Analysis on ultra-metric and fractal spaces. Heat equation approach

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Heat kernels and Dirichlet forms in $\mathbb{R}^n$

The classical Laplace operator $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ in $\mathbb{R}^n$ is associated with the Dirichlet integral

$$\int_{\mathbb{R}^n} |\nabla f|^2 \, dx \tag{1}$$

via the Green formula

$$(f, -\Delta f)_{L^2} = \int_{\mathbb{R}^n} |\nabla f|^2 \, dx.$$

More precisely, the *Dirichlet form* (1) in the domain $f \in W^{1,2}(\mathbb{R}^n)$ has the generator $\mathcal{L} = -\Delta$ that is a non-negative definite self-adjoint operator in $L^2(\mathbb{R}^n)$ with the domain $W^{2,2}(\mathbb{R}^n)$.

The associated heat equation

$$\partial_t u - \Delta u = 0$$

has a fundamental solution

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right),$$

that is also the transition density function of a diffusion process – Brownian motion in $\mathbb{R}^n$.

For any $\beta \in (0,2)$, the operator $(-\Delta)^{\beta/2}$ determines in a similar way the *non-local* Dirichlet form

$$c_{n,\beta} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(f(x) - f(y))^2}{|x-y|^{n+\beta}} dxdy \tag{2}$$

with the domain $B^{\beta/2}_{2,2}(\mathbb{R}^n)$. The associated heat equation

$$\partial_t u + (-\Delta)^{\beta/2} u = 0$$

has a non-negative fundamental solution $p_t^{(\beta)}(x,y)$, that also serves as the transition density function of a symmetric stable Levy process of index β (a Markov process of jump type).

It is known that, in the case $\beta = 1$,

$$p_t^{(1)}(x,y) = \frac{c_n t}{\left(t^2 + |x-y|^2\right)^{\frac{n+1}{2}}}, \tag{3}$$

(that is the Cauchy distribution), while for any $\beta \in (0,2)$ there is an estimate

$$p_t^{(\beta)}(x,y) \simeq \frac{t}{\left(t^{1/\beta} + |x-y|\right)^{n+\beta}} = \frac{1}{t^{n/\beta}} \left(1 + \frac{|x-y|}{t^{1/\beta}}\right)^{-(n+\beta)}. \tag{4}$$

The sign $\simeq$ means that the ratio of two sides is bounded between two positive constants.

Dirichlet forms of jump type in metric measure spaces

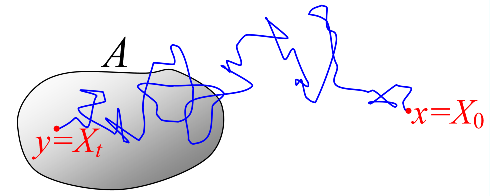
Let (M, d) be a locally compact separable metric space and μ be a Radon measure on M with full support. Consider in $L^2(M, \mu)$ the following quadratic form

$$\mathcal{E}(f,f) = \frac{1}{2}\iint_{M\times M} (f(x) - f(y))^2 J(x,y) d\mu(x) d\mu(y), \tag{5}$$

where $J(x, y)$ is a non-negative symmetric function on $M \times M$ that is called a *jump kernel*. Assume that $\mathcal{E}$ extends to a *regular Dirichlet form* with a domain $\mathcal{F} \subset L^2(M, \mu)$ (that is, $\mathcal{F}$ is a dense subspace of L^2, $\mathcal{F}$ is complete with respect to the norm $\|f\|_{L^2}^2 + \mathcal{E}(f, f)$, and $\mathcal{F} \cap C_0$ is dense both in $\mathcal{F}$ and C_0). The generator of the form (5) is the operator

$$\mathcal{L}f(x) = \int_M (f(x) - f(y)) J(x,y) d\mu(y),$$

that is a non-positive definite self-adjoint operator in $L^2(M, \mu)$. It determines the heat semigroup $\{e^{-t\mathcal{L}}\}_{t\geq 0}$ in $L^2(M, \mu)$ and a certain Hunt process $(\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in M})$ that satisfies the identity



$$\mathbb{P}_x(X_t \in A) = e^{-t\mathcal{L}} 1_A(x)$$

The *heat kernel* $p_t(x, y)$ of $(\mathcal{E}, \mathcal{F})$ is the integral density of the heat semigroup $e^{-t\mathcal{L}}$, if it exists. It is also the transition density function of the Hunt process.

Ultra-metric spaces

Let (M, d) be a metric space. The metric d is called an *ultra-metric* and (M, d) is called an *ultra-metric space* if, for all $x, y, z \in M$,

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}. \tag{6}$$

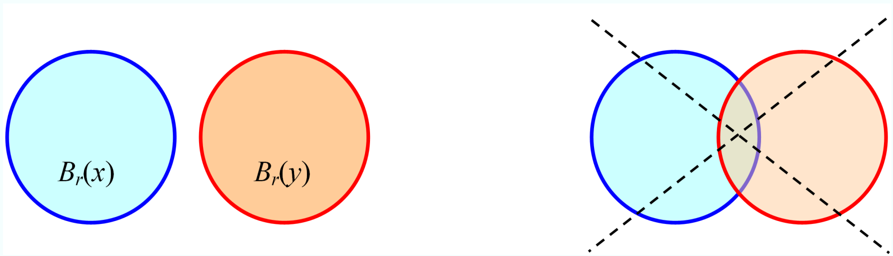
For example, the field $\mathbb{Q}_p$ of p-adic numbers is an ultra-metric space with the p-adic distance

$$d(x, y) = \|x - y\|_p, \quad x, y \in \mathbb{Q}_p.$$

Also, $\mathbb{Q}_p^n$ is an ultra-metric space with the max-distance

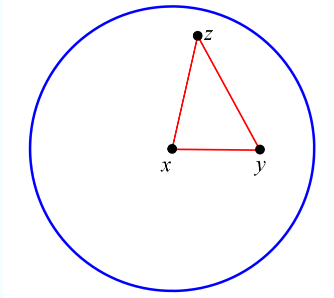
$$d(x, y) = \max(\|x_i - y_i\|_p, \ i = 1, \ldots, n), \quad x, y \in \mathbb{Q}_p^n. \tag{7}$$

On a general ultra-metric space $M, d)$, consider the metric balls $B_r(x) = \{y \in M : d(x, y) \leq r\}$. The ultra-metric property (6) implies that *any two metric balls of the same radius are either disjoint or identical.*



Indeed, let two balls $B_r(x)$ and $B_r(y)$ have a non-empty intersection, say $z \in B_r(x) \cap B_r(y)$. Then $d(x,z) \le r$ and $d(y,z) \le r$ whence it follows $d(x,y) \le r$.

For any point $z \in B_r(x)$ we have $d(x,z) \le r$, which together with $d(x,y) \le r$ implies $d(y,z) \le r$ so that $z \in B_r(y)$. Therefore, $B_r(x) \subset B_r(y)$ and, similarly, $B_r(y) \subset B_r(x)$ whence $B_r(x) = B_r(y)$.
Consequently, the collection of all distinct balls of the same radius r forms a partition of M.



Another consequence: *every point inside a ball is its center*. Indeed, if $y \in B_r(x)$ then the balls $B_r(y)$ and $B_r(x)$ have a non-empty intersection whence $B_r(x) = B_r(y)$.

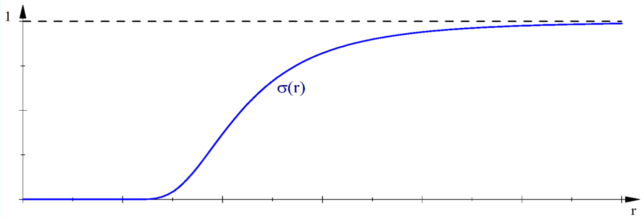
Therefore, all balls are closed and open sets. Consequently, M is *totally disconnected*, that is, M has no connected subsets except for singletons. In particular, an ultra-metric space cannot carry a non-trivial diffusion process.

It follows that any Dirichlet form on an ultra-metric space must be of jump type.

Isotropic Dirichlet forms on ultra-metric spaces

Let (M,d) be an ultra-metric space. We assume throughout that all balls in M are compact. Let μ be a Radon measure on M with full support.

Let $\sigma(r)$ be a cumulative probability distribution function on $(0,\infty)$ that is strictly monotone increasing.



Consider on M the following jump kernel

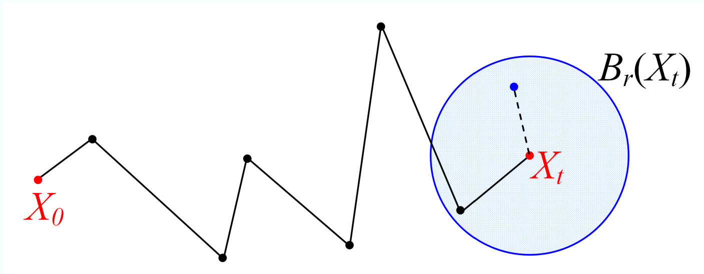
$$J(x,y)=\int_{d(x,y)}^{\infty}\frac{d\log\sigma(r)}{\mu\left(B_r(x)\right)}. \tag{8}$$

Theorem 1 (A.Bendikov, AG, Ch.Pittet, W.Woess, Uspechi, 2014) *The jump kernel* (8) *determines a regular Dirichlet form* $(\mathcal{E},\mathcal{F})$ *in* $L^2(M,\mu)$ *(as in* (5)*), and its heat kernel is*

$$p_t(x,y)=\int_{d(x,y)}^{\infty}\frac{d\sigma^t(r)}{\mu\left(B_r(x)\right)}. \tag{9}$$

The Dirichlet form (5) with the jump kernel (8) is called an *isotropic Dirichlet form.*

The jump process $\{X_t\}_{t\geq 0}$ generated by the Dirichlet form with the jump kernel (8) looks as follows:



At any time t, it jumps from the current position X_t to the next position that is uniformly distributed in $B_r(X_t)$, where r is randomly chosen by using the probability distribution σ.

The ultra-metric property is used in the proof as follows. On an ultra-metric space, the averaging operators

$$Q_r f(x) = \fint_{B_r(x)} f(y) d\mu(y)$$

are bounded in $L^2(M, \mu)$, self-adjoint, and satisfy the identity

$$Q_r Q_s = Q_s Q_r = Q_{\max\{r,s\}} \text{ for all } r, s > 0. \tag{10}$$

In particular, $Q_r^2 = Q_r$, that is, Q_r is an orthoprojector in L^2.

Indeed, for any ball B of radius r, *any* point $x \in B$ is a center of B. Since the value $Q_r f(x)$ is the average of f in B, we see that $Q_r f(x)$ does not depend on $x \in B$. Hence, $Q_r f = \mathrm{const}$ on any ball of radius r.

If $s \leq r$ then the application of Q_s to $Q_r f$ does not change this constant, whence we obtain $Q_s Q_r f = Q_r f$.

If $s > r$ then any ball of radius s is the disjoint union of finitely many balls of radius r. Since the integrals of f and $Q_r f$ over any such ball are the same, we obtain $Q_s Q_r f = Q_s f$.

The property (10) is used to prove that the family of operators

$$P_t = \int_0^\infty Q_r \, d\sigma^t(r), \quad t > 0, \tag{11}$$

is a semigroup and that it coincides with the heat semigroup $e^{-t\mathcal{L}}$ of the isotropic Dirichlet form, which leads to (9).

Since Q_r are orthoprojectors, the identity (11) implies by integration-by-parts the spectral decomposition of P_t, which allows to determine the spectrum of P_t and $\mathcal{L}$.

Let us mention for comparison, that the averaging operator Q_r in $\mathbb{R}^n$ is also bounded and self-adjoint, but it has a non-empty negative part of the spectrum. In particular, it is *not* an orthoprojector.

Isotropic Dirichlet forms on $\mathbb{Q}_p^n$

Consider $M = \mathbb{Q}_p^n$ with the ultra-metric max-distance (7) and with the Haar measure μ normalized to $\mu(B_1(x)) = 1$. One can show that if $p^m \leq r < p^{m+1}$ for some $m \in \mathbb{Z}$ then

$$\mu\left(B_r(x)\right) = p^{nm}. \tag{12}$$

Fix any $\beta > 0$ and consider the distribution function

$$\sigma(r) = \exp\left(-\left(\frac{p}{r}\right)^{\beta}\right) \tag{13}$$

(a Fréchet distribution). Substituting (12) and (13) into (8), we obtain that

$$J(x,y) = c_{p,\beta,n}\, d(x,y)^{-(n+\beta)}, \tag{14}$$

where $c_{p,\beta,n} = \frac{p^{\beta}-1}{1-p^{-n-\beta}}$. Hence, the generator of the corresponding Dirichlet form is

$$\mathcal{L}f(x) = c_{p,\beta,n} \int_M \frac{f(x)-f(y)}{d(x,y)^{n+\beta}} d\mu(y).$$

Miraculously, $\mathcal{L}$ coincides with the Taibleson-Vladimirov operator $\mathcal{D}^{\beta}$ that was originally introduced by means of the Fourier transform in $\mathbb{Q}_p^n$:

$$\widehat{\mathcal{D}^{\beta} f}(\xi) = \|\xi\|_p^{\beta}\, \widehat{f}(\xi). \tag{15}$$

Substituting (12) and (13) into (9), we obtain that the heat kernel of the operator $\mathcal{D}^\beta = \mathcal{L}$ satisfies the identity

$$p_t(x,y) = \int_{d(x,y)}^{\infty} \frac{d \exp\left(-t\left(\frac{p}{r}\right)^\beta\right)}{\mu\left(B_r(x)\right)}.$$

Estimating of the integral yields the following bound of the heat kernel:

$$p_t(x,y) \simeq \frac{1}{t^{n/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(n+\beta)},$$

which is clearly similar to the estimate (4) of the heat kernel of $(-\Delta)^{\beta/2}$ in $\mathbb{R}^n$.

Let us emphasize that $\mathcal{D}^\beta$ generates a Markov jump process in $\mathbb{Q}_p^n$ for any $\beta > 0$ unlike the case of $\mathbb{R}^n$ where $(-\Delta)^{\beta/2}$ generates such a process only if $\beta \in (0,2)$.

Jump kernels on α-regular ultra-metric spaces

Let an ultra-metric space (M,d) with measure μ be α-regular for some $\alpha>0$, that is, for any metric ball $B_r(x)$,

$$\mu\left(B_r(x)\right)\simeq r^{\alpha}. \tag{16}$$

Fix some $\beta>0$ and consider a jump kernel on M such that

$$J(x,y)\simeq d(x,y)^{-(\alpha+\beta)}. \tag{17}$$

The associated Dirichlet form is not necessarily isotropic, and the above approach does not work. A much more elaborate method allows to prove the following.

Theorem 2 (A.Bendikov, AG, E. Hu, J.Hu, Ann. Scuola Norm. Sup. Pisa, 2021) *Assume that* (16) *and* (17) *are satisfied. Then the quadratic form*

$$\mathcal{E}\left(f,f\right)=\frac{1}{2}\iint_{M\times M}\left(f(x)-f(y)\right)^2 J(x,y)d\mu(x)d\mu(y)$$

determines a regular Dirichlet form in $L^2\left(M,\mu\right)$*. Moreover, its heat kernel* $p_t(x,y)$ *exists, is continuous in* (t,x,y)*, Hölder continuous in* (x,y) *and satisfies the estimate*

$$p_t(x,y)\simeq\frac{1}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}, \tag{18}$$

for all $x,y\in M$ *and* $t>0$*. In fact,* (16)+(17)$\Leftrightarrow$(18).

Walk dimension on arbitrary regular metric spaces

Let now (M,d) be an arbitrary metric space that is regular in the following sense: there exists a Radon measure μ on M that is α-regular for some $\alpha>0$. It follows that $\alpha=\dim_H M$ and $\mu\simeq\mathcal{H}_\alpha$, where $\mathcal{H}_\alpha$ denotes the Hausdorff measure of dimension α.

Assume in what follows that (M,d) is regular and that $\mu=\mathcal{H}_\alpha$ with $\alpha=\dim_H M$.

Consider for any $\beta>0$ the quadratic form

$$\mathcal{E}_\beta(f,f)=\frac{1}{2}\iint_{M\times M}\frac{(f(x)-f(y))^2}{d(x,y)^{\alpha+\beta}}d\mu(x)d\mu(y),$$

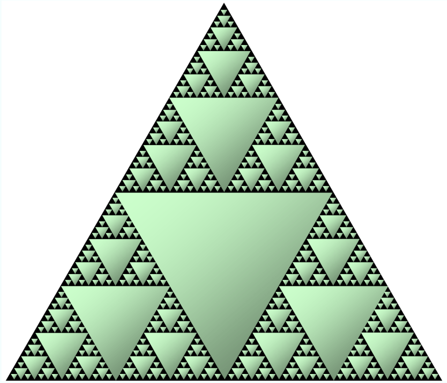
and define the *walk dimension* β^* of (M,d) as follows:

$$\boxed{\beta^*=\sup\left\{\beta>0:\exists\mathcal{F}_\beta\subset L^2(M,\mu)\text{ such that }(\mathcal{E}_\beta,\mathcal{F}_\beta)\text{ is a regular Dirichlet form in }L^2(M,\mu)\right\}}.$$

The point is that with increase of β the set of functions f with $\mathcal{E}_\beta(f,f)<\infty$ shrinks and may become non-dense in L^2. It is easy to show if $\beta<2$ then $\mathcal{E}_\beta(f,f)<\infty$ for all $f\in\operatorname{Lip}_0(M)$, which implies that $\beta^*\geq 2$.

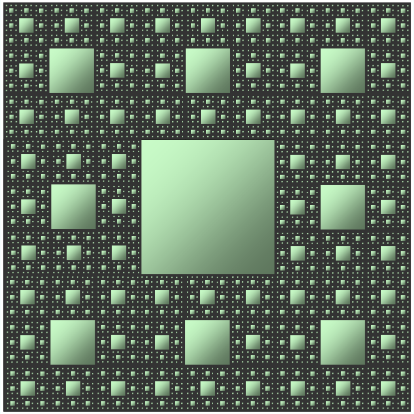
For example:

- in $\mathbb{R}^n$ we have $\beta^*=2$;
- on regular ultra-metric spaces $\beta^*=\infty$ (by Theorem 2);
- on typical *fractal spaces* $2<\beta^*<\infty$.



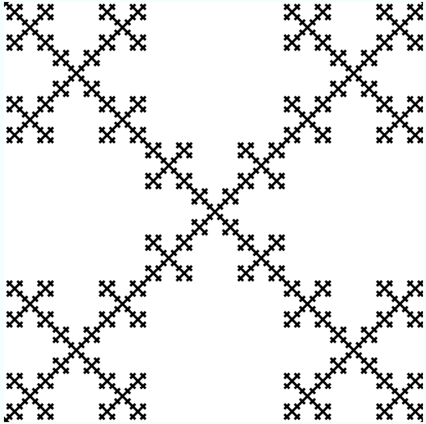
Sierpinski gasket (SG)

$\alpha = \frac{\log 3}{\log 2}, \ \beta^* = \frac{\log 5}{\log 3} \approx 2.32$



Sierpinski carpet (SC)

$\alpha = \frac{\log 8}{\log 3}, \ \beta^* \approx 2.10$



Vicsek snowflake (VS)

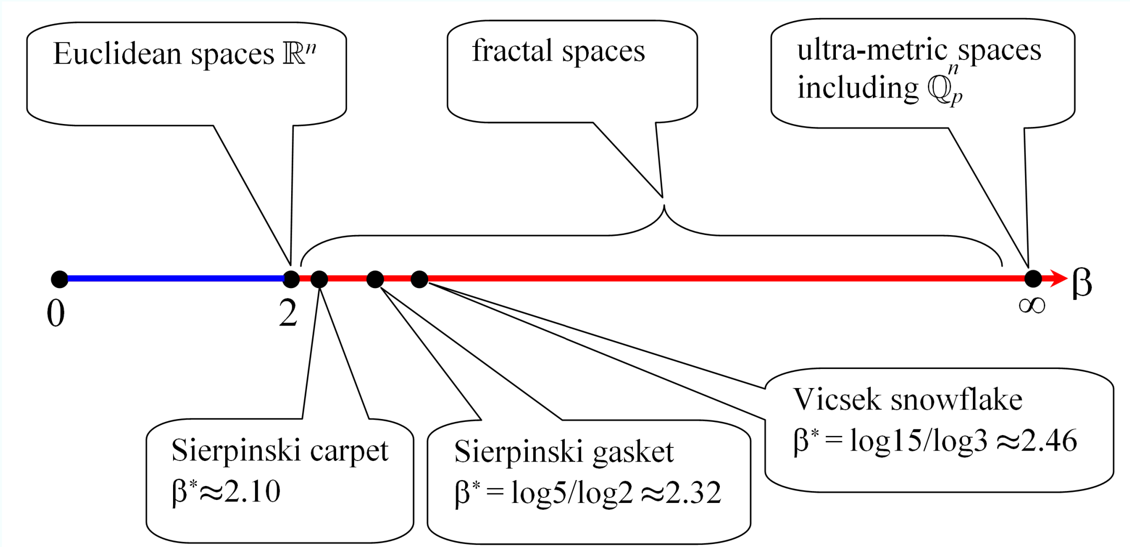
$\alpha = \frac{\log 5}{\log 3}, \ \beta^* = \frac{\log 15}{\log 3} \approx 2.46$

On many fractal spaces (including SG, SC, VS), there exists a *local* regular Dirichlet form (and associated diffusion), whose heat kernel satisfies *sub-Gaussian* estimate

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\gamma}} \exp\left(-c\left(\frac{d^\gamma(x,y)}{t}\right)^{\frac{1}{\gamma-1}}\right) \tag{19}$$

for some $\alpha > 0$ and $\gamma > 1$ (Barlow, Bass, Chen, Hambly, Kigami, Kumagai, Kusuoka, Perkins, et al.). If (19) is satisfied then the metric measure space is necessarily α-regular and $\gamma = \beta^*$. Consequently, we have $\gamma \geq 2$. As M.Barlow showed, any $\gamma \geq 2$ can be realized in (19) on some fractal space.

On the diagram below, we represent graphically a classification of regular metric spaces according to the value of the walk dimension β^*. The Euclidean spaces $\mathbb{R}^n$ and p-adic spaces $\mathbb{Q}_p^n$ lie at the opposite boundaries of this scale, while the entire interior is filled with fractal spaces.



Parameter α is responsible for *integration* on M as it determines measure $\mu = \mathcal{H}_\alpha$, while β^* is responsible for *differentiation* on M as in many cases it determines the generator $\mathcal{L}$ of a local Dirichlet form on M that is a natural Laplacian on M.