \leq \|f\|_2$ for all $f \in L^2$ and $t > 0$.
- $\{P_t\}_{t\geq 0}$ is *strongly continuous*, that is, for every $f \in L^2$,

$$P_t f \xrightarrow{L^2} f \text{ as } t \to 0+.$$

- $\{P_t\}_{t\geq 0}$ is *symmetric*, that is,

$$(P_t f, g) = (f, P_t g) \quad \text{for all } f, g \in L^2.$$

- $\{P_t\}_{t\geq 0}$ is *Markovian*, that is, for any $t > 0$,

$$\text{if } f \geq 0 \text{ then } P_t f \geq 0, \text{ and if } f \leq 1 \text{ then } P_t f \leq 1.$$

Here and below the identities and inequalities between L^2-functions are understood μ-almost everywhere in M.

The form $\mathcal{E}(f)$ can be recovered from the heat semigroup as follows. For any $f \in L^2(M,\mu)$, the function

$$t \mapsto \frac{1}{t}(f - P_t f, f)$$

is increasing as t is decreasing. In particular, it has a limit as $t \to 0$. It turns out that the limit is finite if and only if $f \in \mathcal{F}$, and

$$\lim_{t\to 0+} \frac{1}{t}(f - P_t f, f) = \mathcal{E}(f). \tag{2.3}$$

The Markovian property of the heat semigroup implies that the operator P_t preserves the inequalities between functions, which allows to use monotone limits to extend P_t from L^2 to L^∞ and, in fact, to any L^q, $1 \le q \le \infty$. Moreover, the extended operator P_t is a contraction on any L^q and preserves positivity (cf. [15, p.33]).

The form $(\mathcal{E},\mathcal{F})$ is called *conservative* if $P_t 1 = 1$ for every $t > 0$.

The form $(\mathcal{E},\mathcal{F})$ is called *local* if $\mathcal{E}(f,g) = 0$ for any couple $f,g \in \mathcal{F}$ with disjoint compact supports. The form $(\mathcal{E},\mathcal{F})$ is called *strongly local* if $\mathcal{E}(f,g) = 0$ for any couple $f,g \in \mathcal{F}$ with compact supports, such that $f \equiv \text{const}$ in an open neighborhood of $\operatorname{supp} g$.

The Dirichlet form is called *regular* if $\mathcal{F} \cap C_0(M)$ is dense both in $\mathcal{F}$ and in $C_0(M)$, where $C_0(M)$ is the space of all continuous functions with compact support in M, endowed with sup-norm.

For a non-empty open $\Omega \subset M$, let $\mathcal{F}(\Omega)$ be the closure of $\mathcal{F} \cap C_0(\Omega)$ in the norm of $\mathcal{F}$. It is known that if $(\mathcal{E},\mathcal{F})$ is regular, then $(\mathcal{E},\mathcal{F}(\Omega))$ is also a regular Dirichlet form in $L^2(\Omega,\mu)$. Denote by P_t^Ω the heat semigroup of $(\mathcal{E},\mathcal{F}(\Omega))$. It is known that if $\{\Omega_i\}_{i=1}^\infty$ is an increasing sequence of open subsets of M and $\Omega = \bigcup_{i=1}^\infty \Omega_i$ then, for any $t > 0$ and any $0 \le f \in L^2(\Omega)$, the sequence $P_t^{\Omega_i} f$ increases and converges to $P_t^\Omega f$ as $i \to \infty$ almost everywhere (see [15], [21, L.4.17]).

2.3. **Analytic conditions (Theorem 2.1).** For any set $A \subset M$, write A^c for $M \setminus A$. Fix a ball $B = B(x,r)$ on M. In what follows we frequently consider expressions like $P_t 1_B$ and $P_t 1_{B^c}$. By definition, these functions are from $L^2(M,\mu)$ so that they are defined almost everywhere rather than pointwise. In particular, the values $P_t 1_B(x)$ and $P_t 1_{B^c}(x)$ are not well-defined where x is the center of B. However, in the presence of the heat kernel, one can give meaning to these functions for almost all x as follows. Fix $t > 0$, choose a pointwise version $p_t(x,y)$ of the heat kernel and consider the integral

$$\int_{B^c(x,r)} p_t(x,y)\, d\mu(y). \tag{2.4}$$

Since the function $y \mapsto p_t(x,y)$ is measurable for almost all x, this integral is also defined for almost all $x \in M$. We claim that, for any other pointwise version $\widetilde{p}_t(x,y)$ of the heat kernel, the following identity holds for almost all $x \in M$:

$$\int_{B^c(x,r)} p_t(x,y)\, d\mu(y) = \int_{B^c(x,r)} \widetilde{p}_t(x,y)\, d\mu(y). \tag{2.5}$$

That is, the expression (2.4) is well-defined for almost all $x \in M$. Indeed, observe that

$$\int_{B^c(x,r)} p_t(x,y)\, d\mu(y) = \int_M p_t(x,y)\, 1_{B^c(x,r)}(y)\, d\mu(y) \tag{2.6}$$

and that the following identity

$$p_t(x,y)\, 1_{B^c(x,r)}(y) = \widetilde{p}_t(x,y)\, 1_{B^c(x,r)}(y) \tag{2.7}$$

holds for almost all $(x,y) \in M \times M$. Since the both functions in (2.7) are measurable in $(x,y) \in M \times M$, we obtain by Fubini's theorem that, for almost all $x \in M$,

$$\int_M p_t(x,y)\, 1_{B^c(x,r)}(y)\, d\mu(y) = \int_M \widetilde{p}_t(x,y)\, 1_{B^c(x,r)}(y)\, d\mu(y),$$

whence (2.5) follows.

Fix a constant $\beta > 1$ and consider the following conditions.

$(T_{\exp})$: (*The exponential tail estimate*) The heat kernel p_t exists and satisfies the estimate

$$\int_{B(x,r)^c} p_t(x,y)\, d\mu(y) \le C \exp\left(-c\left(\frac{r}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right), \tag{2.8}$$

for some constants $C, c > 0$, all $t > 0, r > 0$ and μ-almost all $x \in M$. It is easy to show that (2.8) is equivalent to the following inequality: for any ball $B = B(x_0, r)$ and $t > 0$,

$$P_t \mathbf{1}_{B^c}(x) \le C \exp\left(-c\left(\frac{r}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right) \text{ for } \mu\text{-almost all } x \in \frac{1}{4}B$$

(see [21, Remark 3.3]).

(T_β) : (*The tail estimate*) There exist $0 < \varepsilon < \frac{1}{2}$ and $C > 0$ such that, for all $t > 0$ and all balls $B = B(x_0, r)$ with $r \ge Ct^{1/\beta}$,

$$P_t \mathbf{1}_{B^c}(x) \le \varepsilon \quad \text{for } \mu\text{-almost all } x \in \frac{1}{4}B. \tag{2.9}$$

(S_β) : (*The survival estimate*) There exist $0 < \varepsilon < 1$ and $C > 0$ such that, for all $t > 0$ and all balls $B = B(x_0, r)$ with $r \ge Ct^{1/\beta}$,

$$1 - P_t^B \mathbf{1}_B(x) \le \varepsilon \quad \text{for } \mu\text{-almost all } x \in \frac{1}{4}B. \tag{2.10}$$

(FK_β) : (*The Faber-Krahn inequality*) There exist positive constants ν, c such that, for all balls $B \subset M$ of radius r and for all non-empty open sets $\Omega \subset B$,

$$\lambda_{\min}(\Omega) \ge \frac{c}{r^\beta}\left(\frac{\mu(B)}{\mu(\Omega)}\right)^\nu, \tag{2.11}$$

where $\lambda_{\min}(\Omega)$ is the bottom of the spectrum of the (positive definite) generator of $(\mathcal{E}, \mathcal{F}(\Omega))$, that is,

$$\lambda_{\min}(\Omega) = \inf_{f \in \mathcal{F}(\Omega)\setminus\{0\}} \frac{\mathcal{E}(f)}{\|f\|_2^2}. \tag{2.12}$$

Note that since $\mu(B) \ge \mu(\Omega)$, the value of ν in (2.11) can be chosen to be arbitrarily small. We will frequently assume that $\nu < 1$.

(ΦUE_β) : (*Upper estimate with* Φ*-term*) The heat kernel $p_t(x,y)$ exists and admits the following estimate

$$p_t(x,y) \le \frac{C}{V(x,t^{1/\beta})}\Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right), \tag{2.13}$$

for some constant C, all $t>0$ and μ-almost all $x,y\in M$, where Φ is a decreasing positive function on $[0,+\infty)$ such that

$$\int_0^\infty s^{\alpha-1}\Phi(s)\,ds<\infty, \tag{2.14}$$

and α is the same exponent as in (2.2).

Clearly, (UE_β) is a particular case of (ΦUE_β). Observe also that (ΦUE_β) implies the following estimate (cf. Section 5.5):

(DUE_β) : (*On-diagonal upper bound*) The heat kernel p_t exists and satisfies the estimate

$$p_t(x,y) \le \frac{C}{\sqrt{V(x,t^{1/\beta})V(y,t^{1/\beta})}}, \tag{2.15}$$

for some constant C, all $t>0$ and μ-almost all $x,y\in M$.

In the case of a continuous heat kernel, (2.15) is equivalent to the estimate

$$p_t(x,x) \le \frac{C}{V(x,t^{1/\beta})},$$

which explains the term "on-diagonal". Now we can state our first main result.

Theorem 2.1. *Let* (M,d,μ) *be a metric measure space and let* μ *satisfy* (VD) *and* (RVD). *Let* $(\mathcal{E},\mathcal{F})$ *be a regular, local, conservative Dirichlet form in* $L^2(M,\mu)$. *Then, the following conditions are equivalent:*

$$\begin{aligned}(UE_\beta) &\Leftrightarrow (\Phi UE_\beta)\\ &\Leftrightarrow (FK_\beta)+(S_\beta)\Leftrightarrow(FK_\beta)+(T_\beta)\\ &\Leftrightarrow (DUE_\beta)+(S_\beta)\Leftrightarrow(DUE_\beta)+(T_\beta)\\ &\Leftrightarrow (DUE_\beta)+(T_{\exp}).\end{aligned}$$

Let us make some comments on the statement.

1. The following two equivalences

$$(UE_\beta)\Leftrightarrow(\Phi UE_\beta)$$

and

$$(UE_\beta)\Leftrightarrow(FK_\beta)+(S_\beta)\Leftrightarrow(FK_\beta)+(T_\beta)$$

are new and have not been previously known in any setting except for the case when (M,d,μ) is a Riemannian manifold and $\beta=2$ (in the latter case, one has $(UE_2)\Leftrightarrow(DUE_2)\Leftrightarrow(FK_2)$ so that the conditions (S_β) and (T_β) can be omitted – cf. [17]).

2. The equivalences

$$(UE_\beta)\Leftrightarrow(DUE_\beta)+(S_\beta)\Leftrightarrow(DUE_\beta)+(T_\beta)\Leftrightarrow(DUE_\beta)+(T_{\exp}) \tag{2.16}$$

were proved in [21, Theorems 3.1, 3.4, 310] under an additional hypothesis that all metric balls in (M,d) are precompact. We have been able here to drop this hypothesis thanks to the comparison estimates of Section 4.3 (cf. Proposition 4.7 and Remark 4.8). Note also that the implication $(T_\beta) \Rightarrow (S_\beta)$ is the only place where the conservativeness of $(\mathcal{E}, \mathcal{F})$ is used.

3. The implication $(\Phi UE_\beta) \Rightarrow (UE_\beta)$ in Theorem 2.1 may fail, if the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is not local. Indeed, let $M = \mathbb{R}^n$ and μ be the Lebesgue measure. For every $0 < \beta < 2$, the heat kernel p_t generated by $(-\Delta)^{\beta/2}$, where Δ is the Laplace operator, admits the estimate

$$p_t(x, y) \simeq \frac{1}{t^{n/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(n+\beta)}.$$

The corresponding Dirichlet form is non-local. It is clear that (ΦUE_β) holds with $\Phi(s) = (1+s)^{-(n+\beta)}$, whereas (UE_β) is not true.

4. In order to prove the implication

$$(FK_\beta) + (S_\beta) \Rightarrow (DUE_\beta), \tag{2.17}$$

we use the locality of $(\mathcal{E}, \mathcal{F})$, although it is not clear whether the locality is really essential or just technical. Note for comparison that if the volume doubling (VD) is replaced by a stronger condition

$$V(x, r) \simeq r^\alpha \quad \text{for all } x \in M \text{ and all } r > 0,$$

then one can easily prove the implication

$$(FK_\beta) \Rightarrow (DUE_\beta)$$

without assuming the locality of $(\mathcal{E}, \mathcal{F})$ (see Lemma 5.5 as well as the argument in [7], [17]).

5. The reverse volume doubling (RVD) is used only to prove the implication

$$(DUE_\beta) \Rightarrow (FK_\beta).$$

Without (RVD) this implication does not hold in general. Indeed, let M be a compact Riemannian manifolds, for example, $\mathbb{S}^n$. Then (DUE_2) holds while for the ball $B = M$ we have $\lambda_{\min}(B) = 0$ so that (FK_2) fails. Note also that (RVD) follows from (VD) provided M is connected and unbounded (see Corollary 5.3). Of course, in this case (RVD) can be dropped from the hypotheses of Theorem 2.1.

6. We regard the following implication as a central and most interesting part of Theorem 2.1:

$$(FK_\beta) + (T_\beta) \Rightarrow (UE_\beta). \tag{2.18}$$

The proof of (2.18) uses the following ingredients:

- (i) Lemma 5.5: (FK_β) implies on-diagonal upper bound (5.48) for the heat kernels in balls.
- (ii) Theorem 5.8: $(T_\beta) \Leftrightarrow (S_\beta) \Leftrightarrow (T_{\exp})$.
- (iii) Lemma 5.6: (5.48) and (S_β) imply (DUE_β).
- (iv) $(DUE_\beta) + (T_{\exp}) \Rightarrow (UE_\beta)$

The proof of Lemma 5.5 follows [17]: one first obtains a Nash type inequality in balls (Lemma 5.4) and then uses Nash's argument to estimate the heat kernels in balls.

The idea of Theorem 5.8 in the setting of fractals goes back to Barlow [1] using a probabilistic approach. In the general setting but still assuming that the metric balls are precompact, Theorem 5.8 was proved by the authors in [21]. Here we have been able to drop this assumption at the expense of using a quite involved Proposition 4.7. Also, we have simplified other parts of the proof by using the argument of Hebisch and Saloff-Coste [26] (cf. Theorem 5.7).

The idea of Lemma 5.6 goes back to Kigami [28] who introduced the method called now Kigami's iterations. We had to overcome significant difficulties to adapt this method to the present setting, due to the lack of a priori continuity and boundedness of the heat kernel. For that we use a new comparison estimate for the heat kernels in different domains (Theorem 4.6).

Finally, the proof of the implication $(DUE_\beta)+(T_{exp}) \Rightarrow (UE_\beta)$ uses a new computational argument. For the full proof of Theorem 2.1 we refer the reader to Section 5.5.

2.4. **Probabilistic conditions (Theorem 2.2).** For any regular Dirichlet form $(\mathcal{E}, \mathcal{F})$, there is an associated *Hunt process*[2]. Denote by $X_t, t \geq 0$, the trajectories of a process and by $\mathbb{P}_x, x \in M$, the probability measure in the space of trajectories emanating from the point x. Denote by $\mathbb{E}_x$ the expectation of the probability measure $\mathbb{P}_x$. Then the relation between the Dirichlet form and the associated Hunt process is given by the following identity:

$$P_t f(x) = \mathbb{E}_x f(X_t), \tag{2.19}$$

which holds for any bounded Borel function f, for every $t > 0$, and for μ-almost all $x \in M$ (note that $P_t f$ is a function from L^∞ and, hence, is defined up to a set of measure zero, whereas $\mathbb{E}_x f(X_t)$ is defined pointwise for all $x \in M$). By [15, Theorem 7.2.1, p.302], such a process always exists but, in general, is not unique. Let us fix one of such processes once and for all. Note that if $(\mathcal{E}, \mathcal{F})$ is local, then the Hunt process X_t is a *diffusion*, that is, the sample path $t \mapsto X_t$ is continuous almost surely.

We say that the process $\{X_t\}$ is *stochastically complete* if

$$\mathbb{P}_x (X_t \in M) \equiv 1 \text{ for all } t > 0 \text{ and } x \in M.$$

If the process is not stochastically complete then the state space M is added an ideal point ∞, which is called a *cemetery* and which is assigned a complimentary probability to ensure that $\mathbb{P}_x$ has the total mass 1. Applying (2.19) with $f \equiv 1$, we obtain that $\{X_t\}$ is stochastically complete if and only if $(\mathcal{E}, \mathcal{F})$ is conservative.

For any open set $\Omega \subset M$, define the *first exit time* τ_Ω as follows:

$$\tau_\Omega = \inf \{t > 0 : X_t \notin \Omega\}, \tag{2.20}$$

where $X_t \notin \Omega$ means that either $X_t \in M \setminus \Omega$, or $X_t = \infty$.

A Borel set $N \subset M$ is called *invisible* if $\mu(N) = 0$ and

$$\mathbb{P}_x (X_t \in N \text{ or } X_{t-} \in N \text{ for some } t > 0) = 0 \quad \text{for all } x \in M \setminus N.$$

For a fixed parameter $\beta > 1$, consider the following conditions.

[2]Loosely speaking, a *Hunt process* is a strong Markov process whose sample paths are right continuous and have left limit almost surely – see [6], [14], [15] for a detailed definition.

(P_β) : *The exit probability estimate.* There exist an invisible set $N \subset M$ and constants $\varepsilon \in (0,1)$, $\delta > 0$ such that, for all $x \in M \setminus N$ and $r > 0$,

$$\mathbb{P}_x\left(\tau_{B(x,r)} \le \delta r^\beta\right) \le \varepsilon. \tag{2.21}$$

Some equivalent conditions to (P_β) are proved in Section 6.4. In particular, if $\{X_t\}$ is stochastically complete, then $(P_\beta) \Leftrightarrow (S_\beta) \Leftrightarrow (T_\beta)$.

(E_β) : *The mean exit time estimate.* There exist an invisible set $N \subset M$ and positive constants C, c such that, for all $x \in M \setminus N$ and $r > 0$,

$$cr^\beta \le \mathbb{E}_x\left(\tau_{B(x,r)}\right) \le Cr^\beta. \tag{2.22}$$

For example, one has $\mathbb{E}_x\left(\tau_{B(x,r)}\right) = cr^2$ in $\mathbb{R}^n$, and so (E_β) holds with $\beta = 2$. The condition (E_2) is satisfied also for any complete non-compact manifold of non-negative Ricci curvature. For fractal spaces, one usually obtains (E_β) with $\beta > 2$. It is true that

$$(E_\beta) \Rightarrow (P_\beta) \quad \text{and} \quad (P_\beta) \Rightarrow (E_\beta \ge),$$

where $(E_\beta \ge)$ stands for the lower bound in (2.22) (see Section 6.4).

$(E\Omega_\beta)$: *Isoperimetric inequality for the mean exit time.* There exist an invisible set $N \subset M$ and positive constants C, ν such that, for any ball B in M of radius r and for any non-empty open set $\Omega \subset B$,

$$\sup_{x \in \Omega \setminus N} \mathbb{E}_x(\tau_\Omega) \le Cr^\beta \left(\frac{\mu(\Omega)}{\mu(B)}\right)^\nu. \tag{2.23}$$

For example, it is known that for any bounded open set $\Omega \subset \mathbb{R}^n$,

$$\sup_{x \in \Omega} \mathbb{E}_x(\tau_\Omega) \le \sup_{x \in \Omega_*} \mathbb{E}_x(\tau_{\Omega_*})$$

where Ω_* is a ball of the same volume as Ω. If its radius is r, then

$$\sup_{x \in \Omega_*} \mathbb{E}_x(\tau_{\Omega_*}) = c_n r^2 = C\mu(\Omega_*)^{2/n} = C\mu(\Omega)^{2/n},$$

whence (2.23) follows with $\beta = 2$, $\nu = 2/n$, and $N = \emptyset$. Hence, $(E\Omega_2)$ holds in $\mathbb{R}^n$. It follows from a result in [16] that $(E\Omega_2)$ holds also on any complete non-compact Riemannian manifold of non-negative Ricci curvature.

Theorem 2.2. *Let (M, d, μ) be a metric measure space, and let μ satisfy (VD) and (RVD). Let $(\mathcal{E}, \mathcal{F})$ be a regular, local, conservative Dirichlet form in $L^2(M, \mu)$. Then the following equivalences hold:*

$$\begin{aligned}(UE_\beta) \quad &\Leftrightarrow \quad (DUE_\beta) + (P_\beta) \Leftrightarrow (DUE_\beta) + (E_\beta)\\ &\Leftrightarrow \quad (FK_\beta) + (P_\beta) \Leftrightarrow (FK_\beta) + (E_\beta)\\ &\Leftrightarrow \quad (E\Omega_\beta) + (P_\beta) \Leftrightarrow (E\Omega_\beta) + (E_\beta).\end{aligned}$$

Theorem 2.2 will be proved in Section 6.4. Let us give some comments on the statement.

1. We consider the equivalence

$$(UE_\beta) \Leftrightarrow (E\Omega_\beta) + (E_\beta) \tag{2.24}$$

as the most interesting part of Theorem 2.2. Let a ball $B = B(x,r)$ be centered at a point $x \in M \setminus N$. Taking in (2.23) $\Omega = B$, we obtain $\mathbb{E}_x \tau_{B(x,r)} \le Cr^\beta$, which gives the upper bound in condition (E_β) that is,

$$(E\Omega_\beta) \Rightarrow (E_\beta \le). \tag{2.25}$$

Hence, the equivalence (2.24) can be also stated as follows:

$$(UE_\beta) \Leftrightarrow (E\Omega_\beta) + (E_\beta \ge), \tag{2.26}$$

where $(E_\beta \ge)$ stands for the lower bound in (E_β). Condition $(E\Omega_\beta)$ can be considered as an *isoperimetric inequality* for the mean exit time $\mathbb{E}_x \tau_\Omega$. From this point of view, the condition $(E_\beta \ge)$ means that the upper bound for $\mathbb{E}_x \tau_\Omega$ in (2.23) is sharp (up to a constant multiple) and is attained when Ω is a ball and x is its center. Hence, we can shortly state (2.26) as follows:

The heat kernel upper estimate (UE_β) is equivalent to the fact that the isoperimetric inequality $(E\Omega_\beta)$ for the mean exit time holds, and it is sharp for balls (up to a constant multiple).

2. The importance of the condition (E_β) for heat kernel estimates was revealed by Barlow [1, Theorem 3.11]. He proved, in particular, that

$$(DUE_\beta) + (E_\beta) \Rightarrow (DUE_\beta) + (P_\beta) \Rightarrow (UE_\beta)$$

although in a more restricted setting of a regular volume growth and a continuous heat kernel. In our proof of Theorem 2.2 we use the implication $(E_\beta) \Rightarrow (P_\beta)$ from [1]. Barlow [1, Lemma 3.9] also showed that (E_β) follows from two-sided estimates of the heat kernel.

3. Kigami [28] was the first to prove that $(UE_\beta) \Rightarrow (E_\beta)$ although in a more restricted setting than ours. We give here a different proof that goes through a sequence of implications:

$$(UE_\beta) \Rightarrow (S_\beta) \Rightarrow (P_\beta) \Rightarrow (E_\beta \ge)$$

and $(UE_\beta) \Rightarrow (E\Omega_\beta) \Rightarrow (E_\beta \le)$.

4. Note that the hypothesis of conservativeness is used only in the proof of the implication $(P_\beta) \Rightarrow (E_\beta \ge)$ via Theorem 5.8. Without the stochastic completeness, the implication

$$(UE_\beta) \Rightarrow (E_\beta \ge)$$

is not true. Indeed, let $M = \mathbb{R}$ with the Euclidean distance, μ be the Lebesgue measure, and let $\{X_t\}$ be the diffusion process generated by the operator $H = -\frac{d^2}{dx^2} + q(x)$ where q is a positive smooth function on $\mathbb{R}$. The associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ is given by $\mathcal{F} = W_0^1(\mathbb{R})$ and

$$\mathcal{E}(f,g) = \int_{\mathbb{R}} (f'g' + qfg)\, d\mu.$$

Since $q > 0$, the heat kernel of this process satisfies the upper bound (UE_2). Clearly, the stochastic completeness fails because of the killing term. Let us verify that $(E_2 \ge)$ also fails, for example, in the case $q(x) = x^2$. Indeed, in this case the heat kernel of $\{X_t\}$ is given by the explicit expression

$$p_t(x,y) = \frac{1}{(2\pi \sinh 2t)^{1/2}} \exp\left(-\frac{(x-y)^2}{2\sinh 2t} - \frac{1}{2}x^2 \tanh t - \frac{1}{2}y^2 \tanh t\right)$$

(see for example [10]). In particular, noticing that $\frac{1}{\sinh 2t}+\tanh t\geq 1$, we obtain

$$p_t(0,x)\leq\frac{1}{(2\pi\sinh 2t)^{1/2}}\exp\left(-\frac{1}{2}x^2\right)$$

whence

$$\int_0^\infty\int_{\mathbb{R}}p_t(0,x)\,dx\,dt<\infty.$$

By (6.7), the function $r\mapsto\mathbb{E}_0\left(\tau_{B(0,r)}\right)$ is bounded, which makes the lower bound $\mathbb{E}_0\left(\tau_{B(0,r)}\right)\geq cr^2$ impossible.

5. The reverse volume doubling property (RVD) is only used to prove the implication

$$(UE_\beta)\Rightarrow(E_\beta\leq),$$

which is not true without (RVD); in this case, $(UE_\beta)\Rightarrow(E\Omega_\beta)$ is not true either. Indeed, let M be any compact Riemannian manifold with non-negative Ricci curvature (for example, just a sphere). Clearly, M satisfies all the hypotheses[3] of Theorem 2.2 with $\beta=2$, except for (RVD), because $\mu(M)<\infty$. However, condition $(E_\beta\leq)$ fails because the exit time from balls with large radii is ∞.

3. Basics of heat semigroups

In this section, we prove some basic facts about the heat semigroups and heat kernels. One of the most important results is the existence of the heat kernel under the assumption that the heat semigroup is ultracontractive (see Lemma 3.7). It is traditionally deduced from the following abstract result.

Proposition 3.1. *If $K:L^1(M)\to L^\infty(M)$ is a bounded linear operator then it has a measurable integral kernel $k(x,y)$, that is,*

$$Kf(x)=\int_M k(x,y)\,f(y)\,d\mu(y)\tag{3.1}$$

for all $f\in L^1(M)$ and almost all $x\in M$. Moreover,

$$\operatorname*{esup}_{x,y}|k(x,y)|=\|K\|_{1\to\infty}.\tag{3.2}$$

A short proof of this result can be found in [12, T.2.2.7] although under an additional hypothesis that requires the existence of a certain partition of the space M into disjoint subsets similar to the partition of $\mathbb{R}^n$ into dyadic cubes. It is not clear whether such a partition exists in our setting.

A more general setting, that requires only the separability of $L^1(M)$, was considered in [13, Ch.VI, Sect.8, Th.6 and Cor.7]. However, the latter states the existence of a so called vector-valued kernel, that is, of a mapping $k:M\to L^\infty(M)$ that associates with any $y\in M$ a function $k_y\in L^\infty$, such that, for all $f,g\in L^1$,

$$(Kf,g)=\int_M(k_y,g)\,f(y)\,d\mu(y)$$

and $\operatorname{esup}_y\|k_y\|_\infty=\|K\|$. If the function $k_y(x)$ has a jointly measurable in x,y version $k(x,y)$ then one obtains (3.1) and (3.2). However, the existence of a jointly

[3]For example, (UE_2) holds by [30].

measurable realization of $k_y(x)$ requires an additional argument and is not at all automatic. Since the question of a joint measurability of the heat kernel has been invariably neglected in the literature, we have decided to include a complete proof of the existence and the measurability of the heat kernel for ultracontractive semigroups. A key ingredient of the proof is Lemma 3.3 that is an L^2-version of Proposition 3.1. Proposition 3.1 can be then deduced from Lemma 3.3, but we skip the details of that because we need only Lemma 3.3.

3.1. **Bounded operators $L^2 \to L^\infty$.** We say that a couple (X, μ) is a measure space if X is an arbitrary set and μ is a complete σ-finite measure on X. Let (X, μ) and (Y, ν) be two measure spaces. In this section, we assume in addition that the space $L^2(Y)$ is separable.

Let any $x \in X$ be associated with a function $k_x \in L^2(Y)$. Define the following operator K for all $f \in L^2(Y)$:

$$Kf(x) = (k_x, f) := \int_Y k_x f d\nu \tag{3.3}$$

so that Kf is a function on X.

Lemma 3.2. *If, for any non-negative $f \in L^2(Y)$, $Kf(x) \geq 0$ for almost all $x \in X$ then, for almost all $x \in X$, we have $k_x \geq 0$ almost everywhere on Y.*

Proof. For any non-negative function $f \in L^2(Y)$, there is a null set $\mathcal{N}_f \subset X$ such that

$$Kf(x) \geq 0 \quad \text{for all } x \in X \setminus \mathcal{N}_f.$$

Let S be a countable family of non-negative functions on $L^2(Y)$ that is dense in the cone of all non-negative functions in $L^2(Y)$, and set

$$\mathcal{N} = \bigcup_{f \in S} \mathcal{N}_f$$

so that $\mathcal{N}$ is a null set in X. Then, for any $f \in S$ and all $x \in X \setminus \mathcal{N}$, we have $Kf(x) \geq 0$. If f is any other non-negative function in $L^2(Y)$, then f is an L^2-limit of a sequence $\{f_k\} \subset S$, whence, for any $x \in X \setminus \mathcal{N}$,

$$(k_x, f) = \lim_{k\to\infty} (k_x, f_k) = \lim_{k\to\infty} Kf_k(x) \geq 0.$$

Therefore, for any $x \in X \setminus \mathcal{N}$, we have that $k_x(y) \geq 0$ for almost all $y \in Y$, which finishes the proof. $\square$

Lemma 3.3. *Let $K : L^2(Y) \to L^\infty(X)$ be a bounded linear operator, with the norm bounded by c, that is, for any $f \in L^2(Y)$,*

$$\operatorname*{esup}_X |Kf| \leq c \|f\|_2 . \tag{3.4}$$

Then there exists a mapping $k_x : X \to L^2(Y)$ (that is, $k_x \in L^2(Y)$ for any $x \in X$) such that, for all $f \in L^2(Y)$,

$$Kf(x) = (k_x, f) \quad \text{for almost all } x \in X.$$

Moreover, for all $x \in X$,

$$\|k_x\|_{L^2(Y)} \leq c.$$

Furthermore, there is a function $k(x,y)$ *that is jointly measurable in* $(x,y) \in M \times M$ *such that, for almost all* $x \in X$, $k(x,\cdot) = k_x$ *almost everywhere on* Y.

Consequently, we see that, for any $f \in L^2(Y)$ and almost all $x \in X$,

$$Kf(x) = \int_M k(x,y)\, f(y)\, d\nu(y).$$

Proof. For any $f \in L^2(Y)$, Kf is an element of $L^\infty(X)$ and, hence, is defined for μ-*almost* all $x \in X$. We would like to choose a *pointwise* realization of $Kf(x)$ while keeping the linearity of the mapping $f \to Kf(x)$. Denote by $\mathcal{L}^\infty(X)$ the set of all bounded measurable functions on X defined pointwise. Then $\mathcal{L}^\infty(X)$ is a Banach space with the sup-norm (in contrast to $L^\infty(X)$ where the norm is the essential supremum).

We claim that there exists a linear operator $\mathcal{K} : L^2(Y) \to \mathcal{L}^\infty(X)$ such that, for any $f \in L^2(Y)$,

$$\mathcal{K}f(x) = Kf(x) \quad \text{for almost all } x \in X, \tag{3.5}$$

and

$$\sup|\mathcal{K}f| \le c\,\|f\|_2. \tag{3.6}$$

For any measurable function φ on X, which is defined pointwise, consider the following set

$$\mathcal{N}(\varphi) := \{x \in X : |\varphi(x)| > \operatorname{esup}|\varphi|\},$$

which has μ-measure 0 by the definition of the essential supremum. Modifying φ by setting it to be 0 on $\mathcal{N}(\varphi)$ (or on any null set containing $\mathcal{N}(\varphi)$), one achieves that $\sup|\varphi| = \operatorname{esup}|\varphi|$. We use this idea to construct an operator $\mathcal{K} : L^2(Y) \to \mathcal{L}^\infty(X)$ as follows. Let $\{v_j\}_{j=1}^\infty$ be an orthonormal basis in $L^2(Y)$, and let $\mathcal{V}$ be the set of all finite linear combinations of functions v_j with rational coefficients. First define $\mathcal{K}v_j$ to be any pointwise realization of Kv_j, then extend $\mathcal{K}$ to the whole space $\mathcal{V}$ by linearity. Hence, (3.5) holds for all $f \in \mathcal{V}$. Since the set $\mathcal{V}$ is countable and each set $\mathcal{N}(\mathcal{K}f)$ has measure 0, the union $\mathcal{N}_0$ of all sets $\mathcal{N}(\mathcal{K}f)$ over all $f \in \mathcal{V}$ has also measure 0. Now we modify the definition of $\mathcal{K}f$ for every $f \in \mathcal{V}$ by setting $\mathcal{K}f$ to be zero on $\mathcal{N}_0$ (and not changing it outside $\mathcal{N}_0$). Clearly, the linearity of $\mathcal{K}$ and (3.5) are preserved, but we acquire in addition that

$$\sup|\mathcal{K}f| = \operatorname{esup}|Kf| \text{ for all } f \in \mathcal{V},$$

which together with (3.4) implies

$$\sup|\mathcal{K}f| \le c\|f\|_2 \quad \text{for all } f \in \mathcal{V}.$$

Hence, $\mathcal{K}$ is a bounded linear mapping from $(\mathcal{V}, \|\cdot\|_2)$ to $\mathcal{L}^\infty(X)$. Since $\mathcal{V}$ is dense in $L^2(Y)$, $\mathcal{K}$ uniquely extends to a bounded linear mapping from $L^2(Y)$ to $\mathcal{L}^\infty(X)$, which then satisfies (3.5) and (3.6).

Now fix $x \in M$ and observe that by (3.6)

$$|\mathcal{K}f(x)| \le c\,\|f\|_2, \tag{3.7}$$

that is, the mapping $f \mapsto \mathcal{K}f(x)$ is a bounded linear functional in $L^2(Y)$. By the Riesz representation theorem, there exists a function $k_x \in L^2(Y)$ such that, for any $f \in L^2(Y)$,

$$\mathcal{K}f(x) = (k_x, f). \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\|k_x\|_2 \le c \quad \text{for all } x \in X. \tag{3.9}$$

We are left to prove the existence of a jointly measurable in x, y version of $k_x(y)$. Note that the mapping $x \mapsto k_x$ is weakly measurable as a mapping from X to $L^2(Y)$ because for any $f \in L^2(Y)$, the function $(k_x, f) = Kf(x)$ is measurable in x. Since $L^2(Y)$ is separable, by Pettis's measurability theorem (see [37, Ch.V, Sect.4]) the mapping $x \mapsto k_x$ is strongly measurable. Since the norms $\|k_x\|_2$ are uniformly bounded by (3.9), by a Bochner theorem (see [37, Ch.V, Sect.5]) the mapping $x \mapsto k_x$ is Bochner integrable on subsets of X of finite measure. Finally, by [13, Ch.III, Sect.11, Th.17], any Bochner integrable mapping admits a jointly measurable version. □

3.2. **A norm estimate of a bilinear functional.** Let (X, μ) and (Y, ν) be two measure spaces and $\varphi : X \times Y \to \mathbb{R}$ be a measurable function on $X \times Y$ (the measure on $X \times Y$ is $\mu \times \nu$). By a *rectangle* in $X \times Y$ we mean any set R of the form $R = A \times B$ where $A \subset X$ and $B \subset Y$ are measurable sets with finite measures. We write $\varphi \in L^1_{rec}(X \times Y)$ if φ is integrable on any rectangle. For example, $L^1_{rec}(X \times Y)$ contains all $L^p(X \times Y)$, $1 \le p \le \infty$.

Let us use the notation

$$\Phi(f, g) := \int_{X \times Y} \varphi(x, y) f(x) g(y) \, d(\mu \times \nu), \tag{3.10}$$

for those functions f and g for which the integral makes sense. Also, we use notation $a_- = \max\{-a, 0\}$.

Lemma 3.4. *Assume that φ is a measurable function on $X \times Y$ and*

$$\varphi_- \in L^1_{rec}(X \times Y) \tag{3.11}$$

Then

$$\sup_{f \in \mathcal{T}_X, g \in \mathcal{T}_Y} \Phi(f, g) = \operatorname{esup}_{X \times Y} \varphi \tag{3.12}$$

where $\mathcal{T}_X$ is the set of test functions of the form $\frac{1}{\mu(A)} \mathbf{1}_A$ for an arbitrary measurable subset A of X with $0 < \mu(A) < \infty$, and $\mathcal{T}_Y$ is defined similarly.

The proof of this lemma can be found in Appendix at the end of the paper. If $\varphi \in L^\infty(X \times Y)$ then (3.12) is well known. However, it is important for us that φ is not assumed a priori bounded because Lemma 3.4 can be used to prove the boundedness (and upper bounds) of φ using upper bounds of the functional Φ.

Note that, for any function $f \in \mathcal{T}_X$, we have $\|f\|_1 = 1$. Observe also that, due to the hypothesis (3.11), the integral (3.10) is well defined for all $f \in \mathcal{T}_X$ and $g \in \mathcal{T}_Y$ and takes values in $(-\infty, +\infty]$. It is obvious that if

$$f, g \ge 0, \ \ \|f\|_1 = \|g\|_1 = 1, \ \text{and } \Phi(f, g) \ \text{is well defined}, \tag{3.13}$$

then $\Phi(f, g) \le \operatorname{esup} \varphi$. Therefore, for any class of test functions f and g, satisfying (3.13), we have

$$\sup_{f, g} \Phi(f, g) \le \operatorname{esup} \varphi.$$

Hence, the main point of Lemma 3.4 is to ensure the opposite inequality. Once it is established for some classes of test functions f, g, these classes can be enlarged arbitrarily with the only restrictions that they satisfy (3.13). For example, if $\varphi \geq 0$ then (3.12) holds provided $\mathcal{T}_X$ is a class of non-negative functions f with $\|f\|_1 = 1$, and $\mathcal{T}_Y$ is defined similarly.

Let us restate Lemma 3.4 in terms of the operator Φ that is defined by

$$\Phi g(x) = \int_Y \varphi(x,y)\, g(y)\, d\nu(y),$$

for all $g \in \mathcal{T}_Y$.

Corollary 3.5. *Let φ be a measurable function on $X \times Y$ that satisfies (3.11). Assume that, for some $a \in \mathbb{R}$ and for all $g \in \mathcal{T}_Y$,*

$$\Phi g(x) \leq a \text{ for almost all } x \in X.$$

Then $\varphi(x,y) \leq a$ for almost all $(x,y) \in X \times Y$.

Proof. Indeed, for any $f \in \mathcal{T}_X$, we have $\Phi(f,g) \leq a$ whence $\operatorname{esup} \varphi \leq a$ by (3.12). □

3.3. **The notion of a heat kernel.** We are back in the setting of a locally compact separable metric space M with a Radon measure μ with full support. Note that $L^2(M)$ is separable so that the results of Section 3.1 apply.

Let $p_t(x,y)$ be a function of $(t,x,y) \in \mathbb{R}_+ \times M \times M$. We say that $p_t(x,y)$ is a *heat kernel* if it satisfies the following properties:

(i) Measurability: for any $t > 0$, $p_t(x,y)$ is $\mu \times \mu$-measurable in $(x,y) \in M \times M$.

(ii) The Markovian properties: for any $t > 0$, $p_t(x,y) \geq 0$ for μ-almost all $x, y \in M$, and

$$\int_M p_t(x,y) d\mu(y) \leq 1, \tag{3.14}$$

for μ-almost all $x \in M$.

(iii) Symmetry: for any $t > 0$, $p_t(x,y) = p_t(y,x)$ for μ-almost all $x, y \in M$.

(iv) The semigroup property: for all $t, s > 0$,

$$p_{t+s}(x,y) = \int_M p_t(x,z) p_s(z,y) d\mu(z) \tag{3.15}$$

for μ-almost all $x, y \in M$.

(v) The approximation of identity: for any $f \in L^2$,

$$\int_M p_t(x,y)\, f(y)\, d\mu(y) \stackrel{L^2}{\to} f(x) \text{ as } t \to 0+. \tag{3.16}$$

Note that a heat kernel is effectively defined for any $t > 0$ and for *almost* all $x, y \in M$ since changing it at a null-set in $M \times M$ does not affect the properties (i)-(v).

Let $\{P_t\}_{t \geq 0}$ be the heat semigroup in L^2 associated with a Dirichlet form $(\mathcal{E}, \mathcal{F})$. A a function $p_t(x,y)$ on $\mathbb{R}_+ \times M \times M$ is called the *integral kernel* of P_t if $p_t(x,y)$ is non-negative, measurable in $(x,y) \in M \times M$ for any $t > 0$, and the following identity holds

$$P_t f(x) = \int_M p_t(x,y)\, f(y)\, d\mu(y) \tag{3.17}$$

for all $f \in L^2$, $t > 0$, and μ-a.a. $x \in M$.

Lemma 3.6. *(a) If $p_t(x,y)$ and $q_t(x,t)$ are two integral kernels of P_t then, for any $t>0$,*

$$p_t(x,y)=q_t(x,y) \tag{3.18}$$

for almost all $(x,y)\in M\times M$.

(b) If $p_t(x,y)$ is the integral kernel of P_t then $p_t(x,y)$ is a heat kernel.

Hence, the integral kernel of P_t will be referred to as *the heat kernel* of P_t.

Proof. (a) Note that $p_t\in L^1_{rec}(M\times M)$ because for any measurable set $A\subset M$ of finite measure,

$$\int_A p_t(x,y)\,d\mu(y)=P_t1_A(x)\le 1 \tag{3.19}$$

for almost all $x\in M$. For all non-negative $f,g\in L^2$, we have by Fubini's theorem[4],

$$(P_tf,g)=\int_{M\times M} p_t(x,y)\,f(y)\,g(x)\,d\mu(y)d\mu(x)\,. \tag{3.20}$$

Using a similar identity for $q_t(x,y)$, we obtain that

$$\int_{M\times M}(q_t(x,y)-p_t(x,y))\,f(y)\,g(x)\,d\mu(y)d\mu(x)=0.$$

By Lemma 3.4 we conclude that

$$\operatorname*{esup}_{x,y}(q_t(x,y)-p_t(x,y))=0$$

that is $q_t\le p_t$ almost everywhere. In the same way, we have $p_t\le q_t$ whence the identity $p_t=q_t$ follows.

(b) Applying (3.19) to $A=A_n$ where $\{A_n\}$ is an exhaustive sequence of subsets of M with finite measures, we obtain (3.14). Let f,g be two non-negative functions from L^2. Using the symmetry of P_t, we obtain

$$\begin{aligned}(P_tf,g)=(f,P_tg)&=\int_M P_tg(y)\,f(y)\,d\mu(y)\\&=\int_{M\times M}p_t(y,x)\,f(y)\,g(x)\,d\mu(y)d\mu(x).\end{aligned} \tag{3.21}$$

Comparing (3.20) and (3.21) and arguing as in part (a), we obtain $p_t(x,y)=p_t(y,x)$ for almost all x,y.

[4]For the future applications, observe that the identity (3.20) holds not only for non-negative $f,g\in L^2$ but for all $f,g\in L^2$. Indeed, if f,g are signed functions from L^2 then applying (3.20) to $|f|$ and $|g|$, we obtain that the function $p_t(x,y)\,f(y)\,g(x)$ is integrable on $M\times M$, which allows to use Fubini's theorem and to conclude that

$$\begin{aligned}\int_{M\times M}p_t(x,y)\,f(y)\,g(x)\,d\mu(y)\,d\mu(x)&=\int_M\left(\int_M p_t(x,y)\,f(y)\,d\mu(y)\right)g(x)\,d\mu(x)\\&=(P_tf,g)\,.\end{aligned}$$

Using the semigroup identity $P_{t+s} = P_t(P_s)$ and Fubini's theorem, we obtain that, for any non-negative $f \in L^2$ and for μ-a.a. $x \in M$,

$$\begin{aligned} P_{t+s}f(x) &= P_t(P_s f)(x) \\ &= \int_M p_t(x,z)\left(\int_M p_s(z,y)\, f(y)\, d\mu(y)\right) d\mu(z) \\ &= \int_M \left(\int_M p_t(x,z)p_s(z,y)d\mu(z)\right) f(y)\, d\mu(y). \end{aligned}$$

Hence, for any non-negative $g \in L^2$,

$$(P_{t+s}f, g) = \int_{M\times M}\left(\int_M p_t(x,z)p_s(z,y)d\mu(z)\right) f(y)\, g(x)\, d\mu(y)d\mu(x).$$

Comparing with

$$(P_{t+s}f, g) = \int_{M\times M} p_{t+s}(x,y)\, f(y)\, g(x)\, d\mu(y)d\mu(x),$$

and using again Lemma 3.4, we obtain (3.15).

Finally, (3.16) follows immediately from (3.17) and $P_t f \overset{L^2}{\to} f$ as $t \to 0$. $\square$

3.4. **Ultracontractivity and the existence of the heat kernel.** Fix some p, q such that $1 \le p \le q \le +\infty$. A semigroup $\{P_t\}$ in L^2 is said to be $L^p \to L^q$ *ultracontractive* if there exists a positive decreasing function γ on $(0,+\infty)$, called the *rate* function, such that, for each $t > 0$ and for all $f \in L^p \cap L^2$,

$$\|P_t f\|_q \le \gamma(t)\, \|f\|_p. \tag{3.22}$$

Note that if P_t is $L^p \to L^q$ ultracontractive, then P_t is also $L^{q^*} \to L^{p^*}$ ultracontractive with the same rate function, where p^* and q^* are the Hölder conjugates to p and q, respectively. This is because for any $t \ge 0$, the operator $T := P_t$ is symmetric and $\|T^*\| = \|T\|$. In particular, we see that P_t is $L^1 \to L^2$ ultracontractive if and only if it is $L^2 \to L^\infty$ ultracontractive. In this case, we simply say that $\{P_t\}$ is *ultracontractive*.

The next lemma relates the ultracontractivity of $\{P_t\}$ with the existence of a heat kernel satisfying a uniform upper bound. This fact is well known but there hardly exists a reference with a detailed proof matching our setting (see [1], [4], [7], [11], [20], [23], [36] for the proofs in various settings).

Lemma 3.7. *The heat semigroup $\{P_t\}$ is $L^1 \to L^2$ ultracontractive with a rate function γ, if and only if $\{P_t\}$ has the heat kernel p_t satisfying the estimate*

$$\operatorname*{esup}_{x,y\in M} p_t(x,y) \le \gamma(t/2)^2, \tag{3.23}$$

for all $t > 0$.

Proof. If the heat kernel exists and satisfies (3.23), then we have, by (3.20) and (3.23),

$$(P_{2t}f, g) \le \operatorname*{esup}_{x,y\in M} p_{2t}(x,y)\, \|f\|_1 \|g\|_1 \le \gamma(t)^2\, \|f\|_1 \|g\|_1,$$

for all $f, g \in L^1 \cap L^2$. Taking $f = g$ and noticing that $(P_{2t}f, f) = \|P_t f\|_2^2$, we obtain

$$\|P_t f\|_2 \le \gamma(t)\, \|f\|_1,$$

that is, the semigroup $\{P_t\}$ is $L^1 \to L^2$ ultracontractive with the rate function $\gamma(t)$.

Conversely, if P_t is $L^1 \to L^2$ ultracontractive, then P_t is also $L^2 \to L^\infty$ ultracontractive, that is, for any $f \in L^2$ and $t > 0$,

$$\|P_t f\|_\infty \leq \gamma(t)\|f\|_2. \tag{3.24}$$

By Lemma 3.3, the operator P_t has the integral kernel $p_t(x, y)$ (it is non-negative by Lemma 3.2) that satisfies the estimate

$$\operatorname*{esup}_x \|p_t(x, \cdot)\|_2 \leq \gamma(t).$$

By Lemma 3.6, $p_t(x, y)$ is the heat kernel of P_t. Using the semigroup identity (3.15) and the symmetry of the heat kernel, we obtain that, for almost all $x, y \in M$,

$$p_{2t}(x, y) = (p_t(x, \cdot), p_t(y, \cdot)) \leq \|p_t(x, \cdot)\|_2 \|p_t(y, \cdot)\|_2, \tag{3.25}$$

whence (3.23) follows. □

For the sake of completeness, let us prove also a similar result for $L^1 \to L^\infty$ ultracontractivity, although we will not use it.

Corollary 3.8. *The heat semigroup $\{P_t\}$ is $L^1 \to L^\infty$ ultracontractive with a rate function γ, if and only if $\{P_t\}$ has the heat kernel p_t satisfying the estimate*

$$\operatorname*{esup}_{x,y\in M} p_t(x, y) \leq \gamma(t), \tag{3.26}$$

for all $t > 0$.

Proof. If the heat kernel exists and satisfies (3.26) then we have, for any $f \in L^1 \cap L^2$,

$$\|P_t f\|_\infty = \operatorname*{esup}_{x\in M} \left| \int_M p_t(x, y) f(y) d\mu(y) \right| \leq \operatorname*{esup}_{x,y\in M} p_t(x, y) \|f\|_1 \leq \gamma(t) \|f\|_1.$$

Conversely, if the semigroup $\{P_t\}$ is $L^1 \to L^\infty$ ultracontractive then we have, for any $f \in L^2$, that $f^2 \in L^1$ and, hence

$$\left\|P_t\left(f^2\right)\right\|_\infty \leq \gamma(t) \left\|f^2\right\|_1 = \gamma(t) \|f\|_2^2.$$

Since $(P_t f)^2 \leq P_t(f^2)$, we obtain

$$\|P_t f\|_\infty \leq \sqrt{\gamma(t)} \|f\|_2$$

so that P_t is $L^2 \to L^\infty$ ultracontractive. By Lemma 3.7, we conclude that the heat kernel exists. Finally, by the $L^1 \to L^\infty$ ultracontractivity, we have, for all non-negative $f, g \in L^1 \cap L^2$,

$$\int_M \int_M p_t(x, y) f(x) g(y) d\mu(x) d\mu(y) = (P_t f, g) \leq \gamma(t) \|f\|_1 \|g\|_1,$$

whence (3.26) follows by Lemma 3.4. □

3.5. **Decay of** $\operatorname{esup}_U p_t$**.** For any subset $U \subset M$, set

$$\operatorname*{esup}_U p_t := \operatorname*{esup}_{x,y\in U} p_t(x,y).$$

Lemma 3.9. *If the heat kernel p_t exists, then for any set $U \subset M$, the function $t \mapsto \operatorname{esup}_U p_t$ is non-increasing on $(0,+\infty)$. Also, for any two sets $U,V \subset M$,*

$$\operatorname*{esup}_{x\in V, y\in U} p_t(x,y) \le \left(\operatorname*{esup}_V p_t \cdot \operatorname*{esup}_U p_t\right)^{1/2}. \tag{3.27}$$

Proof. For an open set U, define the class of test functions $\mathcal{T}(U)$ by

$$\mathcal{T}(U) = \left\{f \in L^1(U) \cap L^2(U) : \|f\|_1 = 1\right\}.$$

By Lemma 3.4, we have

$$\begin{aligned}\operatorname*{esup}_{x\in V, y\in U} p_t(x,y) &= \sup_{\substack{f\in\mathcal{T}(V)\\ g\in\mathcal{T}(U)}} \int_U\int_V p_t(x,y) f(x) g(y)\, d\mu(x)\, d\mu(y)\\ &= \sup_{\substack{f\in\mathcal{T}(V)\\ g\in\mathcal{T}(U)}} (P_t f, g). \end{aligned}\tag{3.28}$$

The symmetry of P_t and the semigroup property imply

$$(P_t f, g) = \left(P_{t/2} f, P_{t/2} g\right) \le \|P_{t/2} f\|_2 \|P_{t/2} g\|_2 = (P_t f, f)^{1/2} (P_t g, g)^{1/2},$$

whence

$$\sup_{\substack{f\in\mathcal{T}(V)\\ g\in\mathcal{T}(U)}} (P_t f, g) \le \left(\sup_{f\in\mathcal{T}(V)} (P_t f, f) \sup_{g\in\mathcal{T}(U)} (P_t g, g)\right)^{1/2}. \tag{3.29}$$

Applying this to the case $U = V$, we obtain

$$\sup_{f,g\in\mathcal{T}(U)} (P_t f, g) \le \sup_{f\in\mathcal{T}(U)} (P_t f, f).$$

Since the opposite inequality is trivial, we have in fact the identity

$$\sup_{f,g\in\mathcal{T}(U)} (P_t f, g) = \sup_{f\in\mathcal{T}(U)} (P_t f, f),$$

which implies

$$\operatorname*{esup}_{x,y\in U} p_t(x,y) = \sup_{f\in\mathcal{T}(U)} (P_t f, f). \tag{3.30}$$

Combining (3.28), (3.29), and (3.30), we obtain (3.27).

Finally, let us show that $(P_t f, f) = \|P_{t/2} f\|_2^2$ is non-increasing in $t > 0$, which will finish the proof. It follows from (2.3) that, for all $f, g \in \mathcal{F}$,

$$\frac{d}{dt}(P_t f, g)|_{t=0} = -\mathcal{E}(f,g).$$

Since $P_t f \in \mathcal{F}$ for any $t > 0$ and $f \in L^2$, it follows that

$$\frac{d}{dt}(P_t f, g) = -\mathcal{E}(P_t f, g)$$

for all $t > 0$. Therefore, we obtain

$$\frac{d}{dt}\|P_t f\|_2^2 = \frac{d}{dt}(P_t f, P_t f) = -2\mathcal{E}(P_t f, P_t f) \le 0, \tag{3.31}$$

which was to be proved. $\square$

4. Comparison of heat semigroups in different domains

In this section we prove comparison inequalities for heat semigroups and heat kernels in different domains (Corollaries 4.4, 4.5 and Theorem 4.6). The main tool in the proofs is the parabolic maximum principle of [21, Proposition 4.11].

4.1. A maximum principle and its applications.

Let I be an interval in $\mathbb{R}$, and let Ω be an open subset of M. A function $u : I \to \mathcal{F}$ is said to be a *weak subsolution* (resp. a *weak supersolution*) of the heat equation in $I \times \Omega$ if the derivative $\frac{\partial u}{\partial t}$ of u exists in I in the norm topology of $L^2(\Omega)$ and, for any $t \in I$ and any non-negative function $\psi \in \mathcal{F}(\Omega)$,

$$\left(\frac{\partial u}{\partial t}(t,\cdot),\psi\right) + \mathcal{E}\left(u(t,\cdot),\psi\right) \le 0 \quad \text{(resp. } \ge 0\text{)}. \tag{4.1}$$

If the inequality in (4.1) is replaced by equality, then u is called a *weak solution* of the heat equation in $I \times \Omega$. It is known that $P_t f$ is a weak solution in $(0,\infty)\times\Omega$ for any open $\Omega \subset M$ (cf. [21, Example 4.10]).

Proposition 4.1 (parabolic maximum principle [21])**.** *Let u be a weak subsolution of the heat equation in $(0,T)\times\Omega$, where $T \in (0,+\infty]$ and Ω is an open subset of M. Assume in addition that u satisfies the following boundary and initial conditions:*

- *$u_+(t,\cdot) \in \mathcal{F}(\Omega)$ for any $t \in (0,T)$;*
- *$u_+(t,\cdot) \overset{L^2(\Omega)}{\longrightarrow} 0$ as $t \to 0$.*

Then $u(t,x) \le 0$ for any $t \in (0,T)$ and μ-almost all $x \in \Omega$.

Remark 4.2. It was shown in [21, Lemma 4.4] that, for a regular Dirichlet form $(\mathcal{E},\mathcal{F})$, if $u \in \mathcal{F}$ and if $u \le v$ for some $v \in \mathcal{F}(\Omega)$, then $u_+ \in \mathcal{F}(\Omega)$. We will frequently use this result later on.

We use the maximum principle to prove the following lemma.

Lemma 4.3. *Assume that $(\mathcal{E},\mathcal{F})$ is regular and local. Let $u(t,x)$ be a weak subsolution of the heat equation in $(0,\infty)\times U$, where U is an open subset of M. Assume further, for any $t>0$, $u(t,\cdot)$ is bounded in M and is non-negative in U. If*

$$u(t,\cdot) \overset{L^2(U)}{\longrightarrow} 0 \text{ as } t \to 0 \tag{4.2}$$

then the following inequality hold for all $t>0$ and almost all $x \in U$:

$$u(t,x) \le \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{0<s\le t} \|u(s,\cdot)\|_{L^\infty(U)}. \tag{4.3}$$

Proof. We first assume that U is precompact. Choose an open set W such that $W \Subset U$. Fix a real $T>0$ and set

$$m := \sup_{0<s\le T} \|u(s,\cdot)\|_{L^\infty(U)}. \tag{4.4}$$

We show that, for all $0<t\le T$ and μ-almost all $x \in W$,

$$u(t,x) \le m\left(1 - P_t^W \mathbf{1}_W(x)\right). \tag{4.5}$$

Let ζ and η be cut-off functions[5] of the couples (W,U) and (U,M), respectively. Consider the function

$$w := \zeta u - m\left[\eta - P_t^W \mathbf{1}_W\right]. \tag{4.6}$$

Then (4.5) will follow if we prove that $w \le 0$ in $(0,T]\times W$.

Claim 1. The function w is a weak subsolution of the heat equation in $(0,\infty)\times W$.

Clearly, $P_t^W \mathbf{1}_W$ is a weak solution of the heat equation in $(0,\infty)\times W$. Let us show that so is ζu. Indeed, the product ζu belongs to $\mathcal{F}$ because both ζ and u are in $L^\infty \cap \mathcal{F}$. For any test function $\psi \in \mathcal{F}(W)$, we have, using $\zeta\psi \equiv \psi$,

$$\begin{aligned}\left(\frac{\partial(\zeta u)}{\partial t},\psi\right) &= \left(\zeta\frac{\partial}{\partial t}u,\psi\right) = \left(\frac{\partial}{\partial t}u,\psi\right)\\ &= -\mathcal{E}(u,\psi) = -\mathcal{E}(\zeta u,\psi) + \mathcal{E}((\zeta-1)u,\psi)\\ &= -\mathcal{E}(\zeta u,\psi),\end{aligned}$$

where we have used also that $(\zeta - 1)u = 0$ in W and, hence,

$$\mathcal{E}((\zeta-1)u,\psi) = 0,$$

by the locality of $(\mathcal{E},\mathcal{F})$. Thus, ζu is a weak solution in $(0,\infty)\times W$.

Finally, the function $\eta(x)$ considered as a function of (t,x), is a weak supersolution of the heat equation in $(0,\infty)\times W$, since for any non-negative $\psi \in \mathcal{F}(W)$

$$\mathcal{E}(\eta,\psi) = \lim_{t\to 0} t^{-1}(\eta - P_t\eta,\psi) = \lim_{t\to 0} t^{-1}(1 - P_t\eta,\psi) \ge 0,$$

whence it follows that w is a weak subsolution.

Claim 2. For every $t \in (0,T]$, we have $(w(t,\cdot))_+ \in \mathcal{F}(W)$.

By Remark 4.2, it suffices to prove that in $(0,T]\times M$

$$w(t,\cdot) \le m P_t^W \mathbf{1}_W, \tag{4.7}$$

because $mP_t^W\mathbf{1}_W \in \mathcal{F}(W)$. In $M\setminus U$, inequality (4.7) holds trivially because

$$\zeta = 0 = P_t^W \mathbf{1}_W \quad \text{in } M\setminus U$$

and, hence, $w = -m\eta \le 0$. To prove (4.7) in U, observe that $\eta = 1$ in U and $0 \le u \le m$ in $(0,T]\times U$, whence

$$w = \zeta u - m + mP_t^W\mathbf{1}_W \le u - m + mP_t^W\mathbf{1}_W \le mP_t^W\mathbf{1}_W,$$

which was to be proved.

Claim 3. The function w satisfies the initial condition

$$w(t,\cdot) \overset{L^2(W)}{\longrightarrow} 0 \text{ as } t\to 0. \tag{4.8}$$

Noticing that $\eta = 1$ in W, we see that

$$\eta - P_t^W\mathbf{1}_W = \mathbf{1}_W - P_t^W\mathbf{1}_W \overset{L^2(W)}{\longrightarrow} 0 \text{ as } t\to 0.$$

Combining with (4.2), we obtain (4.8).

By the parabolic maximum principle (cf. Prop. 4.1), we obtain from Claims 1-3 that $w \le 0$ in $(0,T]\times W$, thus proving (4.5).

[5] A cut-off function of the couple (W,U) is a function $\zeta \in \mathcal{F}\cap C_0(M)$ such that $0 \le \zeta \le 1$ in M, $\zeta = 1$ on an open neighborhood of $\overline{W}$, and $\operatorname{supp}\zeta \subset U$. If $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form then a cut-off function exists for any couple (W,U) provided U is open and $\overline{W}$ is a compact subset of U (cf. [15, p.27]).

Finally, let U be an arbitrary open subset of M. Let $\{W_i\}_{i=1}^{\infty}$ and $\{U_i\}_{i=1}^{\infty}$ be two increasing sequences of precompact open sets, both of which exhaust U, and such that $W_i \Subset U_i$ for all i. For each i, we have by (4.5) with $t = T$ that in W_i

$$u \leq \left[1 - P_t^{W_i} \mathbf{1}_{W_i}\right] \sup_{0<s\leq t} \|u(s, \cdot)\|_{L^\infty(U_i)}. \tag{4.9}$$

Replacing by the monotonicity in the right hand side U_i by U, and noticing that

$$P_t^{W_i} \mathbf{1}_{W_i} \xrightarrow{\text{a.e.}} P_t^U \mathbf{1}_U \text{ as } i \to \infty,$$

we obtain (4.3) by letting $i \to \infty$ in (4.9). □

Corollary 4.4. *Assume that $(\mathcal{E}, \mathcal{F})$ is regular and local. Let $U \subset \Omega$ be two open subsets of M. Then, for any non-negative function $f \in L^2(\Omega)$, for all $t > 0$ and μ-almost all $x \in U$,*

$$P_t^\Omega f(x) - P_t^U f(x) \leq \left[1 - P_t^U \mathbf{1}_U(x)\right] \sup_{0<s\leq t} \left\| P_s^\Omega f \right\|_{L^\infty(U)}. \tag{4.10}$$

Proof. It suffices to prove (4.10) for non-negative functions $f \in L^2 \cap L^\infty(\Omega)$ because for a general non-negative $f \in L^2(\Omega)$, one first applies (4.10) to bounded functions $f_n = f \wedge n$ for any $n = 1, 2, \ldots$ and then let $n \to \infty$.

Fix such a function f, set

$$u(t, x) = P_t^\Omega f(x) - P_t^U f(x)$$

and observe that u is a non-negative bounded weak solution of the heat equation in $(0, \infty) \times U$ satisfying the initial condition (4.2). By Lemma 4.3, we conclude that u satisfies (4.3), whence (4.10) follows. □

Corollary 4.5. *Assume that $(\mathcal{E}, \mathcal{F})$ is regular and strongly local. Let $U \subset \Omega$ be two open subsets of M. Then the following inequality holds for all $t > 0$ and μ-almost all $x \in U$:*

$$1 - P_t^\Omega \mathbf{1}_\Omega(x) \leq \left(1 - P_t^U \mathbf{1}_U(x)\right) \sup_{0<s\leq t} \left\| 1 - P_s^\Omega \mathbf{1}_\Omega \right\|_{L^\infty(U)}. \tag{4.11}$$

Proof. Approximating U by precompact open subsets, it suffices to prove the claim in the case when $U \Subset \Omega$. Let φ be a cut-off function of the couple (U, Ω). Then we can replace the term $1 - P_t^\Omega \mathbf{1}_\Omega(x)$ in the both sides of (4.11) by the function

$$u(t, x) = \varphi(x) - P_t^\Omega \mathbf{1}_\Omega(x).$$

Clearly, for any $t > 0$, the function $u(t, \cdot)$ is bounded in M, non-negative in U, and satisfies the initial condition (4.2). Let us verify that $u(t, x)$ is a weak solution of the heat equation in $(0, \infty) \times U$. It suffices to show that the function $\varphi(x)$ as a function of (t, x) is a weak solution in $(0, \infty) \times U$. Indeed, since φ is constant in a neighborhood of $\overline{U}$, the strong locality of $(\mathcal{E}, \mathcal{F})$ yields that $\mathcal{E}(\varphi, \psi) = 0$ for any $\psi \in \mathcal{F}(U)$, which finishes the proof. □

4.2. **Comparison of heat kernels.** We now give a comparison inequality of two Dirichlet heat kernels in distinct domains that will be used in Section 5 for obtaining on-diagonal upper estimate of the global heat kernel. For simplicity, write

$$\operatorname*{esup}_U p_t^\Omega := \operatorname*{esup}_{x,y\in U} p_t^\Omega(x, y).$$

Theorem 4.6. *Assume that $(\mathcal{E},\mathcal{F})$ is regular and local. Let $U \subset \Omega$ be two open subsets of M. If the Dirichlet heat kernels p_t^U, p_t^Ω exist, then the following inequality holds*

$$\operatorname*{esup}_{y\in U} p_{t+s}^\Omega(x,y) \le \operatorname*{esup}_{y\in U} p_t^U(x,y) + \left[1 - P_t^U \mathbf{1}_U(x)\right] \operatorname*{esup}_U p_s^\Omega \tag{4.12}$$

for any $t,s>0$ and μ-almost every $x\in U$.

If the indeterminate form $0\cdot\infty$ occurs in the last term in (4.12) then we understand it as 0.

Let V be an open subset of U. Taking in (4.12) esup in $x \in V$, we obtain that, for all $t,s>0$,

$$\operatorname*{esup}_V p_{t+s}^\Omega \le \operatorname*{esup}_U p_t^U + \operatorname*{esup}_{x\in V}\left[1 - P_t^U \mathbf{1}(x)\right] \operatorname*{esup}_U p_s^\Omega. \tag{4.13}$$

This inequality will be used in Section 5.

Proof. Fix $t,s>0$. Assume first that $\operatorname{esup}_U p_s^\Omega < \infty$. Choose a non-negative function $f \in L^1 \cap L^2(U)$. Using the inequality (4.10) of Corollary 4.4 with $P_s^\Omega f$ instead of f, we obtain that, for μ-almost all $x\in U$,

$$P_{t+s}^\Omega f(x) \le P_t^U\left(P_s^\Omega f\right)(x) + \left[1 - P_t^U \mathbf{1}_U(x)\right] \sup_{s\le\lambda\le t+s}\left\|P_\lambda^\Omega f\right\|_{L^\infty(U)}. \tag{4.14}$$

Note that

$$P_t^U\left(P_s^\Omega f\right)(x) = \int_U p_t^U(x,z) P_s^\Omega f(z)\,d\mu(z) = \int_U q(x,y)\, f(y)\, d\mu(y),$$

where

$$q(x,y) := \int_U p_t^U(x,z)\, p_s^\Omega(z,y)\, d\mu(z). \tag{4.15}$$

By Lemma 3.9, we have

$$\operatorname*{esup}_U p_\lambda^\Omega \le \operatorname*{esup}_U p_s^\Omega$$

for any $s\le\lambda$, and so

$$\sup_{s\le\lambda\le t+s}\left\|P_\lambda^\Omega f\right\|_{L^\infty(U)} \le \|f\|_1 \sup_{s\le\lambda\le t+s} \operatorname*{esup}_U p_\lambda^\Omega \le \|f\|_1 \operatorname*{esup}_U p_s^\Omega.$$

Therefore, it follows from (4.14) that, for almost all $x\in U$,

$$\int_U p_{t+s}^\Omega(x,y)\, f(y)\, d\mu(y) \le \int_U q(x,y)\, f(y)\, d\mu(y) + K\left[1 - P_t^U \mathbf{1}_U(x)\right] \|f\|_1, \tag{4.16}$$

where $K = \operatorname{esup}_U p_s^\Omega$. Next, we will apply Corollary 3.5 with function

$$\varphi(x,y) = p_{t+s}^\Omega(x,y) - q(x,y) - K\left[1 - P_t^U \mathbf{1}_U(x)\right].$$

Observe that $\varphi_- \in L^1_{rec}(U\times U)$ (cf. Lemma 3.4), which follows from

$$q(x,y) \le \int_\Omega p_t^\Omega(x,z)\, p_s^\Omega(z,y)\, d\mu(z) = p_{t+s}^\Omega(x,y)$$

and $p_{t+s}^\Omega \in L^1_{rec}(\Omega\times\Omega)$ (cf. the proof of Lemma 3.6). Hence, by Corollary 3.5, (4.16) implies that, for almost all $x,y\in U$,

$$p_{t+s}^\Omega(x,y) \le q(x,y) + K\left[1 - P_t^U \mathbf{1}_U(x)\right]. \tag{4.17}$$

Taking the essential supremum in $y \in U$ and noticing that by (4.15)

$$\operatorname*{esup}_{y\in U} q(x,y) \le \operatorname*{esup}_{z\in U} p_t^U(x,z), \tag{4.18}$$

we obtain (4.12).

Assume now that $\operatorname{esup}_U p_s^\Omega = \infty$. Fix some pointwise version of $P_t^U \mathbf{1}_U$ and consider the set

$$W = \left\{ x \in U : P_t^U \mathbf{1}_U(x) = 1 \right\}.$$

If $x \in U \setminus W$ then the last term in (4.12) is ∞ and (4.12) is trivially satisfied. In order to prove (4.12) in W, it suffices to show that, for almost all $x \in W$,

$$\operatorname*{esup}_{y\in U} p_{t+s}^\Omega(x,y) \le \operatorname*{esup}_{y\in U} p_t^U(x,y). \tag{4.19}$$

Indeed, for any measurable set $A \subset \Omega$, we have that, in W,

$$\begin{aligned} P_t^U \mathbf{1}_A \le P_t^\Omega \mathbf{1}_A &= P_t^\Omega \mathbf{1}_\Omega - P_t^\Omega \mathbf{1}_{\Omega\setminus A} \\ &\le 1 - P_t^U \mathbf{1}_{U\setminus A} = P_t^U \mathbf{1}_U - P_t^U \mathbf{1}_{U\setminus A} = P_t^U \mathbf{1}_A, \end{aligned}$$

which implies that $P_t^\Omega \mathbf{1}_A = P_t^U \mathbf{1}_A$ in W. It follows by an approximation argument that, for any $h \in L^2(\Omega)$,

$$P_t^\Omega h = P_t^U h \quad \text{in } W. \tag{4.20}$$

Choosing $h = P_s^\Omega f$ where f is as above, we obtain that, for μ-almost all $x \in W$,

$$P_{t+s}^\Omega f(x) = P_t^U P_s^\Omega f(x) = \int_U q(x,y) f(y) \, d\mu(y),$$

where q is given by (4.15). Applying Corollary 3.5 and (4.18), we obtain (4.19). $\square$

4.3. **Comparison estimate with localization in space.**

Proposition 4.7. *Assume that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form. Let U,Ω be two open subsets of M, and let K be a closed subset of M with $K \subset U$. Then, for any non-negative function $f \in L^2 \cap L^\infty(\Omega)$, for all $t > 0$ and μ-almost all $x \in \Omega$,*

$$P_t^\Omega f(x) - P_t^U f(x) \le \sup_{0<s\le t} \left\| P_s^\Omega f \right\|_{L^\infty(\Omega\setminus K)}. \tag{4.21}$$

Remark 4.8. For the case when K is compact, this statement was proved in [21, Lemma 4.18]. The extension to an arbitrary closed set K is rather non-trivial as one can see from the proof below.

Before we proceed to the proof of Proposition 4.7, we prove two auxiliary statements. We use a sign '$\rightharpoonup$' to denote the weak convergence in a Hilbert space.

Proposition 4.9. *Let $\{u_k\}$ be a sequence of functions from $\mathcal{F}$ such that $u_k \overset{L^2}{\rightharpoonup} u \in \mathcal{F}$ as $k \to \infty$. If in addition the numerical sequence $\{\mathcal{E}(u_k)\}$ is bounded then $u_k \overset{\mathcal{F}}{\rightharpoonup} u$ as $k \to \infty$.*

Proof. Renaming $u_k - u$ to u_k, we can assume that $u = 0$. We need to prove that u_k converges to 0 weakly in $\mathcal{F}$, that is, for any $\varphi \in \mathcal{F}$,

$$\mathcal{E}_1(u_k, \varphi) \to 0 \text{ as } k \to \infty. \tag{4.22}$$

Let Δ be the generator of the form $(\mathcal{E},\mathcal{F})$ and $\mathcal{D}$ be its domain. It is easy to verify that (4.22) holds for any $\varphi\in\mathcal{D}$. Indeed, since $\varphi,\Delta\varphi\in L^2$ and $u_k\overset{L^2}{\rightharpoonup}0$, it follows that

$$\mathcal{E}_1(u_k,\varphi)=\mathcal{E}(u_k,\varphi)+(u_k,\varphi)=-(u_k,\Delta\varphi)+(u_k,\varphi)\to 0.$$

Since $\mathcal{D}$ is dense in $\mathcal{F}$, for any function $\varphi\in\mathcal{F}$, there exists a sequence $\{\varphi_j\}\subset\mathcal{D}(\Delta)$ such that $\varphi_j\to\varphi$ in $\mathcal{F}$ as $j\to\infty$. Let C be a constant that bounds all $\mathcal{E}(u_k)$. By the Cauchy-Schwarz inequality, we have

$$\left|\mathcal{E}_1\left(u_k,\varphi-\varphi_j\right)\right|\le\sqrt{\mathcal{E}_1\left(u_k\right)\mathcal{E}_1\left(\varphi-\varphi_j\right)}\le C\sqrt{\mathcal{E}_1\left(\varphi-\varphi_j\right)},$$

whence

$$\begin{aligned}\left|\mathcal{E}_1\left(u_k,\varphi\right)\right|&=\left|\mathcal{E}_1\left(u_k,\varphi_j\right)+\mathcal{E}_1\left(u_k,\varphi-\varphi_j\right)\right|\\&\le\left|\mathcal{E}_1\left(u_k,\varphi_j\right)\right|+C\sqrt{\mathcal{E}_1\left(\varphi-\varphi_j\right)}.\end{aligned}$$

Letting $k\to\infty$ and then letting $j\to\infty$, we obtain (4.22). □

Proposition 4.10. *Let Ω be a precompact open subset of M, U be an open subset of M, and K be a closed subset of U. If $g\in\mathcal{F}(\Omega)\cap L^\infty$ and $g\le\psi$ in $\Omega\setminus K$ for some $0\le\psi\in\mathcal{F}\cap C_0(M)$, then*

$$(g-\psi)_+\in\mathcal{F}\left(\Omega\cap U\right).$$

Proof. Assume first that $g\in\mathcal{F}\cap C_0(\Omega)$. Since $\psi\ge0$ and $g\le\psi$ in $\Omega\setminus K$, we see that

$$\operatorname{supp}(g-\psi)_+\subset(\operatorname{supp}g)\cap K\Subset\Omega\cap U.$$

Noting that $(g-\psi)_+\in\mathcal{F}\cap C_0(M)$, we obtain that

$$(g-\psi)_+\in\mathcal{F}\left(\Omega\cap U\right). \tag{4.23}$$

For a general function $g\in\mathcal{F}(\Omega)\cap L^\infty$, choose a sequence $\{g_k\}$ from $\mathcal{F}\cap C_0(\Omega)$ such that $g_k\overset{\mathcal{F}}{\to}g$ as $k\to\infty$. Since g is bounded, say by a constant C, we can assume all g_k are also bounded by C; otherwise, we can replace g_k by $(-C)\vee g_k\wedge C$, using the fact that

$$(-C)\vee g_k\wedge C\overset{\mathcal{F}}{\longrightarrow}g$$

(cf. [15, Theorem 1.4.2 (v)]). Choose a cut-off function φ of the couple $(K\cap\overline{\Omega},U)$ and set

$$\widetilde{g}_k:=\left(g_k\wedge\psi\right)\left(1-\varphi\right)+g_k\varphi. \tag{4.24}$$

Since $\widetilde{g}_k\in\mathcal{F}\cap C(M)$, and $\operatorname{supp}\widetilde{g}_k\subset\operatorname{supp}g_k\Subset\Omega$, we see that $\widetilde{g}_k\in\mathcal{F}\cap C_0(\Omega)$. Define the set

$$\widetilde{K}:=K\cup\operatorname{supp}\varphi$$

(see Fig. 1).

Clearly $\widetilde{K}$ is a closed subset of U. Observe that $\widetilde{g}_k\le\psi$ in $\Omega\setminus\widetilde{K}$. Applying the above argument to the closed set $\widetilde{K}$, we obtain that

$$\left(\widetilde{g}_k-\psi\right)_+\in\mathcal{F}\left(\Omega\cap U\right). \tag{4.25}$$

By the weak closedness of $\mathcal{F}\left(\Omega\cap U\right)$ in $\mathcal{F}$, it remains to show that

$$\left(\widetilde{g}_k-\psi\right)_+\overset{\mathcal{F}}{\rightharpoonup}\left(g-\psi\right)_+.$$

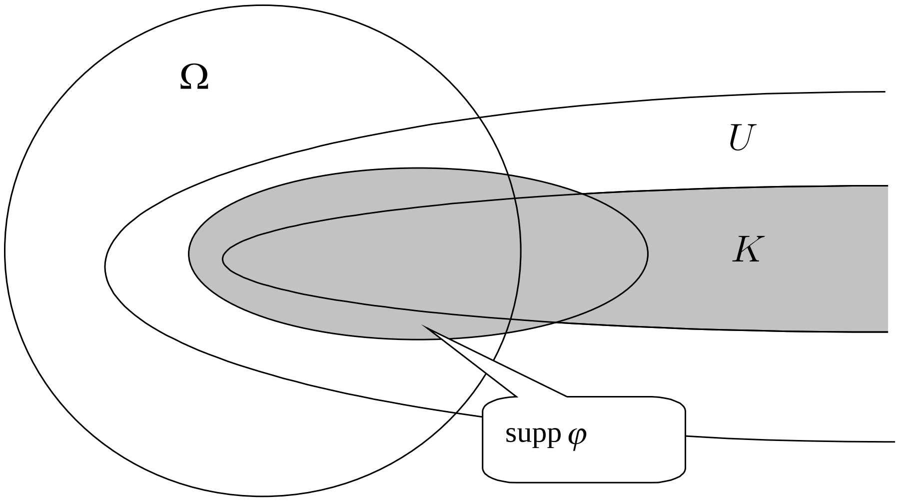

FIGURE 1. Set Ω, U, K and $\widetilde{K} = K \cup \operatorname{supp} \varphi$

By Proposition 4.9, it suffices to verify that

$$(\widetilde{g}_k - \psi)_+ \xrightarrow{L^2} (g - \psi)_+ \quad \text{as } k \to \infty, \tag{4.26}$$

$$\mathcal{E}\left((\widetilde{g}_k - \psi)_+\right) \le C \quad \text{for some constant } C \text{ and for all } k. \tag{4.27}$$

To prove (4.26), it is enough to show that

$$\widetilde{g}_k \xrightarrow{L^2} g \quad \text{as } k \to \infty.$$

Since $g_k \overset{L^2}{\to} g$, we see that

$$\widetilde{g}_k = (g_k \wedge \psi)(1 - \varphi) + g_k \varphi \xrightarrow{L^2} (g \wedge \psi)(1 - \varphi) + g\varphi =: \widetilde{g}.$$

Let us verify that $g \equiv \widetilde{g}$. Indeed, on K we have $\varphi = 1$ in K and hence $\widetilde{g} = g$; in $\Omega \setminus K$ we have $g \le \psi$ and hence

$$\widetilde{g} = g(1 - \varphi) + g\varphi = g;$$

finally, outside Ω, we have $g = 0$ whence also $\widetilde{g} = 0$. Therefore, (4.26) is proved.

To prove (4.27), note that $(\widetilde{g}_k - \psi)_+$ is obtained from g_k, ψ by using a finite number of the following operations on functions from $\mathcal{F} \cap L^\infty$:

(1) addition and subtraction;
(2) taking a positive part;
(3) taking minimum of two functions;
(4) multiplication by $\varphi \in \mathcal{F} \cap C_0$.

In each of these operations, the energy of the outcome is controlled by the energy of the input entries as follows.

(1) If $u, v \in \mathcal{F}$ then

$$\mathcal{E}(u \pm v) \le 2\mathcal{E}(u) + 2\mathcal{E}(v)$$

(2) If $u \in \mathcal{F}$, then $\mathcal{E}(u_+) \le \mathcal{E}(u)$.

(3) If $u, v \in \mathcal{F}$ then

$$\begin{aligned}\mathcal{E}(u \wedge v) &= \frac{1}{4}\mathcal{E}(u+v-|u-v|) \\ &\leq \frac{1}{2}(\mathcal{E}(u+v)+\mathcal{E}(u-v)) \\ &\leq 2(\mathcal{E}(u)+\mathcal{E}(v)).\end{aligned}$$

(4) If $u \in \mathcal{F} \cap L^\infty$ then

$$\mathcal{E}(u\varphi)^{1/2} \leq \|\varphi\|_\infty \mathcal{E}(u)^{1/2} + \|u\|_\infty \mathcal{E}(\varphi)^{1/2}.$$

It follows that $\mathcal{E}\left((\widetilde{g}_k - \psi)_+\right)$ is bounded in terms of $\mathcal{E}(g_k), \mathcal{E}(\psi), \mathcal{E}(\varphi), \|g_k\|_\infty, \|\varphi\|_\infty$, which implies that $\mathcal{E}\left((\widetilde{g}_k - \psi)_+\right)$ is uniformly bounded for all k, which proves (4.27). □

We are now in a position to prove Proposition 4.7.

Proof of Proposition 4.7. We can assume that Ω is precompact, because in the general case one can exhaust Ω by a sequence of precompact open subsets $\{\Omega_k\}$, apply (4.21) to each Ω_k and then pass to the limit as $k \to \infty$.

In $\Omega \setminus U$ the inequality (4.21) is trivially satisfied so that it suffices to verify it in $\Omega \cap U$. Fix $T > 0$, set

$$m := \sup_{0<s\leq T} \left\|P_s^\Omega f\right\|_{L^\infty(\Omega\setminus K)}, \tag{4.28}$$

and prove that

$$P_t^\Omega f - P_t^U f \leq m \text{ in } (0,T] \times (\Omega \cap U), \tag{4.29}$$

which will imply (4.21). Let us first state the idea of the proof assuming that M is a Riemannian manifold and functions $P_t^\Omega f$ and $P_t^U f$ are continuous up to the boundaries of Ω and U, respectively. Then the function $v = P_t^\Omega f - P_t^U f$ satisfies in $\Omega \cap U$ the heat equation, the initial condition $v(t, \cdot) \to 0$ as $t \to 0$ and the boundary condition

$$v \leq m \text{ on } (0,T] \times \partial(\Omega \cap U),$$

because $v \leq 0$ on $\partial\Omega$ and $v \leq m$ on $\partial U \cap \Omega$ by (4.28). By the classical parabolic maximum principle, we obtain $v \leq m$ in $(0,T] \times (\Omega \cap U)$.

In the general case, let η be a cut-off function of the couple (Ω, M). Set

$$u := P_t^\Omega f - P_t^U f - m\eta$$

and show that u is non-positive in $(0,T] \times (\Omega \cap U)$, which will settle (4.29). It is easy to see that u is a weak subsolution of the heat equation in $(0,\infty) \times (\Omega \cap U)$ (cf. the proof of Lemma 4.3) and that

$$u(t,\cdot)_+ \overset{L^2(\Omega\cap U)}{\longrightarrow} 0 \text{ as } t \to 0.$$

We are left to verify that

$$u(t,\cdot)_+ \in \mathcal{F}(\Omega \cap U) \text{ for all } t \in (0,T] \tag{4.30}$$

because then the parabolic maximum principle yields $u \leq 0$. First observe that, for $t \in (0, T]$,

$$\begin{aligned} P_t^\Omega f &\in \mathcal{F}(\Omega) \cap L^\infty, \\ 0 &\leq m\eta \in \mathcal{F} \cap C_0(M), \\ P_t^\Omega f &\leq m\eta \quad \text{in } \Omega \setminus K, \end{aligned}$$

where the latter holds by the definition of m. Applying Proposition 4.10 with $g = P_t^\Omega f$ and $\psi = m\eta$, we conclude that

$$\left(P_t^\Omega f - m\eta\right)_+ \in \mathcal{F}(\Omega \cap U).$$

Since $u \leq \left(P_t^\Omega f - m\eta\right)_+$, we see that (4.30) follows by Remark 4.2. □

5. Analytic characterization of (UE_β)

In this section we present the proof of the main Theorem 2.1, which is preceded by a number of auxiliary results.

5.1. **Volume doubling.** Let (M, d) be a metric space, and let μ be a Borel measure on M. The following lemmas are well-known in the setting of complete manifolds, see for example [24], [16], [33].

Proposition 5.1. *If (VD) holds on M, then there exists $\alpha > 0$ depending only on the doubling constant C_D, such that for all $x, y \in M$ and $0 < r \leq R$,*

$$\frac{V(x,R)}{V(y,r)} \leq C_D \left(\frac{R + d(x,y)}{r}\right)^\alpha. \tag{5.1}$$

Proof. If $x = y$, then $R \leq 2^n r$ where

$$n = \left\lceil \log_2 \frac{R}{r} \right\rceil \leq \log_2 \frac{R}{r} + 1,$$

whence, it follows from (VD) that

$$\frac{V(x,R)}{V(x,r)} \leq \frac{V(x, 2^n r)}{V(x,r)} \leq (C_D)^n \leq (C_D)^{\log_2 \frac{R}{r} + 1} = C_D \left(\frac{R}{r}\right)^{\log_2 C_D}. \tag{5.2}$$

If $x \neq y$, then $B(x,R) \subset B(y, R + r_0)$ where $r_0 = d(x,y)$. It follows from (5.2) that

$$\frac{V(x,R)}{V(y,r)} \leq \frac{V(y, R + r_0)}{V(y,r)} \leq C_D \left(\frac{R + r_0}{r}\right)^{\log_2 C_D},$$

which finishes the proof. □

Proposition 5.2. *If (M, d) is connected and μ satisfies (VD), then there exist positive constants α' and c such that*

$$\frac{V(x,R)}{V(x,r)} \geq c \left(\frac{R}{r}\right)^{\alpha'} \quad \textit{for all } x \in M \textit{ and } 0 < r \leq R, \tag{RVD}$$

provided $B(x,R)^c$ is non-empty.

Proof. The condition $B\left(x,R\right)^{c}\neq\emptyset$ implies that

$$B\left(x,\rho'\right)\setminus B\left(x,\rho\right)\neq\emptyset \tag{5.3}$$

for all $0<\rho<R$ and $\rho'>\rho$. Indeed, otherwise M splits into disjoint union of two open sets: $B\left(x,\rho\right)$ and $\overline{B\left(x,\rho\right)}^{c}$. Since M is connected, the set $\overline{B\left(x,\rho\right)}^{c}$ must be empty, which contradicts the non-emptiness of $B\left(x,R\right)^{c}$.

If $0<\rho\leq R/2$, then we have by (5.3)

$$B\left(x,\frac{5}{3}\rho\right)\setminus B\left(x,\frac{4}{3}\rho\right)\neq\emptyset.$$

Let y be a point in this annulus. It follows from (5.1) that

$$V\left(x,\rho\right)\leq CV\left(y,\rho/3\right)$$

for some constant $C>0$, whence

$$V\left(x,2\rho\right)\geq V\left(x,\rho\right)+V\left(y,\rho/3\right)\geq\left(1+\varepsilon\right)V\left(x,\rho\right), \tag{5.4}$$

where $\varepsilon=C^{-1}$.

For any $0<r\leq R$, we have that $2^{n}r\leq R$ where

$$n:=\left\lceil\log_{2}\frac{R}{r}\right\rceil\geq\log_{2}\frac{R}{r}-1.$$

For any $0\leq k\leq n-1$, we have $2^{k}r\leq R/2$, and whence by (5.4),

$$V\left(x,2^{k+1}r\right)\geq\left(1+\varepsilon\right)V(x,2^{k}r).$$

Iterating this inequality, we obtain

$$\begin{aligned}\frac{V\left(x,R\right)}{V\left(x,r\right)}&\geq\frac{V\left(x,2^{n}r\right)}{V\left(x,r\right)}\geq\left(1+\varepsilon\right)^{n}\\&\geq\left(1+\varepsilon\right)^{\log_{2}\frac{R}{r}-1}=\left(1+\varepsilon\right)^{-1}\left(\frac{R}{r}\right)^{\log_{2}\left(1+\varepsilon\right)},\end{aligned}$$

thus proving (RVD). □

Corollary 5.3. *Assume that (M,d) is connected and μ satisfies (VD). Then*

$$\mu\left(M\right)=\infty\ \Leftrightarrow\ \operatorname{diam}(M)=\infty\ \Leftrightarrow\ \left(RVD\right).$$

Proof. If $\mu\left(M\right)=\infty$, then $\operatorname{diam}(M)=\infty$; indeed, otherwise M would be a ball of a finite radius and its measure would be finite by (VD). If $\operatorname{diam}(M)=\infty$, then $B^{c}\left(x,R\right)\neq\emptyset$ for any ball $B\left(x,R\right)$, and (RVD) holds by Proposition 5.2. Finally, (RVD) implies $\mu\left(M\right)=\infty$ by letting $R\to\infty$ in (RVD). □

In the case when all balls in M are precompact, the statement of Corollary 5.3 can be complemented as follows[6]:

$$\mu\left(M\right)=\infty\Leftrightarrow M\text{ is non-compact}\Leftrightarrow\left(RVD\right).$$

Indeed, if $\operatorname{diam}M<\infty$ then M is a ball and, hence, M is compact. If $\operatorname{diam}M=\infty$ then M is non-compact as an unbounded set. Let us emphasize that we never assume in this paper that balls are precompact.

[6]As the referee pointed it out, this equivalence is due to Martell.

5.2. **An estimate of the Dirichlet heat kernel.** The results of this Subsection are used in the proof of the implication $(FK_\beta) + (S_\beta) \Rightarrow (DUE_\beta)$ of Theorem 2.1. The arguments are known in the setting of manifolds, see for example [24], [17], [18], [9]. Here we have modified them to adjust to the present setting.

Lemma 5.4. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in $L^2(M,\mu)$. Let $U \subset M$ be an open set with $\mu(U) < \infty$. Assume that there exist positive constants a and ν such that, for all non-empty open sets $\Omega \subset U$,*

$$\lambda_{\min}(\Omega) \geq a\mu(\Omega)^{-\nu}. \tag{5.5}$$

Then there exists a constant $c_\nu > 0$ depending only on ν such that

$$\mathcal{E}(u) \geq c_\nu\, a\|u\|_2^{2+2\nu}\|u\|_1^{-2\nu}, \tag{5.6}$$

for any function $u \in \mathcal{F}(U) \setminus \{0\}$.

Proof. Assume first that $u \in \mathcal{F} \cap C_0(U)$ and $u \geq 0$. By the Markov property, we have that, for any $t \geq 0$,

$$\mathcal{E}(u) \geq \mathcal{E}\left((u-t)_+\right). \tag{5.7}$$

The set

$$U_s := \{x \in U : u(x) > s\}$$

is open for every $s > 0$. For any $t > s$, we have that $U_t \Subset U_s$. Since $(u-t)_+$ vanishes outside U_t, we see that $(u-t)_+ \in \mathcal{F}(U_s)$. It follows from (2.12) that

$$\mathcal{E}\left((u-t)_+\right) \geq \lambda_{\min}(U_s)\int_{U_s}(u-t)_+^2\,d\mu. \tag{5.8}$$

For simplicity, let $A = \|u\|_1$ and $B = \|u\|_2^2$. Since $u \geq 0$, we have

$$(u-t)_+^2 \geq u^2 - 2tu \quad \text{in } M,$$

which implies that

$$\int_{U_s}(u-t)_+^2 d\mu = \int_M (u-t)_+^2 d\mu \geq B - 2tA. \tag{5.9}$$

On the other hand, we have

$$\mu(U_s) \leq \frac{1}{s}\int_{U_s} u\,d\mu \leq \frac{A}{s},$$

which combines with (5.5) to yield that

$$\lambda_{\min}(U_s) \geq a\mu(U_s)^{-\nu} \geq a\left(\frac{s}{A}\right)^{\nu}. \tag{5.10}$$

Combining (5.7), (5.8), (5.9), and (5.10), we obtain

$$\mathcal{E}(u) \geq \lambda_{\min}(U_s)\int_{U_s}(u-t)_+^2\,d\mu \geq a\left(\frac{s}{A}\right)^{\nu}(B-2tA).$$

Letting $t \to s+$ and then choosing $s = \frac{B}{4A}$, we obtain that (5.6) holds for any non-negative $u \in \mathcal{F} \cap C_0(U)$.

Since $\mathcal{E}(|u|) \leq \mathcal{E}(u)$ by the Markov property, we see that (5.6) holds also for any signed $u \in \mathcal{F} \cap C_0(U)$.

Consider now a general function $u \in \mathcal{F}(U)$. By the definition of $(\mathcal{E}, \mathcal{F}(U))$, there exists a sequence $\{u_n\} \in \mathcal{F} \cap C_0(U)$ such that

$$\|u_n - u\|_2 \longrightarrow 0 \quad \text{and} \quad \mathcal{E}(u_n - u) \longrightarrow 0. \tag{5.11}$$

Since $\mu(U) < \infty$, the Cauchy-Schwarz inequality yields

$$\|u_n - u\|_1 \le \sqrt{\mu(U)}\, \|u_n - u\|_2 \to 0. \tag{5.12}$$

By the first part of the proof, (5.6) holds for each function u_n. Passing to the limit as $n \to \infty$ and using (5.11), (5.12), we obtain (5.6) for any $u \in \mathcal{F}(U)$. □

The next lemma is a modification of the Nash argument [31], which allows to obtain an upper bound of the Dirichlet heat kernel from the Nash type inequality (5.6) (see also [7], [17, Theorem 2.1]).

Lemma 5.5. *Under the hypotheses of* Lemma 5.4, *the heat kernel p_t^U exists, and satisfies the inequality*

$$\operatorname*{esup}_{x,y\in U} p_t^U(x,y) \le C(at)^{-1/\nu} \tag{5.13}$$

for all $t > 0$, where $C = C(\nu)$.

Proof. Let $f \in L^2(U,\mu)$ be non-negative with $\|f\|_1 = 1$. Set $u_t = P_t^U f$ for $t > 0$ so that $u_t \in \mathcal{F}(U)$. Denote $J(t) = \|u_t\|_2^2$. Similarly to (3.31), we have that

$$\frac{dJ}{dt} = -2\mathcal{E}(u_t). \tag{5.14}$$

Because P_t^U is a contraction in L^1, we have $\|u_t\|_1 \le 1$. Applying (5.6) for u_t and then using (5.14), we obtain the differential inequality

$$\frac{dJ}{dt} \le -caJ^{1+\nu},$$

whence $J(t) \le C(at)^{-1/\nu}$. That is,

$$\|P_t^U f\|_2^2 \le C(at)^{-1/\nu},$$

which means that the semigroup $\{P_t^U\}_{t>0}$ is $L^1(U) \to L^2(U)$ ultracontractive. By Lemma 3.7, we conclude that P_t^U has a heat kernel p_t^U satisfying (5.13). □

Note that the hypothesis $\mu(U) < \infty$ can be dropped in the context of Lemma 5.5 using an exhaustion of U by precompact open sets.

5.3. **From heat kernels in balls to the global heat kernel.** The following lemma is used in the proof of the implication $(FK_\beta) + (S_\beta) \Rightarrow (DUE_\beta)$,which is a part of Theorem 2.1.

Lemma 5.6. *Assume that $(\mathcal{E}, \mathcal{F})$ is regular and local. Let $Q_t(B)$ be a positive function defined for any $t > 0$ and any ball $B \subset M$ such that, for some constant L,*

$$Q_s(\lambda B) \le LQ_t(B) \tag{5.15}$$

for all balls B, $\frac{t}{2} \le s \le t$, and $1 \le \lambda \le 4$. Assume that the heat kernel p_t^B exists for any ball B and let $\rho : [0, \infty) \to [0, \infty)$ be an increasing function. Assume that, for any ball B of radius $r \ge \rho(t)$, the following conditions both are satisfied:

$$\operatorname*{esup}_B p_t^B \le Q_t(B), \tag{5.16}$$

$$1 - P_t^B \mathbf{1}_B \le \frac{1}{2L} \quad in \ \frac{1}{4}B. \tag{5.17}$$

Then the global heat kernel p_t exists and satisfies the following estimate, for any $t > 0$ and any ball B of radius $r = \rho(t)$:

$$\operatorname*{esup}_B p_t \le 2L^2 Q_t(B).$$

The proof below is an elaborated version of Kigami's iteration argument [28, proof of Theorem 2.9]. Unlike [28], we use neither ultracontractivity nor continuity of the heat kernel.

Proof. Choose some $x_0 \in M$, $R > r > 0$, and set

$$U = B(x_0, r) \quad \text{and} \quad \Omega = B(x_0, R).$$

It follows from (5.17) that if $0 < s < t$ and

$$r \ge \rho(t - s)$$

then

$$\operatorname*{esup}_{\frac{1}{4}U} \left(1 - P_{t-s}^U \mathbf{1}_U\right) \le \varepsilon := \frac{1}{2L}.$$

By Theorem 4.6 (cf. (4.13)), we have

$$\operatorname*{esup}_{\frac{1}{4}U} p_t^\Omega \le \operatorname*{esup}_U p_{t-s}^U + \varepsilon \operatorname*{esup}_U p_s^\Omega, \tag{5.18}$$

which implies together with (5.16) that

$$\operatorname*{esup}_{\frac{1}{4}U} p_t^\Omega \le Q_{t-s}(U) + \varepsilon \operatorname*{esup}_U p_s^\Omega. \tag{5.19}$$

Fix $t > 0$ and define for $k = 0, 1, 2, ...$ the sequences

$$t_k = \frac{1}{2}\left(1 + 2^{-k}\right) t, \quad r_k = 4^k \rho(t), \quad B_k = B(x_0, r_k).$$

(see Fig. 2). In particular, we have $t_0 = t$ and $B_0 = B(x_0, \rho(t))$.

Note that, for any $k \ge 0$,

$$r_{k+1} = 4^{k+1}\rho(t) \ge \rho\left(2^{-(k+2)}t\right) = \rho(t_k - t_{k+1}).$$

Assuming that $B_{k+1} \subset \Omega$ and applying (5.19) with $U = B_{k+1}$, we obtain

$$\operatorname*{esup}_{B_k} p_{t_k}^\Omega \le Q_{2^{-(k+2)}t}(B_{k+1}) + \varepsilon \operatorname*{esup}_{B_{k+1}} p_{t_{k+1}}^\Omega. \tag{5.20}$$

On the other hand, we have by (5.15)

$$\begin{aligned} Q_{2^{-(k+2)}t}(B_{k+1}) &\le L Q_{2^{-(k+1)}t}(B_k) \le L^2 Q_{2^{-k}t}(B_{k-1}) \\ &\le \cdots \le L^{k+1} Q_{t/2}(B_0) \le L^{k+2} Q_t(B_0). \end{aligned}$$

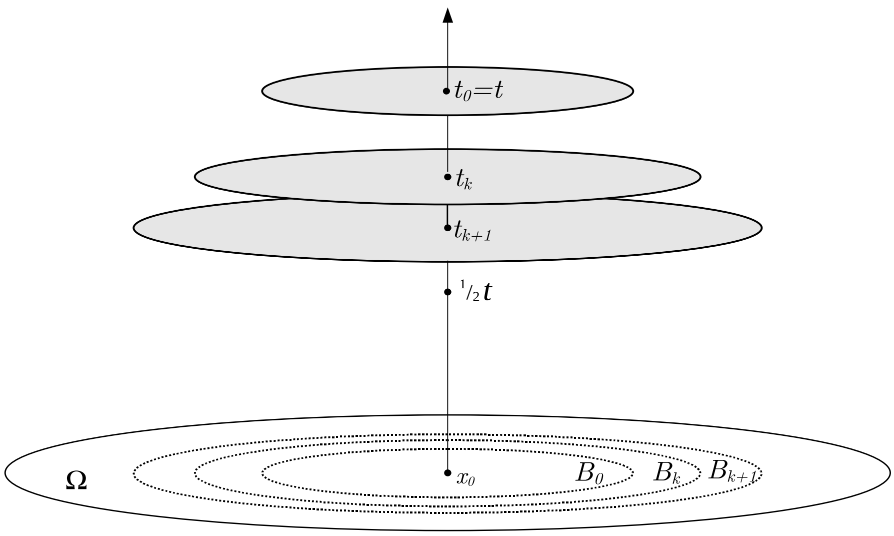

FIGURE 2. The sequences of times t_k and balls B_k

Hence, it follows from (5.20) that

$$\operatorname*{esup}_{B_k} p^{\Omega}_{t_k} \leq L^{k+2} Q_t\left(B_0\right) + \varepsilon \operatorname*{esup}_{B_{k+1}} p^{\Omega}_{t_{k+1}},$$

which gives by iteration that, for any positive integer n,

$$\begin{aligned} \operatorname*{esup}_{B_0} p^{\Omega}_{t} &\leq L^2\left(1 + L\varepsilon + (L\varepsilon)^2 + \dots\right) Q_t\left(B_0\right) + \varepsilon^n \operatorname*{esup}_{B_n} p^{\Omega}_{t_n} \\ &= 2L^2 Q_t\left(B_0\right) + \varepsilon^n \operatorname*{esup}_{B_n} p^{\Omega}_{t_n}, \end{aligned} \tag{5.21}$$

as long as $B_n \subset \Omega$. Set here $\Omega = B_n$ and let $n \to \infty$. Observe that by (5.16) and (5.15),

$$\operatorname*{esup}_{B_n} p^{B_n}_{t_n} \leq Q_{t_n}\left(B_n\right) \leq L^n Q_t\left(B_0\right),$$

and, hence,

$$\lim_{n\to\infty} \varepsilon^n \operatorname*{esup}_{B_n} p^{B_n}_{t_n} \leq \lim_{n\to\infty} \left(\varepsilon L\right)^n Q_t\left(B_0\right) = 0.$$

It follows from (5.21) that

$$\lim_{n\to\infty} \operatorname*{esup}_{B_0} p^{B_n}_{t} \leq 2L^2 Q_t\left(B_0\right). \tag{5.22}$$

On the other hand, the sequence $\left\{p^{B_n}_t\right\}_{n=1}^{\infty}$ increases as $n \to \infty$ and converges almost everywhere on $M \times M$ to a function p_t . This function is finite almost everywhere because of the uniform estimate

$$\int_{B_n} p^{B_n}_t\left(x, y\right)\, d\mu\left(y\right) \leq 1.$$

For any non-negative function $f \in L^2\left(M\right)$, we have as $n \to \infty$

$$\int_{B_n} p^{B_n}_t\left(x, y\right)\, f\left(y\right) d\mu\left(y\right) \to \int_{B_n} p_t\left(x, y\right)\, f\left(y\right) d\mu\left(y\right)$$

and
$$\int_{B_n} p_t^{B_n}(x,y)\ f(y)\,d\mu(y) = P_t^{B_n} f(x) \to P_t f(x)\,.$$
Hence, $p_t(x,y)$ is the heat kernel of P_t. It follows from (5.22) that
$$\operatorname*{esup}_{B_0} p_t = \lim_{n\to\infty} \operatorname*{esup}_{B_0} p_t^{B_n} \le 2L^2 Q_t(B_0)\,,$$
which was to be proved. ☐

5.4. **Tail estimates.** The main result of this Subsection is Theorem 5.8, which implies the equivalences
$$(T_{exp}) \Leftrightarrow (T_\beta) \Leftrightarrow (S_\beta)\,.$$
The most non-trivial part of Theorem 5.8 is contained in the following statement.

Theorem 5.7. *Let $(\mathcal{E},\mathcal{F})$ be a regular, strongly local Dirichlet form in $L^2(M,\mu)$. Let $\rho:[0,\infty)\to[0,\infty)$ be an increasing function. Assume that there exist $\varepsilon\in(0,1)$ and $\delta>0$ such that, for any ball B of radius $r>0$ and for any positive t such that $\rho(t)\le\delta r$,*
$$1 - P_t^B \mathbf{1}_B \le \varepsilon \quad in \ \ \frac{1}{4}B. \tag{5.23}$$
Then, for any $t>0$ and any ball B of radius $r>0$,
$$P_t \mathbf{1}_{B^c} \le C\exp\left(-c' t\,\Psi\left(\frac{cr}{t}\right)\right) \ in \ \frac{1}{2}B, \tag{5.24}$$
where $C,c,c'>0$ are constants depending on ε,δ, and function Ψ is defined by
$$\Psi(s) := \sup_{\lambda>0}\left\{\frac{s}{\rho(1/\lambda)} - \lambda\right\} \tag{5.25}$$
for all $s\ge0$.

Before the proof, let us give some comments.

1. Letting $\lambda\to0$ in (5.25), one sees that $\Psi(s)\ge0$ for all $s\ge0$. It is also obvious from (5.25) that $\Psi(s)$ is increasing in s.

2. If $\rho(t)=t^{1/\beta}$ for $\beta>1$, then
$$\Psi(s) = \sup_{\lambda>0}\left\{s\lambda^{1/\beta} - \lambda\right\} = c_\beta s^{\beta/(\beta-1)}$$
for all $s\ge0$, where $c_\beta>0$ depends only on β (the supremum is attained for $\lambda=(s/\beta)^{\frac{\beta}{\beta-1}}$). The estimate (5.24) becomes
$$P_t \mathbf{1}_{B^c} \le C\exp\left(-c\left(\frac{r^\beta}{t}\right)^{\frac{1}{\beta-1}}\right) \quad \text{in } \frac{1}{2}B.$$

3. Substituting (5.25) into (5.24), we can rewrite the latter in the form
$$P_t \mathbf{1}_{B^c} \le C\exp\left(c'\lambda t - \frac{c'' r}{\rho(1/\lambda)}\right) \ \text{in } \frac{1}{2}B, \tag{5.26}$$
for any $\lambda>0$, which is sometimes more convenient than (5.24).

4. A similar Theorem was proved in [21, Theorem 3.4], using a more complicated method. A relatively short proof presented below is an adaptation of the argument of Hebisch and Saloff-Coste [26], although with two modifications: firstly, we do

not assume the existence of the heat kernel, secondly, we do not use the associated diffusion process.

Proof. Step 1. Assume that

$$\rho(t) \le \delta r \tag{5.27}$$

and that B is a ball of radius r. For any positive integer k, set $B_k = kB$. We will prove that

$$1 - P_t^{B_k} \mathbf{1}_{B_k} \le \varepsilon^k \text{ in } \frac{1}{4}B. \tag{5.28}$$

Since M is separable, there is a dense countable set of points in B_k. Let $\{b_j\}$ be a sequence of balls of radii r centered at those points. Clearly, $b_j \subset B_{k+1}$ and the family $\left\{\frac{1}{4}b_j\right\}$ covers B_k (see Fig. 3).

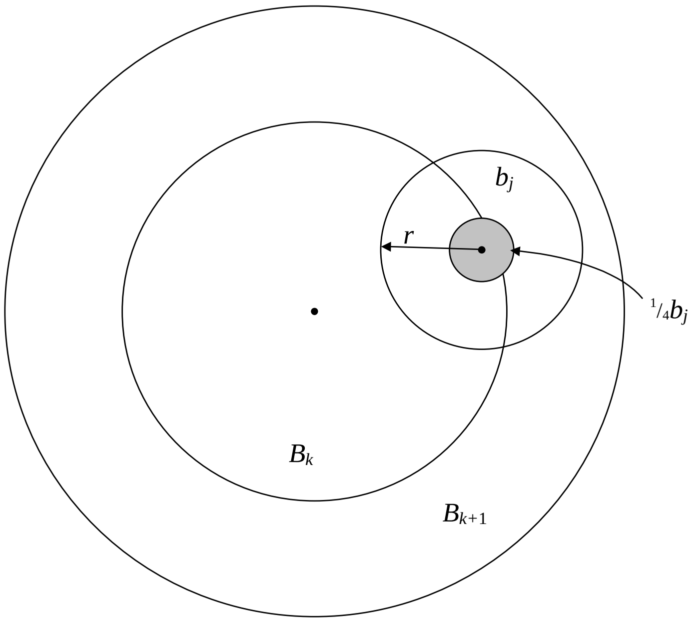

FIGURE 3. Balls B_k and b_j

Due to (5.27), inequality (5.23) is valid for any ball b_j, that is, for all $0 < s \le t$,

$$P_s^{B_{k+1}} \mathbf{1}_{B_{k+1}} \ge P_s^{b_j} \mathbf{1}_{b_j} \ge 1 - \varepsilon \text{ in } \frac{1}{4} b_j.$$

It follows that

$$P_s^{B_{k+1}} \mathbf{1}_{B_{k+1}} \ge 1 - \varepsilon \text{ in } B_k.$$

Applying the inequality (4.11) of Corollary 4.5 with $\Omega = B_{k+1}$ and $U = B_k$, we obtain that the following inequality holds in B_k:

$$\begin{aligned} 1 - P_t^{B_{k+1}} \mathbf{1}_{B_{k+1}} &\le \left(1 - P_t^{B_k} \mathbf{1}_{B_k}\right) \sup_{0<s\le t} \left\|1 - P_s^{B_{k+1}} \mathbf{1}_{B_{k+1}}\right\|_{L^\infty(B_k)} \\ &\le \varepsilon \left(1 - P_t^{B_k} \mathbf{1}_{B_k}\right). \end{aligned}$$

Iterating in k and using (5.23), we obtain (5.28).

It follows from (5.28) that

$$P_t \mathbf{1}_{B_k^c} \leq 1 - P_t \mathbf{1}_{B_k} \leq 1 - P_t^{B_k} \mathbf{1}_{B_k} \leq \varepsilon^k \text{ in } \frac{1}{4}B. \tag{5.29}$$

Although (5.29) has been proved for any integer $k \geq 1$, it is trivially true also for $k = 0$, if we define $B_0 := \emptyset$.

Step 2. Fix $t > 0$, $x \in M$ and consider the function

$$E_{t,x} = \exp\left(c\frac{d(x,\cdot)}{\rho(t)}\right), \tag{5.30}$$

where the constant $c > 0$ is to be determined later on. Set

$$r = \delta^{-1}\rho(t),$$

and we will prove that

$$P_t(E_{t,x}) \leq C \text{ in } B(x, r/4), \tag{5.31}$$

where C is a constant depending on ε, δ. Set as before $B_k = B(x, kr)$, $k \geq 1$, and $B_0 = \emptyset$. Using (5.30) and (5.29), we obtain that in $B(x, r/4)$,

$$\begin{aligned} P_t(E_{t,x}) &= \sum_{k=0}^{\infty} P_t\left(\mathbf{1}_{B_{k+1}\setminus B_k} E_{t,x}\right) \leq \sum_{k=0}^{\infty} \|E_{t,x}\|_{L^\infty(B_{k+1})} P_t\left(\mathbf{1}_{B_{k+1}\setminus B_k}\right) \\ &\leq \sum_{k=0}^{\infty} \exp\left(c\frac{(k+1)\,r}{\rho(t)}\right) P_t\left(\mathbf{1}_{B_k^c}\right) \leq \sum_{k=0}^{\infty} \exp\left(c(k+1)\delta^{-1}\right)\varepsilon^k. \end{aligned}$$

Choosing $c < \delta \log \frac{1}{\varepsilon}$ we obtain that this series converges, which proves (5.31).

Step 3. Let us prove that, for all $t > 0$ and $x \in M$,

$$P_t E_{t,x} \leq C_1 E_{t,x}, \tag{5.32}$$

for some constant $C_1 = C(\varepsilon, \delta)$. Observe first that, for all $y, z \in M$, we have by the triangle inequality

$$\begin{aligned} E_{t,x}(y) &= \exp\left(c\frac{d(x,y)}{\rho(t)}\right) \\ &\leq \exp\left(c\frac{d(x,z)}{\rho(t)}\right)\exp\left(c\frac{d(z,y)}{\rho(t)}\right) = E_{t,x}(z)E_{t,z}(y), \end{aligned}$$

which can also be written in the form of a function inequality:

$$E_{t,x} \leq E_{t,x}(z)\, E_{t,z}.$$

It follows that

$$P_t(E_{t,x}) \leq E_{t,x}(z)\, P_t(E_{t,z}). \tag{5.33}$$

By the previous step, we have

$$P_t(E_{t,z}) \leq C \text{ in } B(z, r), \tag{5.34}$$

where $r = \frac{1}{4}\delta^{-1}\rho(t)$. For all $y \in B(z, r)$, we have

$$E_{t,z}(y) \leq \exp\left(\frac{cr}{\rho(t)}\right) = \exp\left(c\delta^{-1}/4\right) =: C',$$

whence

$$E_{t,x}(z) \leq E_{t,x}(y)\, E_{t,z}(y) \leq C' E_{t,x}(y)$$

(see Fig. 4).

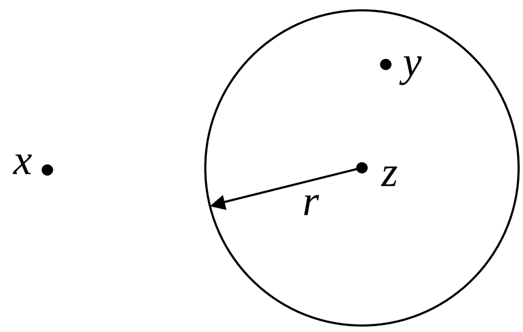

FIGURE 4. Points x, y, z

Letting y vary, we can write

$$E_{t,x}(z) \leq C' E_{t,x} \text{ in } B(z,r).$$

Combining this with (5.33) and (5.34), we obtain

$$P_t(E_{t,x}) \leq CC' E_{t,x} \text{ in } B(z,r).$$

Since z is arbitrary, covering M by a countable sequence of balls like $B(z,r)$, we obtain that (5.32) holds on M with $C_1 = CC'$.

Step 4. Let us prove that, for all $t > 0$, $x \in M$, and for any positive integer k,

$$P_{kt}(E_{t,x}) \leq C_1^k \text{ in } \frac{1}{4}B, \tag{5.35}$$

where $B = \left(x, \delta^{-1}\rho(t)\right)$. Indeed, by (5.32)

$$P_{kt}(E_{t,x}) = P_{(k-1)t} P_t(E_{t,x}) \leq C_1 P_{(k-1)t} E_{t,x}$$

which implies by iteration that

$$P_{kt}(E_{t,x}) \leq C_1^{k-1} P_t E_{t,x}.$$

Combining with (5.31) and noticing that $C \leq C_1$, we obtain (5.35).

Step 5. Fix a ball $B = B(x_0, r)$ and observe that (5.24) is equivalent to the following: for all $t, \lambda > 0$,

$$P_t \mathbf{1}_{B^c} \leq C \exp\left(c'\lambda t - \frac{cr}{\rho(1/\lambda)}\right) \text{ in } \frac{1}{2}B, \tag{5.36}$$

where $C, c, c' > 0$ are constants depending on ε, δ. In what follows, we fix also t and λ.

Observe first that, for any $x \in \frac{1}{2}B$,

$$P_t \mathbf{1}_{B^c} \leq P_t \mathbf{1}_{B(x,r/2)^c}.$$

Hence, it suffices to prove that, for any $x \in \frac{1}{2}B$,

$$P_t \mathbf{1}_{B(x,r/2)^c} \leq C \exp\left(c'\lambda t - \frac{cr}{\rho(1/\lambda)}\right) \tag{5.37}$$

in a (small) ball around x. Covering then $\frac{1}{2}B$ by a countable family of such balls, we then obtain (5.36).

Changing t to t/k in (5.35), we obtain that

$$P_t\left(E_{t/k,x}\right) \leq C_1^k \text{ in } B\left(x,\sigma_k\right)$$

where $\sigma_k = \frac{1}{4}\delta^{-1}\rho\left(t/k\right)$. Since

$$E_{t/k,x} \geq \exp\left(c\frac{r}{\rho\left(t/k\right)}\right) \text{ in } B\left(x,r\right)^c$$

and, hence,

$$\mathbf{1}_{B(x,r)^c} \leq \exp\left(-\frac{cr}{\rho(t/k)}\right)E_{t/k,x},$$

we obtain that the following inequality holds in $B\left(x,\sigma_k\right)$

$$P_t\mathbf{1}_{B(x,r)^c} \leq \exp\left(-\frac{cr}{\rho(t/k)}\right)P_t\left(E_{t/k,x}\right) \leq \exp\left(c'k-\frac{cr}{\rho(t/k)}\right)$$

where $c' = \log C_1$. Given $\lambda > 0$, choose an integer $k \geq 1$ such that

$$\frac{k-1}{t} < \lambda \leq \frac{k}{t}.$$

Then we obtain the following inequality in $B\left(x,\sigma_k\right)$

$$P_t\mathbf{1}_{B(x,r)^c} \leq \exp\left(c'\left(\lambda t+1\right)-\frac{cr}{\rho\left(1/\lambda\right)}\right),$$

which finishes the proof. □

Theorem 5.8. *Let $(\mathcal{E},\mathcal{F})$ be a regular, local, conservative Dirichlet form in $L^2(M,\mu)$. Let $\rho:[0,\infty)\to[0,\infty)$ be an increasing function and let Ψ be defined as in (5.25). Then, the following conditions are equivalent.*

(i) : *There exist $\varepsilon\in(0,1)$ and $\delta>0$ such that, for any ball B of radius r and any positive t such that $\rho\left(t\right)\leq\delta r$,*

$$1-P_t^B\mathbf{1}_B \leq \varepsilon \text{ in } \frac{1}{4}B. \tag{5.38}$$

(ii) : *There exist constants $C,c,c'>0$ such that, for any $t>0$ and any ball B of radius $r>0$,*

$$P_t\mathbf{1}_{B^c} \leq C\exp\left(-c't\,\Psi\left(\frac{cr}{t}\right)\right) \quad \text{in } \frac{1}{4}B. \tag{5.39}$$

(iii) : *There exist $\varepsilon\in(0,\frac{1}{2})$ and $\delta>0$ such that, for any ball B of radius r and any positive t such that $\rho\left(t\right)\leq\delta r$,*

$$P_t\mathbf{1}_{B^c} \leq \varepsilon \text{ in } \frac{1}{4}B. \tag{5.40}$$

Proof. Let us first show that the locality and the conservativeness imply the strong locality. Indeed, by [15, Lemmas 4.5.2, 4.5.3], we have the following identity

$$\lim_{t\to0}\frac{1}{t}\int_M\left(1-P_t1\right)u^2\,d\mu = \int_M\widetilde{u}^2dk$$

for any $u\in\mathcal{F}$ where k is the killing measure of $(\mathcal{E},\mathcal{F})$ and $\widetilde{u}$ is a quasi-continuous version of u. Since $P_t1=1$, it follows that $k=0$. Therefore, by the Beurling-Deny formula [15, Theorem 3.2.1], $(\mathcal{E},\mathcal{F})$ is strongly local. Hence, Theorem 5.7 applies and gives the implication $(i) \Rightarrow (ii)$.

To prove $(ii) \Rightarrow (iii)$ observe first that the function Ψ defined by (5.25) satisfies the following inequality: for all $s \geq 0$ and $A \geq 1$,

$$\Psi(As) \geq A\Psi(s), \tag{5.41}$$

which follows from

$$\Psi(As) = \sup_{\lambda>0}\left\{\frac{As}{\rho(1/\lambda)} - \lambda\right\} \geq A\sup_{\lambda>0}\left\{\frac{s}{\rho(1/\lambda)} - \lambda\right\} = A\Psi(s).$$

If $r \geq 2c^{-1}A\rho(t)$ where a constant A will be specified below, then we have, using (5.41),

$$\begin{aligned}\Psi\left(\frac{cr}{t}\right) &\geq \Psi\left(2A\frac{\rho(t)}{t}\right) \geq A\Psi\left(2\frac{\rho(t)}{t}\right)\\ &= A\sup_{\lambda>0}\left\{\frac{2\rho(t)}{t\rho(1/\lambda)} - \lambda\right\} \geq \frac{A}{t},\end{aligned}$$

where the last inequality holds by letting $\lambda = 1/t$. Hence, we obtain from (5.39) that in $\frac{1}{4}B$

$$P_t\mathbf{1}_{B^c} \leq C\exp(-c'A) \leq \varepsilon,$$

provided A is sufficiently large.

The implication $(iii) \Rightarrow (i)$ was proved in [21, Theorem 3.1] under the assumption that all the balls are precompact (a probabilistic prototype of that proof can be found in [1]). Here we repeat the same argument but now we do not need the precompactness of balls due to Proposition 4.7. Indeed, applying estimate (4.21) of Proposition 4.7 with $\Omega = M$, $U = B = B(x_0, r)$, $K = \frac{3}{4}\overline{B}$ and $f = \mathbf{1}_{\frac{1}{2}B}$, we obtain that, for all t and almost everywhere in M,

$$P_t^B\mathbf{1}_{\frac{1}{2}B} \geq P_t\mathbf{1}_{\frac{1}{2}B} - \sup_{0<s\leq t}\left\|P_s\mathbf{1}_{\frac{1}{2}B}\right\|_{L^\infty\left(\left(\frac{3}{4}\overline{B}\right)^c\right)}. \tag{5.42}$$

For any $x \in \frac{1}{4}B$, we have that $B(x, r/4) \subset \frac{1}{2}B$ (see Fig. 5).
Using the identity $P_t 1 = 1$ we obtain, for any $x \in \frac{1}{4}B$,

$$P_t\mathbf{1}_{\frac{1}{2}B} = 1 - P_t\mathbf{1}_{\left(\frac{1}{2}B\right)^c} \geq 1 - P_t\mathbf{1}_{B(x,r/4)^c}.$$

Applying (5.40) for the ball $B(x, r/4)$, we obtain that

$$P_t\mathbf{1}_{B(x,r/4)^c} \leq \varepsilon \text{ in } B(x, r/16),$$

provided t is so small that

$$\rho(t) \leq \delta\frac{r}{4}. \tag{5.43}$$

It follows that, for any $x \in \frac{1}{4}B$,

$$P_t\mathbf{1}_{\frac{1}{2}B} \geq 1 - \varepsilon \text{ in } B(x, r/16),$$

whence

$$P_t\mathbf{1}_{\frac{1}{2}B} \geq 1 - \varepsilon \text{ in } \frac{1}{4}B. \tag{5.44}$$

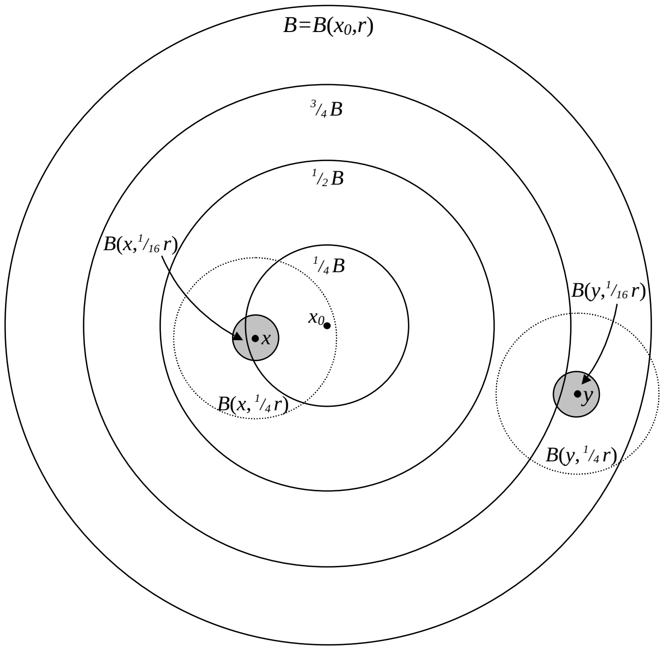

FIGURE 5. Illustration to the proof of the implication $(iii) \Rightarrow (i)$ of Theorem 5.8

On the other hand, for any $y \in \left(\frac{3}{4}\overline{B}\right)^c$, we have $\frac{1}{2}B \subset B(y, r/4)^c$ (see Fig. 5), whence

$$P_s \mathbf{1}_{\frac{1}{2}B} \leq P_s \mathbf{1}_{B(y,r/4)^c}.$$

Applying (5.40) for the ball $B(y, r/4)$ at time s, we obtain if (5.43) holds then, for all $0 < s \leq t$,

$$P_s \mathbf{1}_{B(y,r/4)^c} \leq \varepsilon \text{ in } B(y, r/16).$$

It follows that, for any $y \in \left(\frac{3}{4}\overline{B}\right)^c$,

$$P_s \mathbf{1}_{\frac{1}{2}B} \leq \varepsilon \text{ in } B(y, r/16),$$

whence

$$P_s \mathbf{1}_{\frac{1}{2}B} \leq \varepsilon \text{ in } \left(\frac{3}{4}\overline{B}\right)^c. \tag{5.45}$$

Combining (5.42), (5.44) and (5.45), we obtain that, under condition (5.43),

$$P_t^B \mathbf{1}_B \geq P_t^B \mathbf{1}_{\frac{1}{2}B} \geq 1 - 2\varepsilon \text{ in } \frac{1}{4}B, \tag{5.46}$$

which is equivalent to (5.38). □

Remark 5.9. Observe that for $(i) \Rightarrow (ii)$, the strong locality is needed but the conservation property of $(\mathcal{E}, \mathcal{F})$ is not required. For $(iii) \Rightarrow (i)$, the conservation property is needed but the locality is not required. Finally, the implication $(ii) \Rightarrow (iii)$ requires no assumption on $(\mathcal{E}, \mathcal{F})$.

The same argument as in the proof of the implication $(iii) \Rightarrow (i)$ shows that $(T_{\exp})$ implies the following version of the survival estimate: for any ball $B = B(x_0, r)$ and

any $t>0$,
$$1-P_t^B 1_B \le 2C\exp\left(-c\left(\frac{r}{4t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right) \text{ in } \frac{1}{4}B, \tag{5.47}$$
where C, c are the same constants as in $(T_{\exp})$.

5.5. **Proof of Theorem 2.1.** We are now in position to prove Theorem 2.1. Observe that the implication
$$(UE_\beta) \Rightarrow (\Phi UE_\beta)$$
is trivial by taking
$$\Phi(s) = \exp\left(-cs^{\beta/(\beta-1)}\right).$$
Applying Theorem 5.8 with $\rho(t)=t^{1/\beta}$, we obtain
$$(T_{exp}) \Leftrightarrow (T_\beta) \Leftrightarrow (S_\beta).$$
We are left to verify the following four implications:
$$\begin{aligned}
&(FK_\beta)+(S_\beta) \Rightarrow (DUE_\beta)\\
&(\Phi UE_\beta) \Rightarrow (DUE_\beta)+(T_\beta)\\
&(DUE_\beta)+(T_{exp}) \Rightarrow (UE_\beta)\\
&(DUE_\beta) \Rightarrow (FK_\beta).
\end{aligned}$$
Indeed, using the above implications, the proof of Theorem 2.1 can be presented in the following flowchart:
$$\begin{array}{ccccc}
(UE_\beta) & \Rightarrow & (\Phi UE_\beta) & \Rightarrow & (DUE_\beta)+(T_\beta)\\
\Uparrow & & & & \Downarrow\\
(DUE_\beta)+(T_{\exp}) & \Leftarrow & (FK_\beta)+(S_\beta) & \Leftarrow & (DUE_\beta)+(S_\beta)
\end{array}$$
Proof of $(FK_\beta)+(S_\beta) \Rightarrow (DUE_\beta)$. Let B be a ball of radius $r>0$ in M. Let us restate (FK_β) in the following form: for any non-empty open set $\Omega \subset B$,
$$\lambda_{\min}(\Omega) \ge a\,\mu(\Omega)^{-\nu}$$
where
$$a = \frac{c}{r^\beta}\mu(B)^\nu.$$
Therefore, by Lemma 5.5, the heat kernel p_t^B exists, and satisfies that
$$\operatorname*{esup}_B p_t^B \le \frac{C}{\mu(B)}\left(\frac{r^\beta}{t}\right)^{1/\nu}, \tag{5.48}$$
for all $t>0$. Hence, condition (5.16) is satisfied with the function
$$Q_t(B) = \frac{C}{\mu(B)}\left(\frac{r^\beta}{t}\right)^{1/\nu}.$$
On the other hand, it follows from (S_β) that condition (5.17) of Lemma 5.6 is satisfied with
$$\rho(t) = C't^{1/\beta},$$
where C' is a large enough constant. By Lemma 5.6, the global heat kernel p_t exists and satisfies for all $t>0$ the estimate
$$\operatorname*{esup}_{B(x_0,\rho(t))} p_t \le \frac{C}{V(x_0,\rho(t))} \le \frac{C}{V(x_0,t^{1/\beta})}.$$

Using (3.27), we obtain that, for all $x_0, y_0 \in M$ and $t > 0$,

$$\operatorname*{esup}_{\substack{x\in B(x_0,\rho(t))\\ y\in B(y_0,\rho(t))}} p_t(x,y) \le \frac{C}{\sqrt{V(x_0,t^{1/\beta})\,V(y_0,t^{1/\beta})}}.$$

Using (VD), we see that, for any $x \in B(x_0, \rho(t))$,

$$V\left(x,t^{1/\beta}\right) \le C\,V\left(x_0,t^{1/\beta}\right),$$

whence it follows that, for μ-almost all $x \in B(x_0,\rho(t))$ and $y \in B(y_0,\rho(t))$,

$$p_t(x,y) \le \frac{C}{\sqrt{V(x,t^{1/\beta})\,V(y,t^{1/\beta})}}. \tag{5.49}$$

Since M can be covered by a countable family of balls of radius $\rho(t)$, we conclude that (5.49) holds for μ-almost all $x, y \in M$. □

Proof of $(\Phi UE_\beta) \Rightarrow (DUE_\beta) + (T_\beta)$. It follows from (ΦUE_β) that, for all $t > 0$ and μ-almost all $x \in M$,

$$\operatorname*{esup}_{z\in M} p_t(x,z) \le \frac{C}{V(x,t^{1/\beta})}, \tag{5.50}$$

which implies that

$$\int_M p_t(x,z)^2\,d\mu(z) \le \operatorname*{esup}_{z\in M} p_t(x,z) \int_M p_t(x,z)\,d\mu(z) \le \frac{C}{V(x,t^{1/\beta})}.$$

Using the semigroup property, the Cauchy-Schwarz inequality, and the symmetry, we obtain

$$\begin{aligned} p_{2t}(x,y) &= \int_M p_t(x,z)p_t(z,y)\,d\mu(z) \\ &\le \left(\int_M p_t(x,z)^2 d\mu(z)\right)^{1/2}\left(\int_M p_t(y,z)^2 d\mu(z)\right)^{1/2} \\ &\le \frac{c}{\sqrt{V(x,t^{1/\beta})\,V(y,t^{1/\beta})}} \end{aligned}$$

for all $t > 0$ and μ-almost all $x, y \in M$. Replacing $2t$ by t and then using the doubling property of V, we obtain (DUE_β).

In order to verify the implication $(\Phi UE_\beta) \Rightarrow (T_\beta)$, set $r_k = 2^k r$ for some $r > 0$ and $k = 0, 1, 2, \ldots$. It follows from (ΦUE_β) and (2.2) that, for any $t > 0$ and μ-almost all $x \in M$,

$$\int_{B(x,r)^c} p_t(x,y)\,d\mu(y) = \sum_{k=0}^{\infty}\int_{B(x,r_{k+1})\setminus B(x,r_k)} p_t(x,y)\,d\mu(y) \tag{5.51}$$

$$\begin{aligned} &\le \sum_{k=0}^{\infty} V(x,r_{k+1})\frac{C}{V(x,t^{1/\beta})}\Phi\left(\frac{r_k}{t^{1/\beta}}\right) \\ &\le \sum_{k=0}^{\infty} C\left(\frac{r_k}{t^{1/\beta}}\right)^{\alpha}\Phi\left(\frac{r_k}{t^{1/\beta}}\right) \end{aligned}$$

$$\le C\int_{\frac12 rt^{-1/\beta}}^{\infty} s^{\alpha-1}\Phi(s)\,ds. \tag{5.52}$$

Due to (2.14), the integral in (5.52) can be made arbitrarily small provided $rt^{-1/\beta}$ is sufficiently large. Hence, for any ball $B = B(x_0, r)$, for any $\varepsilon > 0$, and for almost all $x \in \frac{1}{4}B$, we have

$$P_t \mathbf{1}_{B^c}(x) \le \int_{B(x,r/2)^c} p_t(x,y)\, d\mu(y) < \varepsilon$$

provided $rt^{-1/\beta}$ is large enough, which proves (T_β). $\square$

Proof of $(DUE_\beta) + (T_{\exp}) \Rightarrow (UE_\beta)$. By the semigroup property, we have that, for all $t > 0$ and μ-almost all $x, y \in M$,

$$\begin{aligned} p_{2t}(x,y) &= \int_M p_t(x,z)p_t(z,y)\, d\mu(z) \\ &\le \int_{B(x,r)^c} p_t(x,z)p_t(z,y)\, d\mu(z) + \int_{B(y,r)^c} p_t(x,z)p_t(z,y)\, d\mu(z), \end{aligned} \tag{5.53}$$

where $r = \frac{1}{2}d(x,y)$. Let us estimate the first term in (5.53) (and the second term can be estimated similarly). By (2.2) we have, for every $z \in M$,

$$\frac{V\left(x, t^{1/\beta}\right)}{V\left(z, t^{1/\beta}\right)} \le C\left(\frac{d(x,z) + t^{1/\beta}}{t^{1/\beta}}\right)^\alpha = C\left(1 + \frac{d(x,z)}{t^{1/\beta}}\right)^\alpha.$$

It follows from (DUE_β) that, for μ-a.a. $z \in B(x, 2^k r)$,

$$\begin{aligned} p_t(z,y) &\le \frac{C}{\sqrt{V(z,t^{1/\beta})\, V(y,t^{1/\beta})}} \\ &= \frac{C}{V(x,t^{1/\beta})}\sqrt{\frac{V(x,t^{1/\beta})}{V(z,t^{1/\beta})}\frac{V(x,t^{1/\beta})}{V(y,t^{1/\beta})}} \\ &\le \frac{C}{V(x,t^{1/\beta})}\left(1 + \frac{2^k r}{t^{1/\beta}}\right)^{\alpha/2}\left(1 + \frac{2r}{t^{1/\beta}}\right)^{\alpha/2} \\ &\le \frac{C}{V(x,t^{1/\beta})}\left(1 + 2^k R\right)^\alpha \end{aligned} \tag{5.54}$$

where

$$R := \frac{r}{t^{1/\beta}}.$$

On the other hand, we have by $(T_{\exp})$

$$\int_{B(x,2^k r)^c} p_t(x,z)\, d\mu(z) \le C\,\Phi\left(2^k R\right), \tag{5.55}$$

where

$$\Phi(s) := \exp\left(-c\, s^{\beta/(\beta-1)}\right).$$

Combining (5.54) and (5.55), we obtain

$$\begin{aligned}\int_{B(x,2^k r)\setminus B(x,2^{k-1}r)} & p_t(x,z)p_t(z,y)\,d\mu(z)\\ &\leq \frac{C\left(1+2^kR\right)^{\alpha}}{V\left(x,t^{1/\beta}\right)}\int_{B(x,2^k r)\setminus B(x,2^{k-1}r)} p_t(x,z)\,d\mu(z)\\ &\leq \frac{C\left(1+2^kR\right)^{\alpha}}{V\left(x,t^{1/\beta}\right)}\Phi\left(2^{k-1}R\right),\end{aligned}$$

whence it follows that

$$\begin{aligned}\int_{B(x,r)^c} p_t(x,z)p_t(z,y)\,d\mu(z) &\leq \frac{C}{V\left(x,t^{1/\beta}\right)}\sum_{k=1}^{\infty}\left(1+2^kR\right)^{\alpha}\Phi\left(2^{k-1}R\right)\\ &\leq \frac{C}{V\left(x,t^{1/\beta}\right)}\sum_{k=1}^{\infty}\left(1+2^kR\right)^{\alpha}\Phi\left(2^{k-2}R\right)\Phi\left(\frac{1}{2}R\right)\\ &\leq \frac{C}{V\left(x,t^{1/\beta}\right)}\left(\int_0^{\infty}s^{\alpha-1}\Phi(s)\,ds\right)\Phi\left(\frac{1}{2}R\right)\\ &\leq \frac{C}{V\left(x,t^{1/\beta}\right)}\Phi\left(\frac{1}{2}R\right),\end{aligned}\tag{5.56}$$

where we have the following obvious inequality

$$\Phi\left(aR\right)\leq\Phi\left(\frac{1}{2}aR\right)\Phi\left(\frac{1}{2}R\right),$$

which is true for all $a\geq 1$ and $R>0$. Hence, it follows from (5.53) and (5.56) that

$$p_{2t}(x,y)\leq\frac{C}{V\left(x,t^{1/\beta}\right)}\Phi\left(\frac{1}{2}R\right).$$

Replacing $2t$ by t and using the doubling property of V, we obtain (UE_β). □

Proof of $(DUE_\beta)\Rightarrow(FK_\beta)$. Let us first show that (DUE_β) implies the following estimate: for any ball $B=B\left(x_0,r\right)$ on M and for all $t>0$,

$$\operatorname*{esup}_B p_t^B\leq\frac{C}{\mu\left(B\right)}\left(\frac{r}{t^{1/\beta}}\right)^{\alpha},\tag{5.57}$$

where α is the same as in (2.2). Observe the following property of the function $t\mapsto\operatorname{esup}_B p_t^B$: if the inequality

$$\operatorname*{esup}_B p_t^B\leq\frac{K}{\mu\left(B\right)}\left(\frac{r}{t^{1/\beta}}\right)^{\alpha}\tag{5.58}$$

holds for $t=s$, then (5.58) holds also for $t=2s$ provided

$$s\geq T:=2K^{\beta/\alpha}r^{\beta},\tag{5.59}$$

where $K>1$ is a constant to be specified below. Indeed, by the semigroup property, we have, for μ-a.a. $x,y\in B$,

$$p_{2s}^B\left(x,y\right)=\int_B p_s^B\left(x,z\right)p_s^B\left(z,y\right)d\mu\left(z\right)\leq\left(\operatorname*{esup}_B p_s^B\right)^2\mu\left(B\right),$$

whence by (5.58) and (5.59),

$$\operatorname*{esup}_B p^B_{2s} \le \frac{K^2}{\mu(B)}\left(\frac{r}{s^{1/\beta}}\right)^{2\alpha} \le \frac{K^2}{\mu(B)}\left(\frac{r}{(T/2)^{1/\beta}}\right)^{\alpha}\left(\frac{r}{(2s)^{1/\beta}}\right)^{\alpha}$$
$$= \frac{K}{\mu(B)}\left(\frac{r}{(2s)^{1/\beta}}\right)^{\alpha},$$

which was claimed.

Assume for a moment that we have proved (5.58) for $t = T$. Then, by induction, we see that (5.58) holds for all $t = 2^n T$, where n is a non-negative integer. Since by Lemma 3.9 the function $t \mapsto \operatorname{esup}_B p^B_t$ is non-increasing, we obtain that, for $2^n T \le t < 2^{n+1} T$,

$$\operatorname*{esup}_B p^B_t \le \operatorname*{esup}_B p^B_{2^n T} \le \frac{K}{\mu(B)}\left(\frac{r}{(2^n T)^{1/\beta}}\right)^{\alpha} \le \frac{K 2^{\alpha/\beta}}{\mu(B)}\left(\frac{r}{t^{1/\beta}}\right)^{\alpha}.$$

Therefore, if we prove that there exists K such that (5.58) holds for $0 < t \le T$ then we can conclude that (5.57) holds for all $t > 0$.

Consider first the case $0 < t \le r^\beta$. By (DUE_β) we have, for μ-a.a. $x, y \in M$ and $t > 0$,

$$p_t(x,y) \le \frac{C_0}{\sqrt{V(x,t^{1/\beta})\,V(y,t^{1/\beta})}} \tag{5.60}$$

(the argument below is sensitive to constant factors, so we use individual notation for different constants such as C_0, C_1, etc). Observe that, by (2.2), for any $x \in B$ and $0 < t \le r^\beta$,

$$\frac{V(x_0,r)}{V(x,t^{1/\beta})} \le C_1\left(\frac{r}{t^{1/\beta}}\right)^{\alpha}. \tag{5.61}$$

Since $p^B_t \le p_t$, we see from (5.60) and (5.61) that, for μ-a.a. $x, y \in B$ and $0 < t \le r^\beta$,

$$p^B_t(x,y) \le \frac{C_0 C_1}{V(x_0,r)}\left(\frac{r}{t^{1/\beta}}\right)^{\alpha}, \tag{5.62}$$

which means that (5.58) holds for $0 < t \le r^\beta$ provided $K \ge C_0 C_1$.

Consider now the remaining case $r^\beta < t \le T$. We have, for any $x \in B$,

$$\frac{1}{V(x,t^{1/\beta})} = \frac{V(x_0,T^{1/\beta})}{V(x,t^{1/\beta})}\,\frac{V(x_0,r)}{V(x_0,T^{1/\beta})}\,\frac{1}{V(x_0,r)}.$$

Since $t \le T$, we obtain that, using (2.2) and (5.59),

$$\frac{V(x_0,T^{1/\beta})}{V(x,t^{1/\beta})} \le C_1\left(\frac{T}{t}\right)^{\alpha/\beta} = C_1 2^{a/\beta} K\left(\frac{r}{t^{1/\beta}}\right)^{\alpha}.$$

Since $r < T^{1/\beta}$, the reverse volume doubling (RVD) yields

$$\frac{V(x_0,r)}{V(x_0,T^{1/\beta})} \le C_2\left(\frac{r}{T^{1/\beta}}\right)^{\alpha'} \le C_2 K^{-\alpha'/\alpha}$$

Hence, it follows from (5.60) that, for μ-a.a. $x, y \in B$,

$$p^B_t(x,y) \le C_0 C_1 C_2 2^{\alpha/\beta} K^{-\alpha'/\alpha}\frac{K}{V(x_0,r)}\left(\frac{r}{t^{1/\beta}}\right)^{\alpha},$$

whence (5.58) follows, provided K is chosen large enough to satisfy

$$C_0 C_1 C_2 2^{\alpha/\beta} K^{-\alpha'/\alpha} \leq 1.$$

Let Ω be an open subset of B. Using (5.57) and the Cauchy-Schwarz inequality, we obtain, for any $f \in \mathcal{F}(\Omega)$ and $t > 0$,

$$\begin{aligned}
\left(P_t^\Omega f, f\right) &= \int_\Omega \int_\Omega p_t^\Omega(x,y) f(x) f(y) d\mu(x) d\mu(y) \\
&\leq \frac{C}{\mu(B)} \left(\frac{r}{t^{1/\beta}}\right)^\alpha \|f\|_1^2 \\
&\leq \frac{C\mu(\Omega)}{\mu(B)} \left(\frac{r}{t^{1/\beta}}\right)^\alpha \|f\|_2^2.
\end{aligned}$$

Using the fact that the function $t^{-1}\left(f - P_t^\Omega f,\ f\right)$ monotone increases and converges to $\mathcal{E}(f)$ when t monotone decreases and goes to 0 (cf. Section 2.2), we obtain that

$$\mathcal{E}(f) \geq \frac{1}{t}\left(f - P_t^\Omega f,\ f\right) = \frac{1}{t}\left(\|f\|_2^2 - \left(P_t^\Omega f, f\right)\right),$$

whence, for a non-zero f,

$$\frac{\mathcal{E}(f)}{\|f\|_2^2} \geq \frac{1}{t}\left(1 - \frac{C\mu(\Omega)}{\mu(B)}\left(\frac{r}{t^{1/\beta}}\right)^\alpha\right). \tag{5.63}$$

Since t in (5.63) is arbitrary , we can choose t to satisfy the identity

$$C\frac{\mu(\Omega)}{\mu(B)}\left(\frac{r}{t^{1/\beta}}\right)^\alpha = \frac{1}{2},$$

that is,

$$\frac{1}{t} = \frac{c}{r^\beta}\left(\frac{\mu(B)}{\mu(\Omega)}\right)^{\beta/\alpha},$$

for some $c > 0$. Substituting this t into (5.63), we obtain

$$\lambda_{\min}(\Omega) = \inf_{f \in \mathcal{F}(\Omega)\setminus\{0\}} \frac{\mathcal{E}(f)}{\|f\|_2^2} \geq \frac{1}{2t} = \frac{c}{2r^\beta}\left(\frac{\mu(B)}{\mu(\Omega)}\right)^{\beta/\alpha},$$

which proves (FK_β). □

6. The exit time

Let $\left(\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in M}\right)$ be a Hunt process of the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$.

6.1. **Quasi-continuous functions.** For any set $E \subset M$, the *capacity* Cap (also called 1-capacity) is defined by

$$\operatorname{Cap}(E) = \inf_\varphi \mathcal{E}_1(\varphi), \tag{6.1}$$

where φ varies over all functions from $\mathcal{F}$ such that $\varphi \geq 1$ in an open neighborhood of E (see [15, p.64]). Clearly, we have $\operatorname{Cap}(E) \geq \mu(E)$. Also, it is obvious from the definition that $\operatorname{Cap}(E)$ is monotone function of E.

A function $u : M \to \mathbb{R}$ is said to be *quasi-continuous* if it is continuous in $M \setminus E$ for some set E of capacity 0.

Proposition 6.1. *Let u be a quasi-continuous function on M such that $u \geq 0$ μ-a.e. Then there is an invisible set N such that $u(x) \geq 0$ for all $x \in M \setminus N$.*

Proof. By [15, Theorem 2.1.2, p.67-68], the inequality $u \geq 0$ holds quasi-everywhere, that is, outside some set E of capacity 0. We are left to show that E is contained in an invisible set N. Since $(\mathcal{E}, \mathcal{F})$ is regular, any compact set has finite capacity. As $\mathrm{Cap}(E) = 0$, it follows from [15, Theorem 4.2.1, p.142] that set E is exceptional. By [15, Theorem 4.1.1, p.137], every exceptional set is contained a Borel, properly exceptional set, which is also an invisible set. $\square$

6.2. **Exit times and transition functions.** For any bounded Borel function f on M, set

$$\mathcal{P}_t f(x) := \mathbb{E}_x f(x_t), \quad t > 0, \ x \in M. \tag{6.2}$$

It follows from (2.19) that $\mathcal{P}_t f = P_t f$ almost everywhere. By [15, Theorem 4.2.3, p.144], the function $\mathcal{P}_t f$ is a quasi-continuous realization of a measurable function $P_t f$.

For any open set $\Omega \subset M$, the Hunt process $\left(\{X_t\}_{t\geq 0}, \{\mathbb{P}_x^{\Omega}\}_{x\in M}\right)$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{F}(\Omega))$ is obtained from X_t by imposing the killing condition outside Ω. The transition function $\mathcal{P}_t^{\Omega}$ of this process is given by

$$\mathcal{P}_t^{\Omega}(x, B) = \mathbb{P}_x^{\Omega}(X_t \in A) = \mathbb{P}_x(t < \tau_\Omega \text{ and } X_t \in B),$$

where τ_Ω is the first exit time of the process X_t from Ω defined by (2.20) (see [15, p.135, eq. (4.1.2)]). Consequently, we have

$$\mathcal{P}_t^{\Omega} f(x) = \mathbb{E}_x^{\Omega} f(X_t) = \mathbb{E}_x\left(\mathbf{1}_{\{t<\tau_\Omega\}} f(X_t)\right), \tag{6.3}$$

for every $x \in M$, $t > 0$, and for every bounded (or non-negative) Borel function f. For the heat semigroup P_t^{Ω} of the form $(\mathcal{E}, \mathcal{F}(\Omega))$, we have then

$$P_t^{\Omega} f(x) = \mathbb{E}_x\left(\mathbf{1}_{\{t<\tau_\Omega\}} f(X_t)\right) \quad \text{for } \mu\text{-a.a. } x \in M. \tag{6.4}$$

Let $\varphi(t)$ be a non-negative continuous function on $[0, +\infty)$. Multiplying (6.3) by $\varphi(t)$ and integrating in t, we obtain, for any open set $\Omega \subset M$ and all $x \in M$,

$$\int_0^\infty \varphi(t)\, \mathcal{P}_t^{\Omega} f(x)\, dt = \mathbb{E}_x\left(\int_0^{\tau_\Omega} \varphi(t)\, f(X_t)\, dt\right). \tag{6.5}$$

In particular, for $\varphi \equiv 1$, we obtain

$$\int_0^\infty \mathcal{P}_t^{\Omega} f(x)\, dt = \mathbb{E}_x\left(\int_0^{\tau_\Omega} f(X_t)\, dt\right), \tag{6.6}$$

whence it follows, for $f \equiv 1$, that

$$\mathbb{E}_x \tau_\Omega = \int_0^\infty \mathcal{P}_t^{\Omega} \mathbf{1}_\Omega(x)\, dt. \tag{6.7}$$

6.3. **Mean exit time and the spectral gap.** Here we prove an inequality between the spectral gap and the mean exit time. For any open set $\Omega \subset M$, set

$$\overline{E}(\Omega) = \operatorname*{esup}_{x\in\Omega} \mathbb{E}_x \tau_\Omega. \tag{6.8}$$

Lemma 6.2. *For any non-empty open set $\Omega \subset M$, we have*

$$\lambda_{\min}(\Omega) \geq \frac{1}{\overline{E}(\Omega)}, \tag{6.9}$$

where $\lambda_{\min}(\Omega)$ is defined by (2.12).

Inequality (6.9) is well known in the setting of random walks on graphs and diffusions on manifolds (see for example [22]). Here we give a proof in the full generality.

Proof. Let $H = H_\Omega$ be the generator of the form $(\mathcal{E}, \mathcal{F}(\Omega))$ in $L^2(\Omega, \mu)$. For any $T > 0$ and consider the following operator

$$G_T = \int_0^T e^{-tH} dt = \varphi_T(H),$$

where

$$\varphi_T(\lambda) = \int_0^T e^{-t\lambda} dt = \frac{1-e^{-T\lambda}}{\lambda}.$$

Since the function φ_T is bounded and continuous on $[0, +\infty)$, the operator G_T is a bounded self-adjoint operator in L^2. Since the function φ_T is decreasing, we obtain by the spectral mapping theorem

$$\varphi_T(\lambda_{\min}(\Omega)) = \varphi_T(\inf \operatorname{spec}(H)) = \sup \operatorname{spec}(G_T).$$

Note that

$$\sup \operatorname{spec}(G_T) = \|G_T\|_{2\to 2},$$

where $\|\cdot\|_{p\to p}$ stands for the norm of an operator in $L^p(\Omega, \mu)$. It remains to prove that, for all $T > 0$,

$$\|G_T\|_{2\to 2} \leq \overline{E}(\Omega). \tag{6.10}$$

Indeed, if (6.10) holds then we obtain from the above three lines that

$$\varphi_T(\lambda_{\min}(\Omega)) \leq \overline{E}(\Omega).$$

Letting $T \to \infty$ and observing that $\varphi_T(\lambda) \to 1/\lambda$, we obtain (6.9).

To verify (6.10), recall that the operator $e^{tH} = P_t^\Omega$ can be extended to a bounded operator in L^∞. Therefore, the operator G_T also extends to a bounded operator in L^∞. Since P_t^Ω and $\mathcal{P}_t^\Omega$ coincide as operators in L^∞, we see that, for any bounded Borel function f,

$$G_T f = \int_0^T \left(\mathcal{P}_t^\Omega f\right) dt \quad \mu\text{-a.e.}.$$

Therefore, for μ-a.a. $x \in \Omega$, we obtain

$$|G_T f(x)| \leq \int_0^\infty \mathcal{P}_t^\Omega |f|(x)\, dt = \mathbb{E}_x \int_0^{\tau_\Omega} |f|(X_t)\, dt \leq (\mathbb{E}_x \tau_\Omega) \sup |f|,$$

that is, using (6.8),

$$\operatorname*{esup}_\Omega |G_T f| \leq \overline{E}(\Omega) \sup |f|.$$

This implies, for any $g \in L^1 \cap L^2(\Omega, \mu)$,

$$\|G_T g\|_1 = \sup_{f \in C_0(\Omega) \setminus \{0\}} \frac{(G_T g, f)}{\|f\|_\infty} = \sup_{f \in C_0(\Omega) \setminus \{0\}} \frac{(g, G_T f)}{\|f\|_\infty} \leq \overline{E}(\Omega) \|g\|_1$$

that is,

$$\|G_T\|_{1 \to 1} \leq \overline{E}(\Omega). \tag{6.11}$$

Since P_t^Ω is a positivity preserving operator, so is G_T, that is $f \geq 0$ implies $G_T f \geq 0$, for any Borel function f. In particular, for any $s \in \mathbb{R}$ we have $G_T (f + s)^2 \geq 0$, that is

$$G_T f^2 + 2s G_T f + s^2 G_T 1 \geq 0,$$

whence

$$(G_T f)^2 \leq G_T 1 G_T f^2 \leq \overline{E}(\Omega) G_T f^2.$$

Integrating this inequality, we obtain

$$\|G_T f\|_2^2 \leq \overline{E}(\Omega) \|G_T f^2\|_1 \leq \overline{E}(\Omega)^2 \|f^2\|_1 = \overline{E}(\Omega)^2 \|f\|_2^2,$$

whence (6.10) follows. □

6.4. **Proof of Theorem 2.2.** Here we prove Theorem 2.2. Recall that by Theorem 5.8 we have

$$(S_\beta) \Leftrightarrow (T_\beta), \tag{6.12}$$

by Theorem 2.1

$$(UE_\beta) \Leftrightarrow (DUE_\beta) + (S_\beta) \Leftrightarrow (FK_\beta) + (S_\beta), \tag{6.13}$$

and, by a remark after Theorem 2.2,

$$(E\Omega_\beta) \Rightarrow (E_\beta \leq). \tag{6.14}$$

Besides, we have by [1, Lemma 3.16]

$$(E_\beta) \Rightarrow (P_\beta). \tag{6.15}$$

We will prove below the following implications:

$$(DUE_\beta) \Rightarrow (E\Omega_\beta), \tag{6.16}$$

$$(E\Omega_\beta) \Rightarrow (FK_\beta), \tag{6.17}$$

$$(S_\beta) \Rightarrow (P_\beta) \Rightarrow (T_\beta), \tag{6.18}$$

$$(P_\beta) \Rightarrow (E_\beta \geq). \tag{6.19}$$

Together all the above implications settle Theorem 2.2. Indeed, it follows from (6.12) and (6.18) that

$$(S_\beta) \Leftrightarrow (P_\beta),$$

which together with (6.13) yields

$$(UE_\beta) \Leftrightarrow (DUE_\beta) + (P_\beta) \Leftrightarrow (FK_\beta) + (P_\beta).$$

Using these implications and (6.16), (6.19), (6.14), (6.17), (6.15), we obtain

$$
\begin{array}{ccccc}
(UE_\beta) & & & & \\
\Downarrow & & & & \\
(DUE_\beta)+(P_\beta) & \Rightarrow & (E\Omega_\beta)+(E_\beta\geq) & & \\
 & & \Downarrow & & \\
 & & (E\Omega_\beta)+(E_\beta) & \Rightarrow & (FK_\beta)+(E_\beta) \\
 & & \Downarrow & & \Downarrow \\
 & & (E\Omega_\beta)+(P_\beta) & \Rightarrow & (FK_\beta)+(P_\beta) \\
 & & & & \Downarrow \\
 & & & & (UE_\beta)
\end{array}
$$

whence also

$$(UE_\beta) \Rightarrow (DUE_\beta)+(E_\beta) \Rightarrow (DUE_\beta)+(P_\beta) \Rightarrow (UE_\beta)\,.$$

Clearly, the above implications contain all the equivalences of Theorem 2.2.

Proof of $(DUE_\beta) \Rightarrow (E\Omega_\beta)$. Let B be a ball of radius r in M and Ω be a non-empty open subset of B. We use the fact that (DUE_β) implies the estimate (5.57) (see the proof of Theorem 2.1). It follows from (5.57) that, for every $t>0$ and μ-almost all $x\in M$,

$$P_t^\Omega \mathbf{1}_\Omega(x)=\int_\Omega p_t^\Omega(x,y)\,d\mu(y)\leq\int_\Omega p_t^B(x,y)\,d\mu(y)\leq\frac{C\mu(\Omega)}{\mu(B)}\left(\frac{r}{t^{1/\beta}}\right)^\alpha$$

Since the function $\mathcal{P}_t^\Omega\mathbf{1}_\Omega$ is a quasi-continuous realization of $P_t^\Omega\mathbf{1}_\Omega$, it follows from Proposition 6.1 that there is an invisible set $N\subset M$ such that the following inequality holds for all $x\in M\setminus N$:

$$\mathcal{P}_t^\Omega\mathbf{1}_\Omega\,(x)\leq\frac{C\mu(\Omega)}{\mu(B)}\left(\frac{r}{t^{1/\beta}}\right)^\alpha. \tag{6.20}$$

Choose a dense countable subset D of M and let S be the set of all balls of rational radii centered at the points of D. Let S' be the set of all finite unions of balls from S. Since countable unions of invisible sets are invisible, we obtain that there exists an invisible set N such that (6.20) holds for all $B\in S$, $\Omega\in S'$, $t\in\mathbb{Q}_+$, and $x\in M\setminus N$.

For an arbitrary ball $B\subset M$ of radius r and any open subset $\Omega\subset B$, choose a ball $B_0\in S$ of radius $<2r$ such that $B_0\supset B$, and an increasing sequence $\{\Omega_k\}$ of the sets from S' such that $\Omega_k\uparrow\Omega$. It follows that, for all $t\in\mathbb{Q}_+$ and $x\in M\setminus N$,

$$\mathcal{P}_t^{\Omega_k}\mathbf{1}_{\Omega_k}(x)\leq\frac{C\mu(\Omega_k)}{\mu(B_0)}\left(\frac{2r}{t^{1/\beta}}\right)^\alpha\leq\frac{2^\alpha C\mu(\Omega)}{\mu(B)}\left(\frac{r}{t^{1/\beta}}\right)^\alpha.$$

Letting $k\to\infty$, we obtain that the same inequality holds for $\mathcal{P}_t^\Omega\mathbf{1}_\Omega\,(x)$, that is, (6.20). Finally, using the fact that $\mathcal{P}_t^\Omega\mathbf{1}_\Omega(x)$ is monotone decreasing in t, we obtain the same inequality for all real $t>0$.

It follows from (6.7) and (6.20) that, for all $T > 0$,

$$\begin{aligned}\mathbb{E}_x \tau_\Omega &= \int_0^T \mathcal{P}_t^\Omega \mathbf{1}_\Omega(x)\,dt + \int_T^\infty \mathcal{P}_t^\Omega \mathbf{1}_\Omega(x)\,dt \\ &\le T + \int_T^\infty \frac{C\mu(\Omega)}{\mu(B)} \left(\frac{r}{t^{1/\beta}}\right)^\alpha dt \\ &\le T + \frac{C\mu(\Omega)}{\mu(B)} r^\alpha T^{1-\alpha/\beta},\end{aligned}$$

where we have assumed that $\alpha > \beta$ (it is clear from (2.2) that α can be taken arbitrarily large). Finally, setting

$$T = \left(\frac{\mu(\Omega)}{\mu(B)}\right)^{\beta/\alpha} r^\beta,$$

we obtain $(E\Omega_\beta)$. □

Proof of $(E\Omega_\beta) \Rightarrow (FK_\beta)$. Hypothesis $(E\Omega_\beta)$ implies that, for any ball B of radius r and for any open set $\Omega \subset B$, we have

$$\overline{E}(\Omega) := \operatorname*{esup}_{x\in\Omega} \mathbb{E}_x \tau_\Omega \le Cr^\beta \left(\frac{\mu(\Omega)}{\mu(B)}\right)^\nu,$$

which together with (6.9) yields

$$\lambda_{\min}(\Omega) \ge \frac{1}{\overline{E}(\Omega)} \ge \frac{C}{r^\beta}\left(\frac{\mu(B)}{\mu(\Omega)}\right)^\nu.$$

□

Proof of $(P_\beta) \Rightarrow (T_\beta)$. Fix $t > 0$ and let B be a ball of radius $r \ge 2\left(\delta^{-1}t\right)^{1/\beta}$. Then, for μ-a.a. $x \in \frac{1}{2}B$, we have by (2.19)

$$\begin{aligned}P_t \mathbf{1}_{B^c}(x) &= \mathbb{E}_x\left(\mathbf{1}_{B^c}(X_t)\right) = \mathbb{P}_x\left(X_t \in B^c\right) \\ &\le \mathbb{P}_x\left(X_t \in B(x, r/2)^c\right) \\ &\le \mathbb{P}_x\left(\tau_{B(x,r/2)} \le t\right) \\ &\le \mathbb{P}_x\left(\tau_{B(x,r/2)} \le \delta(r/2)^\beta\right) \le \varepsilon,\end{aligned}$$

where we have used that $B^c \subset B(x, r/2)^c$ and the hypothesis (P_β). □

Proof of $(S_\beta) \Rightarrow (P_\beta)$. It follows from (6.3) that the following identity holds for all open sets $\Omega \subset M$, all $t > 0$ and $x \in M$:

$$\mathbb{P}_x(\tau_\Omega \le t) = 1 - \mathcal{P}_t^\Omega \mathbf{1}_\Omega(x)$$

Hence, it suffices to prove that there exist constants $\varepsilon \in (0, 1)$, $C, \delta > 0$ and an invisible set $N \subset M$ such that, for any $t > 0$ and any ball $B = B(x, r)$ with $x \in M \setminus N$ and $r \ge Ct^{1/\beta}$, the following inequality holds:

$$\mathcal{P}_t^B \mathbf{1}_B(x) \ge 1 - \varepsilon. \tag{6.21}$$

Since the function $\mathcal{P}_t^B \mathbf{1}_B$ is a quasi-continuous realization of $P_t^B \mathbf{1}_B$, it follows from (2.10) and Proposition 6.1 that, for any $t > 0$ and ball B as above, there is an invisible set N such that

$$\mathcal{P}_t^B \mathbf{1}_B \geq 1 - \varepsilon \quad \text{in } \frac{1}{4} B \setminus N. \tag{6.22}$$

Choose a dense countable subset D of M, and let S be the set of all balls of rational radii centered at the points of D. Since countable unions of invisible sets is invisible, there exists an invisible set N such that (6.22) holds for all $B \in S$ and $t \in \mathbb{Q}_+$ provided $r \geq C t^{1/\beta}$.

For any real $t > 0$ and an arbitrary ball $B = B(x, r)$ of radius $r \geq C' t^{1/\beta}$ (where $C' = 2^{1+1/\beta} C$) choose a ball $B_0 \in S$ of radius $r_0 \geq \frac{1}{2} r$ such that $x \in \frac{1}{4} B_0$, $B_0 \subset B$, and select $s \in \mathbb{Q}_+$ such that $\frac{s}{2} \leq t \leq s$. Since $r_0 \geq C s^{1/\beta}$, we obtain from the previous paragraph that the following inequalities hold in $\frac{1}{4} B_0 \setminus N$:

$$\mathcal{P}_t^B \mathbf{1}_B \geq \mathcal{P}_s^B \mathbf{1}_B \geq \mathcal{P}_s^{B_0} \mathbf{1}_{B_0} \geq 1 - \varepsilon.$$

If $x \notin N$ then $x \in \frac{1}{4} B_0 \setminus N$ whence we obtain (6.21). □

Proof of $(P_\beta) \Rightarrow (E_\beta \geq)$. Fix a ball $B = B(x, r)$ centered at $x \in M \setminus N$, where N is an invisible set from condition (P_β). Writing $\tau = \tau_B$ and using (2.21), we obtain

$$\mathbb{E}_x \tau \geq \mathbb{E}_x \left(\mathbf{1}_{\left\{ \tau > \delta r^\beta \right\}} \tau \right) \geq \mathbb{P}_x \left(\tau > \delta r^\beta \right) \delta r^\beta \geq (1 - \varepsilon) \delta r^\beta,$$

which finishes the proof. □

7. Appendix: proof of Lemma 3.4

It suffices to prove that

$$\sup_{f \in \mathcal{T}_X, g \in \mathcal{T}_Y} \Phi(f, g) \geq \operatorname*{esup}_{X \times Y} \varphi. \tag{7.1}$$

Having proved (7.1) in the case of finite measures μ and ν, one obtains (7.1) for σ-finite measures in an obvious way. In what follows we assume that $\mu(X) < \infty$ and $\nu(Y) < \infty$ and, hence, $\varphi_- \in L^1(X \times Y)$.

Set $Z = X \times Y$ and $\sigma = \mu \times \nu$. Let $S \subset Z$ be a measurable set with $\sigma(S) > 0$. By construction of the product measure, for any $\varepsilon > 0$, there exists a sequence $\{R_i\}_{i=1}^\infty$ of the rectangles $R_i = A_i \times B_i$ such that $S \subset \bigcup_i R_i$ and

$$\sum_i \sigma(R_i) \leq \sigma(S) + \varepsilon.$$

Using the fact that the difference of two rectangles is a finite disjoint union of rectangles, one can make $\{R_i\}$ into a disjoint sequence while keeping all the above properties. Define a new set $\widetilde{S} = \bigcup_i R_i$. It follows that

$$\begin{aligned} \sum_i \int_{R_i} \varphi d\sigma &= \int_{\widetilde{S}} \varphi d\sigma = \int_S \varphi d\sigma + \int_{\widetilde{S} \setminus S} \varphi d\sigma \\ &\geq \int_S \varphi d\sigma - \int_{\widetilde{S} \setminus S} \varphi_- d\sigma. \end{aligned}$$

Restricting the above summation to those i with $\sigma(R_i) > 0$, we obtain

$$\begin{aligned}\sum_i \frac{\sigma(R_i)}{\sigma(\widetilde{S})}\left(\frac{1}{\sigma(R_i)}\int_{R_i}\varphi d\sigma\right) &= \frac{1}{\sigma(\widetilde{S})}\sum_i \int_{R_i}\varphi d\sigma \\ &\geq \frac{1}{\sigma(\widetilde{S})}\int_S \varphi d\sigma - \frac{1}{\sigma(\widetilde{S})}\int_{\widetilde{S}\setminus S}\varphi_- d\sigma\end{aligned}$$

Since $\sum_i \frac{\sigma(R_i)}{\sigma(\widetilde{S})} = 1$, there exists an index i such that

$$\frac{1}{\sigma(R_i)}\int_{R_i}\varphi d\sigma \geq \frac{1}{\sigma(\widetilde{S})}\int_S \varphi d\sigma - \frac{1}{\sigma(\widetilde{S})}\int_{\widetilde{S}\setminus S}\varphi_- d\sigma. \tag{7.2}$$

Fix a real number $t < \operatorname{esup}\varphi$ and consider the set $S = \{z \in Z : \varphi(z) \geq t\}$. Then $\sigma(S) > 0$ and

$$\frac{1}{\sigma(S)}\int_S \varphi d\sigma \geq t.$$

Note that $\sigma(\widetilde{S})$ can be made arbitrarily close to $\sigma(S)$ by taking ε to be sufficiently small. Also, since $\sigma(\widetilde{S}\setminus S) \leq \varepsilon$ and $\varphi_- \in L^1(Z)$, the integral $\int_{\widetilde{S}\setminus S}\varphi_- d\sigma$ can be made arbitrarily small provided ε is small enough. Hence, we see that the right hand side of (7.2) can be made arbitrarily close to t. Since $t < \operatorname{esup}\varphi$ is arbitrary, we conclude that

$$\sup_R \frac{1}{\sigma(R)}\int_R \varphi d\sigma \geq \operatorname*{esup}_Z \varphi, \tag{7.3}$$

where sup is taken over all rectangles $R = A \times B$ of positive measure. Setting $f = \frac{1}{\mu(A)}\mathbf{1}_A$ and $g = \frac{1}{\mu(B)}\mathbf{1}_B$, we obtain (7.1).

8. Appendix: list of conditions

(VD) : *Volume doubling property.* There exists a positive constant C such that, for all $x \in M$ and $r > 0$,

$$V(x, 2r) \leq CV(x, r).$$

Equivalently, there exist positive constants α and C such that, for all $x, y \in M$ and $0 < r \leq R$,

$$\frac{V(x,R)}{V(y,r)} \leq C\left(\frac{d(x,y)+R}{r}\right)^\alpha. \tag{8.4}$$

(RVD) : *Reverse volume doubling property.* There exist positive constants α' and c such that, for all $x \in M$ and $0 < r \leq R$,

$$\frac{V(x,R)}{V(x,r)} \geq c\left(\frac{R}{r}\right)^{\alpha'}.$$

(UE_β) : *Upper estimate.* The heat kernel exists and satisfies the inequality

$$p_t(x,y) \leq \frac{C}{V(x,t^{1/\beta})}\exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right),$$

for some constants $C, c > 0$, all $t > 0$ and μ-almost all $x, y \in M$.

(ΦUE_β) : *Upper estimate with Φ-term.* The heat kernel $p_t(x,y)$ exists and admits the following estimate

$$p_t(x,y) \leq \frac{C}{V(x,t^{1/\beta})}\Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right),$$

for some constant C, all $t>0$ and μ-almost all $x,y \in M$, where Φ satisfies (2.14).

(DUE_β) : *On-diagonal upper estimate.* The heat kernel p_t exists and satisfies the estimate

$$p_t(x,y) \leq \frac{C}{\sqrt{V(x,t^{1/\beta})V(y,t^{1/\beta})}},$$

for some constant C, all $t>0$ and μ-almost all $x,y \in M$.

(S_β) : *The survival estimate.* There exist $0<\varepsilon<1$ and $C>0$ such that, for all $t>0$ and all balls $B=B(x_0,r)$ with $r \geq Ct^{1/\beta}$,

$$P_t^B \mathbf{1}_B(x) \geq 1-\varepsilon \quad \text{for } \mu\text{-almost all } x \in \frac{1}{4}B.$$

(T_β) : *The tail estimate.* There exist $0<\varepsilon<\frac{1}{2}$ and $C>0$ such that, for all $t>0$ and all balls $B=B(x_0,r)$ with $r \geq Ct^{1/\beta}$,

$$P_t \mathbf{1}_{B^c}(x) \leq \varepsilon \quad \text{for } \mu\text{-almost all } x \in \frac{1}{4}B.$$

$(T_{\exp})$: *The exponential tail estimate.* The heat kernel p_t exists and satisfies estimate

$$\int_{B(x,r)^c} p_t(x,y)\,d\mu(y) \leq C\exp\left(-c\left(\frac{r}{t^{1/\beta}}\right)^{\frac{\beta}{\beta-1}}\right),$$

for some constants $C,c>0$, all $t>0, r>0$ and μ-almost all $x \in M$.

(P_β) : *The exit probability estimate.* There exist an invisible set $N \subset M$ and constants $\varepsilon \in (0,1)$, $\delta>0$ such that, for all $x \in M\setminus N$ and $r>0$,

$$\mathbb{P}_x\left(\tau_{B(x,r)} \leq \delta r^\beta\right) \leq \varepsilon.$$

(E_β) : *The mean exit time estimate.* There exist an invisible set $N \subset M$ and positive constants C,c such that, for all $x \in M\setminus N$ and $r>0$,

$$cr^\beta \leq \mathbb{E}_x\left(\tau_{B(x,r)}\right) \leq Cr^\beta.$$

$(E\Omega_\beta)$: *Isoperimetric inequality for the mean exit time.* There exist an invisible set $N \subset M$ and positive constants C, ν such that, for any ball B in M of radius $r>0$ and for any non-empty open set $\Omega \subset B$,

$$\sup_{x\in\Omega\setminus N} \mathbb{E}_x(\tau_\Omega) \leq Cr^\beta\left(\frac{\mu(\Omega)}{\mu(B)}\right)^\nu.$$

(FK_β) : *The Faber-Krahn inequality.* There exist positive constants ν, c such that, for all balls $B \subset M$ of radius $r>0$ and for any non-empty open sets $\Omega \subset B$,

$$\lambda_{\min}(\Omega) \geq \frac{c}{r^\beta}\left(\frac{\mu(B)}{\mu(\Omega)}\right)^\nu,$$

where $\lambda_{\min}(\Omega)$ is defined by (2.12).

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