Random walks on ultra-metric spaces

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1 Ultra-metric spaces

Let (X, d) be a metric space. The metric d is called *ultra-metric* if it satisfies the ultra-metric inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}, \tag{1}$$

that is obviously stronger than the usual triangle inequality. In this case (X, d) is called an ultra-metric space.

A well-known example of an ultra-metric distance is given by the p-adic norm. Given a prime p, the p-adic norm on $\mathbb{Q}$ is defined as follows: if $x = p^n \frac{a}{b}$, where $a, b, n \in \mathbb{Z}$ and a, b are not divisible by p, then

$$\|x\|_p := p^{-n}.$$

If $x = 0$ then $\|x\|_p := 0$. The p-adic norm on $\mathbb{Q}$ satisfies the ultra-metric inequality. Indeed, if $y = p^m \frac{c}{d}$ and $m \leq n$ then $x + y = p^m \left(\frac{p^{n-m}a}{b} + \frac{c}{d}\right)$. Since the denominator bd is not divisible by p, it follows that

$$\|x + y\|_p \leq p^{-m} = \max\left\{\|x\|_p, \|y\|_p\right\}.$$

Hence, $\mathbb{Q}$ with the metric $\|x-y\|_p$ is an ultra-metric space, and so is its completion $\mathbb{Q}_p$ – the field of p-adic numbers.

The next example of an ultra-metric space is the product

$$\mathbb{Q}_p^n = \overbrace{\mathbb{Q}_p \times \ldots \times \mathbb{Q}_p}^{n \text{ times}}$$

where the ultra-metric is given by the vector p-norm

$$\|(x_1, \ldots, x_n)\|_p := \max_{i=1,\ldots,n} \|x_i\|_p \,.$$

Various constructions of Markov processes on $\mathbb{Q}_p$ and on more general locally compact Abelian groups carrying an ultra-metric, were developed by Steven Evans 1989, Albeverio and Karwowski 1991, Kochubei 2001, Del Muto and Figà-Talamanca 2004, Rodríges-Vega and Zúñiga-Galindo 2008 and many others, by means of Fourier transform on such groups.

Analysis on $\mathbb{Q}_p$ was developed by Taibleson 1975 and by Vladimirov and Volovich 1989, also using Fourier transform. They have introduced a class of pseudo-differential operators on $\mathbb{Q}_p$ and on $\mathbb{Q}_p^n$, in particular, a p-adic Laplacian.

Vladimirov et al. studied the corresponding p-adic Schrödinger equation. Among other results, they explicitly computed (as series expansions) certain heat kernels as well as the Green function of the p-adic Laplacian.

In this work we define a natural class of random walks on an ultra-metric space (X, d) that satisfies in addition the following conditions: it is separable, proper (that is, all balls are compact), and non-compact.

Our construction is very easy, takes full advantage of ultra-metric property and uses no Fourier Analysis. In the case of $\mathbb{Q}_p$ this class of processes coincides with the one constructed by Albeverio and Karwowski, and their generators coincide with the operators of Taibleson and Vladimirov.

Let us first discuss some properties of ultra-metric balls

$$B_r(x) = \{y \in X : d(x, y) \leq r\},$$

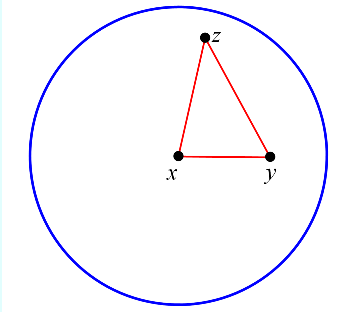
where $x \in X$ and $r > 0$. The ultra-metric property (1) implies that *any two metric balls of the same radius are either disjoint or identical.*

Indeed, let two balls $B_r(x)$ and $B_r(y)$ have a non-empty intersection:

$$\exists z \in B_r(x) \cap B_r(y).$$

Then $d(x, z) \leq r$ and $d(y, z) \leq r$ whence it follows $d(x, y) \leq r$.

Consider now an arbitrary point $z \in B_r(x)$. We have $d(x, z) \leq r$ and $d(x, y) \leq r$ whence $d(y, z) \leq r$ and $z \in B_r(y)$. Therefore, $B_r(x) \subset B_r(y)$ and, similarly, $B_r(y) \subset B_r(x)$ whence $B_r(x) = B_r(y)$.



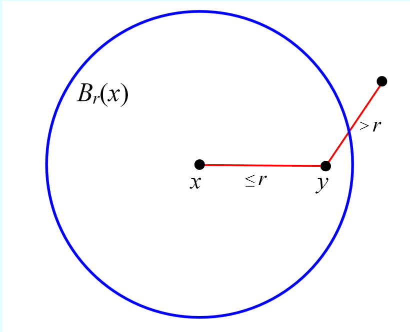
Consequently, a *collection of all distinct balls of the same radius r forms a partition of X*, which is a key property for our construction.

Other properties of ultra-metric spaces:

- *Every point inside a ball is its center.*

  Indeed, if $y \in B_r(x)$ then the balls $B_r(y)$ and $B_r(x)$ have a non-empty intersection whence $B_r(x) = B_r(y)$.

  Consequently, the distance from any point $y \in B_r(x)$ to the complement $B_r(x)^c$ is larger than r.



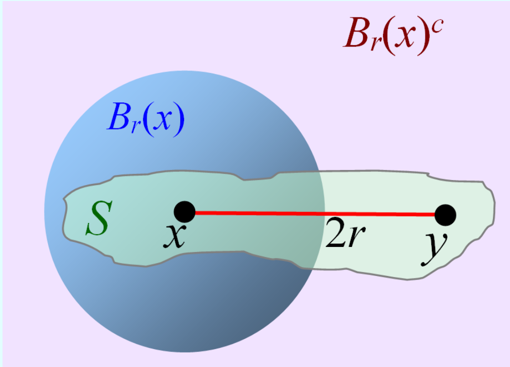
- *Every ball is open and closed as a set.*

  Indeed, any ball $B_r(x)$ is closed by definition, but it is also open because any $y \in B_r(x)$ has a neighborhood $B_r(y) \subset B_r(x)$.

  Consequently, the topological boundary $\partial B_r(x)$ is empty.

- *Any ultrametric space X is totally disconnected*, that is, any non-empty connected subset S of X is an one-point set.

Indeed, if S contains two distinct points, say x and y, set $r = \frac{1}{2} d(x, y)$ and observe that S is covered by two disjoint open sets $B_r(x)$, $B_r(x)^c$ both having non-empty intersection with S. Hence, S is disconnected.



Consequently, X cannot carry any non-trivial diffusion process.

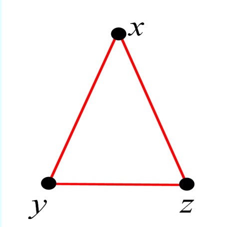
- *Any two balls $B_{r_1}(x)$ and $B_{r_2}(y)$ of arbitrary radii $r_1, r_2 > 0$ are either disjoint or one of them contains the other.*

Indeed, let $r_1 \geq r_2$. If the balls $B_{r_1}(x)$ and $B_{r_2}(y)$ are not disjoint then also the balls $B_{r_1}(x)$ and $B_{r_1}(y)$ are not disjoint, whence

$$B_{r_1}(x) = B_{r_1}(y) \supset B_{r_2}(y).$$

- *Any triangle* $\{x, y, z\} \subset X$ *is isosceles; moreover, the largest two sides of the triangle are equal.*

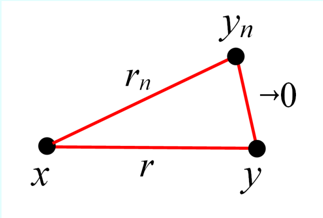
  Indeed, if $d(y, z)$ is smallest among all three distances then we obtain $d(x, y) \leq \max(d(x, z), d(y, z)) = d(x, z)$ and similarly $d(x, z) \leq d(x, y)$ whence $d(x, y) = d(x, z)$.



- *For any* $x \in X$, *a set* $M = \{d(x, y) : y \in X\}$ *has no accumulation point in* $(0, +\infty)$*; in particular,* M *is countable.*

  Let $r \in (0, \infty)$ be an accumulation point of M, i.e. $\exists \{r_n\} \subset M \setminus \{r\}$ such that $r_n \to r$. Choose $y_n \in X$ such that $d(x, y_n) = r_n$.

  By compactness of balls, we can assume that $\{y_n\}$ converges, say $y_n \to y$. Then $d(x, y) = r$. Since $d(y, y_n) \to 0$, it follows that $r = r_n$, which contradicts to the choice of $\{r_n\}$.



  For example, in $\mathbb{Q}_p$ we have $M = \{p^{-m}\}_{m \in Z}$.

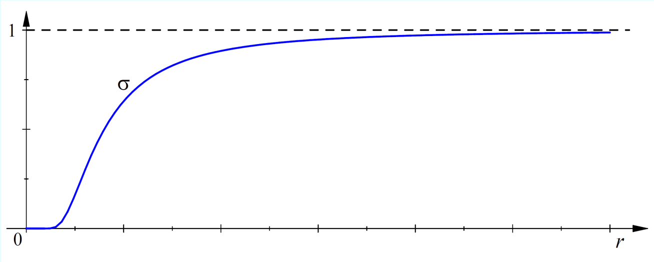
2 Markov operators

Let μ be a Radon measure with full support on an ultra-metric space X. Then $\mu(B_r(x))$ is finite and positive for all $x \in X$ and $r > 0$. Let also $\mu(X) = \infty$.

Define a family $\{Q_r\}_{r>0}$ of averaging operators acting on $f \in L^\infty(X)$:

$$\boxed{Q_r f(x) = \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f \, d\mu}. \tag{2}$$

Clearly, Q_r is a Markov operator. Let $\sigma(r)$ be a cumulative probability distribution function on $(0, \infty)$ that is strictly monotone increasing, left-continuous, and $\sigma(0+) = 0$, $\sigma(\infty-) = 1$.



The following convex combination of Q_r is also a Markov operator:

$$\boxed{Pf = \int_0^\infty Q_r f \, d\sigma(r)}$$

It determines a discrete time Markov chain $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ on X with the following transition rule:

$\mathcal{X}_{n+1}$ is μ-uniformly distributed in $B_r(\mathcal{X}_n)$ where the radius $r > 0$ is chosen at random according to the distribution σ.

We refer to P as an *isotropic Markov* operator associated with (d, μ, σ).

Example. Consider $X = \mathbb{Q}_p$ with the p-adic distance $d(x, y) = \|x - y\|_p$.

Every $x \in \mathbb{Q}_p$ has a presentation in p-adic numeral system:

$$x = \ldots a_k \ldots a_2 a_1 a_0 . a_{-1} a_{-2} \ldots a_{-N} = \sum_{k=-N}^{\infty} a_k p^k,$$

where $N \in \mathbb{N}$ and each a_k is a p-adic digit: $a_k \in \{0, 1, \ldots, p-1\}$. Then $\|x\|_p = p^{-l}$ provided $a_l \neq 0$ and $a_k = 0$ for all $k < l$.

Consider a ball $B_r(x)$ of radius $r = p^{-m}$, where $m \in \mathbb{Z}$. For any

$$y = \ldots b_k \ldots b_2\, b_1\, b_0 . b_{-1}\, b_{-2} \ldots b_{-N} \in B_r(x)$$

we have $\|x - y\|_p \leq p^{-m}$, that is, the first non-zero $a_k - b_k$ occurs for $k \geq m$; that is, $b_k = a_k$ for $k < m$ and b_k are arbitrary for $k \geq m$, so that

$$y = \ldots b_{m+2}\, b_{m+1}\, b_m a_{m-1} a_{m-2} a_{m-3} \ldots$$

Since b_m can take p values, any ball $B_r(x)$ of radius $r = p^{-m}$ consists of p disjoint balls of radii $p^{-(m+1)}$ that are determined by the value of b_m.

Let μ be the Haar measure on $\mathbb{Q}_p$ with the normalization condition

$$\mu\left(B_1(x)\right) = 1.$$

Then we obtain that

$$\mu\left(B_{p^{-m}}(x)\right) = p^{-m}.$$

If $p^{-m} \leq r < p^{-(m-1)}$ then $B_r(x) = B_{p^{-m}}(x)$ which implies

$$\mu\left(B_r(x)\right) = p^{-m} \simeq r.$$

The Markov chain $\{\mathcal{X}_n\}$ with the transition operator P has the following transition rule from $\mathcal{X}_n$ to $\mathcal{X}_{n+1}$. One chooses at random $r > 0$ and, hence, m as above, then changes all the digits a_k of $\mathcal{X}_n$ with $k \geq m$ to b_k, where all b_k are uniformly and independently distributed in $\{0, 1, \ldots, p-1\}$:

$$\begin{aligned}
\mathcal{X}_n &= \ldots a_{m+2} a_{m+1} a_m a_{m-1} a_{m-2} a_{m-3} \ldots \\
\mathcal{X}_{n+1} &= \ldots b_{m+2}\, b_{m+1}\, b_m a_{m-1} a_{m-2} a_{m-3} \ldots
\end{aligned}$$

The averaging operator Q_r on an ultra-metric space X has some unique features arising from ultra-metric properties. We have

$$Q_r f(x) = \frac{1}{\mu(B_r(x))} \int_X \mathbf{1}_{B_r(x)} f d\mu = \int_X q_r(x,y) f(y) d\mu(y),$$

where the kernel

$$q_r(x,y) = \frac{1}{\mu(B_r(x))} \mathbf{1}_{B_r(x)}(y) = \frac{1}{\mu(B_r(y))} \mathbf{1}_{B_r(y)}(x)$$

is symmetric in x, y because $B_r(y) = B_r(x)$ for any $y \in B_r(x)$.

As a Markov operator with symmetric kernel, Q_r extends to a bounded self-adjoint operator in $L^2(X, \mu)$.

Claim. Q_r is an *orthoprojector* in $L^2(X, \mu)$ and $\operatorname{spec} Q_r \subset [0, 1]$.

Proof. For any ball B of radius $r > 0$, any point $x \in B$ is a center of B. The value $Q_r f(x)$ is the average of f in B and, hence, is the same for all $x \in B$; that is, $Q_r f = \text{const}$ on B. A second application of Q_r to $Q_r f$ does not change this constant, whence we obtain $Q_r^2 = Q_r$. Therefore, Q_r is an orthoprojector. It follows that $\operatorname{spec} Q_r \subset [0, 1]$. ∎

Note that general symmetric Markov operators have spectrum in $[-1, 1]$ and the negative part of the spectrum may be non-empty. For example, the stochastic symmetric matrix

$$\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

has eigenvalues 1 and $-\frac{1}{3}$.

The averaging operator Q_r in $\mathbb{R}^n$ is also Markov and symmetric, but it has a non-empty negative part of the L^2-spectrum (and, hence, is not an orthoprojector). For example, the averaging operator in $\mathbb{R}$

$$Q_1 f(x) = \frac{1}{2} \int_{x-1}^{x+1} f(t)\, dt$$

has the Fourier transform

$$\widehat{Q_1 f}(\xi) = \frac{\sin 2\pi\xi}{2\pi\xi} \widehat{f}(\xi)$$

so that its L^2-spectrum consists of all values $\frac{\sin 2\pi\xi}{2\pi\xi}$ $(\xi \in \mathbb{R})$ and, hence, it has a negative part. In fact, $\min \operatorname{spec} Q_1 \approx -0.217$.

Summary of Lecture 1

Let (X,d) be an ultra-metric space, that is, d is a metric that satisfies

$$d(x,y) \leq \max(d(x,z), d(y,z)).$$

Then all metric balls $B_r(x) = \{y \in X : d(x,y) \leq r\}$ (where $x \in X$ and $r > 0$) are open and closed. Also, any point in a ball is its center.

Assume further that all balls are compact, while X is not. Let μ be a Radon measure on X with full support. Consider for any $r > 0$ the averaging operator

$$Q_r f(x) = \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f \, d\mu$$

that is a Markov operator in L^∞. Then Q_r extends to $L^2(X,\mu)$ as an orthoprojector, and $\operatorname{Im} Q_r$ consists of all L^2 functions that are constant on any ball of radius r. Consequently, $\operatorname{spec} Q_r \subset [0,1]$.

It follows also that the family of subspaces $\{\operatorname{Im} Q_r\}_{r>0}$ is monotone decreasing in r in the sense of inclusion.

It is easy to prove that the family of operators $\{Q_r\}_{r>0}$ is right continuous in the strong operator topology and that

$$s\text{-}\lim_{r\to 0} Q_r = \mathrm{id} \quad \text{and} \quad s\text{-}\lim_{r\to\infty} Q_r = 0$$

(for the last claim we need $\mu(X) = \infty$). Hence, $\{Q_r\}_{r>0}$ is a *resolution of identity.*

By a standard convention in spectral theory, a resolution of identity is a family $\{E_\lambda\}_{\lambda>0}$ of orthoprojectors that is monotone *increasing* in λ, left-continuous and $E_{0+} = 0$, $E_{\infty-} = \mathrm{id}$. We obtain such a family by setting $E_\lambda = Q_{1/\lambda}$, but it will be more convenient to work directly with $\{Q_r\}$.

Let σ be a cumulative probability distribution on $(0,\infty)$. It determines a Markov operator P by

$$P := \int_0^\infty Q_r \, d\sigma(r) = -\int_0^\infty \sigma(r) dQ_r.$$

We obtain in the right hand side *explicitly* the spectral resolution of P, where the spectral projectors Q_r are themselves Markov operators!

Consequently, P is a self-adjoint operator in L^2 and $\operatorname{spec} P \subset [0,1]$ because $0 \leq \sigma(r) \leq 1$. By the functional calculus, for any $\varphi \in C[0,1]$,

$$\varphi(P) = -\int_0^\infty \varphi(\sigma(r))\, dQ_r.$$

In particular, the *power* P^t is well-defined for any $t > 0$ and satisfies

$$\boxed{P^t = -\int_0^\infty \sigma^t(r) dQ_r = \int_0^\infty Q_r\, d\sigma^t(r)}. \tag{3}$$

Setting $P^0 = \mathrm{id}$, we obtain a family $\{P^t\}_{t\geq 0}$ that is a symmetric strongly continuous Markov semigroup in $L^2(X)$ (note that $P^t P^s = P^{t+s}$!).

Definition. The semigroup $\{P^t\}_{t\geq 0}$ is referred to as an *isotropic* heat semigroup on X (that is determined by the triple (d, μ, σ)).

Let $\{\mathcal{X}_t\}_{t\geq 0}$ be continuous time Markov process with the transition semigroup $\{P^t\}$. Since the n-step transition operator of the discrete time Markov chain $\{\mathcal{X}_n\}$ is P^n, we see that the discrete time Markov chain $\{\mathcal{X}_n\}$ embeds into the continuous time Markov process $\{\mathcal{X}_t\}$.

3 Heat kernel and isotropic Dirichlet form

Proposition 1 *The isotropic heat semigroup P^t has a density with respect to μ, that is,*

$$P^t f(x) = \int_X p_t(x,y) f(y) d\mu(y),$$

where the heat kernel $p_t(x,y)$ is a continuous function given by

$$\boxed{p_t(x,y) = \int_{d(x,y)}^{\infty} \frac{d\sigma^t(r)}{\mu(B_r(x))}}. \tag{4}$$

Proof. Since by (3)

$$P^t = \int_0^\infty Q_r \, d\sigma^t(r)$$

and Q_r has the kernel

$$q_r(x,y) = \frac{1}{\mu\left(B_r(x)\right)} \mathbf{1}_{B_r(x)}(y),$$

it follows that P^t has the kernel

$$\begin{aligned} p_t(x,y) &= \int_0^\infty q_r(x,y)\, d\sigma^t(r) = \int_0^\infty \frac{1}{\mu\left(B_r(x)\right)} \mathbf{1}_{B_r(x)}(y) d\sigma^t(r) \\ &= \int_{d(x,y)}^\infty \frac{d\sigma^t(r)}{\mu\left(B_r(x)\right)}. \end{aligned}$$

■

As it is well known, any symmetric strongly continuous Markov semigroup in $L^2(X)$ is associated with a Dirichlet form. In particular, the Dirichlet form $(\mathcal{E}, \mathcal{F})$ associated with $\{P^t\}$ is given by

$$\begin{aligned} \mathcal{E}(f,f) &= \lim_{t\to 0} \frac{1}{t}(f - P^t f,\, f)_{L^2} \\ &= \lim_{t\to 0} \frac{1}{2t} \int_X \int_X \left(f(x) - f(y)\right)^2 p_t(x,y) d\mu(x) d\mu(y), \qquad (5) \end{aligned}$$

where the limit always exists in $[0, +\infty]$, and the domain $\mathcal{F}$ consists of functions $f \in L^2(X)$ where the limit is finite.

Proposition 2 *The Dirichlet form $(\mathcal{E}, \mathcal{F})$ associated with $\{P^t\}$ is a jump type Dirichlet form*

$$\mathcal{E}(f,f) = \frac{1}{2}\int_X \int_X (f(x) - f(y))^2 J(x,y) d\mu(x) d\mu(y) \tag{6}$$

with the jump kernel

$$\boxed{J(x,y) = \int_{d(x,y)}^{\infty} \frac{1}{\mu(B_r(x))} d\ln\sigma(r)}. \tag{7}$$

Besides, $(\mathcal{E}, \mathcal{F})$ is regular (that is, $\mathcal{F} \cap C_0(X)$ is dense both in $C_0(X)$ and in $\mathcal{F}$)*.*

We refer to this Dirichlet form $(\mathcal{E}, \mathcal{F})$ as an *isotropic Dirichlet form.*

Proof. Indeed, comparing (5) and (6), as well as using (4), we obtain

$$J(x,y) = \lim_{t\to 0} \frac{1}{t} p_t(x,y) = \lim_{t\to 0} \frac{1}{t} \int_{d(x,y)}^{\infty} \frac{t\sigma^{t-1}(r) d\sigma(r)}{\mu(B_r(x))} = \int_{d(x,y)}^{\infty} \frac{\sigma^{-1} d\sigma(r)}{\mu(B_r(x))}.$$

The regularity of $(\mathcal{E},\mathcal{F})$ follows from the fact that, for any ball B, the indicator function $\mathbf{1}_B$ is continuous in X (because $\partial B=\emptyset$) and $\mathbf{1}_B\in\mathcal{F}$.

Indeed, let $B=B_\rho(z)$. For $f=\mathbf{1}_B$ we have by (6) and (7)

$$\begin{aligned}
\mathcal{E}(f,f) &= \int_{B_\rho(z)}\int_{B^c_\rho(z)} J(x,y)d\mu(x)d\mu(y) \\
&= \iiint_{\left\{x\in B_\rho(z),y\in B^c_\rho(z),r\geq d(x,y)\right\}} \frac{1}{\mu\left(B_r(x)\right)} d\ln\sigma(r)d\mu(x)d\mu(y) \\
&= \int_\rho^\infty \int_{y\in B_r(x)\setminus B_\rho(z)} \left(\int_{x\in B_\rho(z)} \frac{d\mu(x)}{\mu\left(B_r(z)\right)}\right) d\mu(y)d\ln\sigma(r) \\
&= \int_\rho^\infty \int_{y\in B_r(z)\setminus B_\rho(z)} \frac{\mu\left(B_\rho\left(z\right)\right)}{\mu\left(B_r(z)\right)} d\mu(y)d\ln\sigma(r) \\
&= \int_\rho^\infty \mu\left(B_r\left(z\right)\setminus B_\rho\left(z\right)\right)\frac{\mu\left(B_\rho\left(z\right)\right)}{\mu\left(B_r(z)\right)} d\ln\sigma(r) \\
&\leq \mu\left(B_\rho\left(z\right)\right)\ln\frac{1}{\sigma(\rho)}<\infty.
\end{aligned}$$

■

4 Laplacian and Green function

Let $\mathcal{L}$ be the generator of $(\mathcal{E}, \mathcal{F})$ that is a positive definite self-adjoint operator in $L^2(X)$. We refer to $\mathcal{L}$ as an *isotropic Laplacian.*

Since the heat semigroup of $(\mathcal{E}, \mathcal{F})$ is given by $\left\{e^{-t\mathcal{L}}\right\}_{t\geq 0}$, it follows that $e^{-t\mathcal{L}} = P^t$ and, hence,

$$\mathcal{L} = -\ln P = \int_0^\infty \ln \sigma(r) dQ_r. \tag{8}$$

Denote by $\mathcal{C}$ the space of functions $f \in L^2(X)$ satisfying the condition that $\exists r > 0$ such that $f \equiv \text{const}$ on any ball of radius r.

Theorem 3 *The space* $\mathcal{C}$ *is dense in* $L^2(X)$, *is contained in the domain* $\operatorname{dom}(\mathcal{L})$ *of the Laplacian* $\mathcal{L}$, *and, for any* $f \in \mathcal{C}$, *we have*

$$\mathcal{L}f(x) = \int_X (f(x) - f(y))\, J(x, y) d\mu(y). \tag{9}$$

The spectrum of $\mathcal{L}$ is given by

$$\operatorname{spec}\mathcal{L}=\overline{\left\{\ln\frac{1}{\sigma(r)}:r\in\Lambda\right\}}\cup\{0\}\,, \tag{10}$$

where $\Lambda=\{d(x,y):x,y\in X,\ x\neq y\}$. Furthermore, $\mathcal{L}$ has a complete system of eigenfunctions of the form

$$\boxed{f=\tfrac{1}{\mu(B')}\mathbf{1}_{B'}-\tfrac{1}{\mu(B)}\mathbf{1}_{B}}$$

where B is any ball in X and B' is any maximal ball such that $B'\subsetneqq B$. The eigenvalue of f is $\lambda=\ln\frac{1}{\sigma(r)}$ where r is the largest radius of B.

The identity (9) follows from (8) by integration by parts, where one should watch the singularity of $\ln\sigma(r)$ near $r=0$. By (8), the spectrum of $\mathcal{L}$ is determined by the values of $\ln\sigma(r)$ at those r where dQ_r does not vanish, which occurs exactly at $r\in\Lambda$.

Observe that, for any $x\in B$, there exists the maximal ball B' containing x and such that $B'\subsetneqq B$: in fact, $B'=B_{r'}(x)$ where r' is the largest value in $(0,r)$ of $d(x,\cdot)$.

The *Green function* $g(x,y)$ on $X \times X$ is defined by

$$g(x,y) = \int_0^\infty p_t(x,y)dt.$$

It is known that if g finite (which means $g(x,y) < \infty$ for all $x \neq y$) then g is in some sense inverse to $\mathcal{L}$: the minimal non-negative solution to $\mathcal{L}u = f$ (where $f \geq 0$) is given by

$$u(x) = \int_M g(x,y)f(y)d\mu(y).$$

Also, it is known that $\{\mathcal{X}_t\}_{t\geq 0}$ is transient of and only if g is finite.

Proposition 4 *We have*

$$\boxed{g(x,y) = -\int_{d(x,y)}^\infty \frac{1}{\mu(B_r(x))} d\frac{1}{\ln \sigma(r)}}. \tag{11}$$

Proof. Using (4), we obtain

$$\begin{aligned} g(x,y) &= \int_0^\infty \int_{d(x,y)}^\infty \frac{t\sigma^{t-1}(r)}{\mu(B_r(x))} d\sigma(r) dt \\ &= \int_{d(x,y)}^\infty \left(\int_0^\infty t\sigma^t(r)\, dt \right) \frac{d \ln \sigma(r)}{\mu(B_r(x))}. \end{aligned}$$

Since for any $a > 0$

$$\int_0^\infty t e^{-at} dt = \frac{1}{a^2},$$

it follows that

$$\int_0^\infty t\sigma^t(r)\, dt = \frac{1}{(\ln \sigma(r))^2}$$

and

$$g(x,y) = \int_{d(x,y)}^\infty \frac{d \ln \sigma(r)}{\mu(B_r(x)) (\ln \sigma(r))^2} = -\int_{d(x,y)}^\infty \frac{1}{\mu(B_r(x))} d \frac{1}{\ln \sigma(r)}.$$

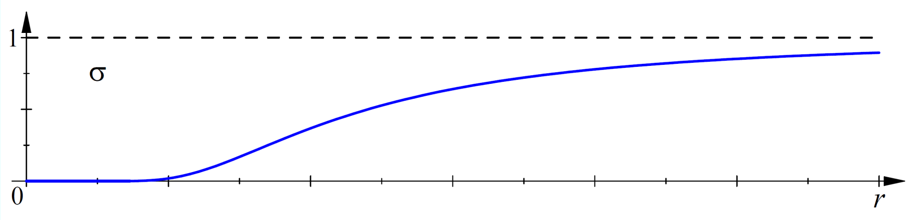
■

Example. Assume that the space (X, d, μ) is α-regular, that is, for all $x \in X$ and $r > 0$,

$$\mu\left(B_r(x)\right) \simeq r^\alpha,$$

where $\alpha > 0$ (in fact, α has to be the Hausdorff dimension of (X, d)).

Choose σ as follows:
$\sigma(r) = \exp\left(-\left(\frac{c}{r}\right)^\beta\right)$
where $c, \beta > 0$
(Fréchet distribution)



By (4) we obtain

$$\begin{aligned} p_t(x,y) &= \int_{d(x,y)}^{\infty} \frac{t\sigma^t(r)\, d\ln\sigma(r)}{\mu\left(B_r(x)\right)} \simeq t\int_{d(x,y)}^{\infty} \exp\left(-\frac{tc^\beta}{r^\beta}\right) r^{-\alpha-\beta-1} dr \\ &\simeq t^{-\alpha/\beta} \int_{d(x,y)/t^{1/\beta}}^{\infty} \exp\left(-\frac{c^\beta}{s^\beta}\right) s^{-\alpha-\beta-1} ds \simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}, \end{aligned}$$

so that

$$p_t(x,y) \simeq \frac{t}{\left(t^{1/\beta} + d(x,y)\right)^{\alpha+\beta}}. \tag{12}$$

The jump kernel admits then the estimate

$$J\left(x,y\right) = \lim_{t\to 0} \frac{p_t\left(x,y\right)}{t} \simeq d(x,y)^{-(\alpha+\beta)}.$$

For the Green function, we have

$$g(x,y) = -\int_{d(x,y)}^{\infty} \frac{d\frac{1}{\ln \sigma(r)}}{\mu(B_r(x))} \simeq \int_{d(x,y)}^{\infty} \frac{dr^{\beta}}{r^{\alpha}} = \begin{cases} \infty, & \alpha \le \beta \\ d(x,y)^{-(\alpha-\beta)}, & \alpha > \beta. \end{cases}$$

Recall for comparison that the symmetric stable process in $\mathbb{R}^n$ of the index $\beta \in (0,2)$ (generated by $(-\Delta)^{\beta/2}$) has the heat kernel

$$p_t(x,y) \simeq \frac{t}{\left(t^{1/\beta} + \|x-y\|\right)^{n+\beta}},$$

while

$$J(x,y) = c_{n,\beta} \left\|x-y\right\|^{-(n+\beta)} \ \text{ and } \ g(x,y) = c'_{n,\beta} \left\|x-y\right\|^{-(n-\beta)} \ (\text{if } n > \beta).$$

5 Isotropic semigroup in $\mathbb{Q}_p$

Set $X = \mathbb{Q}_p$ with the p-adic distance $d(x,y) = \|x-y\|_p$ and with the Haar measure μ normalized so that $\mu(B_1(x)) = 1$. We already know that

$$\mu(B_r(x)) = p^n \quad \text{if} \quad p^n \le r < p^{n+1}, \tag{13}$$

where $n \in \mathbb{Z}$. Fix some $\beta > 0$ and set

$$\sigma(r) = \exp\left(-\left(\frac{p}{r}\right)^{\beta}\right).$$

Knowing $\mu(B_r(x))$ exactly enables us to make a precise computation of $J(x,y)$ as follows. By (7) we have

$$J(x,y) = \int_{d(x,y)}^{\infty} \frac{1}{\mu(B_r(x))} d \ln \sigma(r) = p^{\beta} \int_{\|x-y\|_p}^{\infty} \frac{\beta r^{-\beta-1}\, dr}{\mu(B_r(x))}.$$

Let $\|x-y\|_p = p^k$ for some $k \in \mathbb{Z}$. Using (13), we obtain

$$\begin{aligned}
\int_{p^k}^{\infty} \frac{\beta\, r^{-\beta-1}\, dr}{\mu\left(B_r(x)\right)} &= \sum_{n\geq k} \int_{p^n}^{p^{n+1}} \frac{\beta\, r^{-\beta-1}\, dr}{\mu\left(B_r(x)\right)} \\
&= \sum_{n\geq k} \int_{p^n}^{p^{n+1}} \frac{-dr^{-\beta}}{p^n} = \sum_{n\geq k} \frac{1}{p^n}\left(\frac{1}{p^{n\beta}} - \frac{1}{p^{(n+1)\beta}}\right) \\
&= \left(1-p^{-\beta}\right) \sum_{n\geq k} \frac{1}{p^{n(1+\beta)}} = \left(1-p^{-\beta}\right) \frac{p^{-k(1+\beta)}}{1-p^{-(1+\beta)}} \\
&= \frac{1-p^{-\beta}}{1-p^{-(1+\beta)}} \frac{1}{\|x-y\|_p^{1+\beta}}.
\end{aligned}$$

Hence, we obtain the identity

$$J(x,y) = \frac{p^{\beta}-1}{1-p^{-(1+\beta)}} \frac{1}{\|x-y\|_p^{1+\beta}}. \tag{14}$$

The jump kernel (14) arises from a completely different point of view.

As a locally compact abelian group, $\mathbb{Q}_p$ has the dual group, that is again $\mathbb{Q}_p$, which allows to define Fourier transform. The Fourier transform $f \mapsto \widehat{f}$ of a function f on $\mathbb{Q}_p$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{Q}_p} e^{2\pi i \{x\xi\}} f(x) d\mu(x),$$

where $\xi \in \mathbb{Q}_p$ and $\{x\xi\}$ is the fractional part of the p-adic number $x\theta$, that is, $\{x\xi\} \in \mathbb{Q}$. It is known that $f \mapsto \widehat{f}$ is a linear isomorphism of the space $\mathcal{C}_0$ of locally constant functions on $\mathbb{Q}_p$ with compact support.

Using the Fourier transform, Vladimirov and Volovich introduced in 1989 the following class of *fractional derivatives* $\mathfrak{D}^\beta$ on functions on $\mathbb{Q}_p$.

Definition. For any $\beta > 0$, the operator $\mathfrak{D}^\beta$ is defined on functions $f \in \mathcal{C}_0$ by

$$\widehat{\mathfrak{D}^\beta f}(\xi) = \|\xi\|_p^\beta \, \widehat{f}(\xi), \quad \xi \in \mathbb{Q}_p. \tag{15}$$

They showed that $\mathfrak{D}^{\beta}$ can be written as singular integral operator

$$\mathfrak{D}^{\beta} f(x) = \frac{p^{\beta} - 1}{1 - p^{-(1+\beta)}} \int_{\mathbb{Q}_p} \frac{f(x) - f(y)}{\|x - y\|_p^{1+\beta}} \, d\mu(y). \tag{16}$$

Comparison with (14) shows that $\mathfrak{D}^{\beta}$ coincides with the isotropic Laplacian $\mathcal{L}$ that corresponds to $\sigma(r) = \exp\left(-\left(\frac{p}{r}\right)^{\beta}\right)$. More precisely, we have $\mathfrak{D}^{\beta} = \mathcal{L}$ in $\mathcal{C}_0$ so that $\mathfrak{D}^{\beta}$ is essentially self-adjoint in $L^2(\mathbb{Q}_p)$.

Theorem 5 *The operator $\mathfrak{D}^{\beta}$ generates a heat semigroup in $L^2(\mathbb{Q}_p)$ that admits a continuous heat kernel $p_t(x, y)$ satisfying the estimate*

$$p_t(x, y) \simeq \frac{t}{(t^{1/\beta} + \|x - y\|_p)^{1+\beta}}. \tag{17}$$

The Green function of $\mathfrak{D}^{\beta}$ is finite if and only if $\beta < 1$, and in this case it is given by

$$g(x, y) = \frac{1 - p^{-\beta}}{1 - p^{-(1-\beta)}} \|x - y\|_p^{-(1-\beta)}. \tag{18}$$

Proof. Since $\mathfrak{D}^\beta = \mathcal{L}$, we can apply all the previous results.

The heat kernel estimate (17) follows from (12) because $\mathbb{Q}_p$ is α-regular with $\alpha = 1$.

The identity (18) for the Green function follows by exact integration in (11) similarly to the computation of $J(x, y)$. ■

Despite the fact that the above statement is a simple consequence of the previous results, we call it “Theorem” for the following reason: without knowing the theory of isotropic heat semigroup, the question of estimating the heat kernel of $\mathfrak{D}^\beta$ was very difficult and it remained open for many years. In fact, the full estimate (17) was obtained for the first time in our work in 2014 by using the isotropic Laplacian.

In contrast to that, the identity (18) for the Green function was proved by Vladimirov and Volovich directly from (15).

6 Ultra-metric product spaces

Let $\{(X_i, d_i)\}_{i=1}^n$ be a finite sequence of ultra-metric spaces. Define their *ultra-metric product* (X, d) by $X = X_1 \times \ldots \times X_n$ and

$$d(x, y) = \max_{1 \le i \le n} d_i(x_i, y_i).$$

where $x = (x_1, \ldots, x_n) \in X$ and $y = (y_1, \ldots y_n) \in Y$. Then (X, d) is again an ultra-metric space, and balls in X are products of balls in X_i:

$$B_r(x) = \prod_{i=1}^n B_r^{(i)}(x_i).$$

If there is a Radon measure μ_i on each (X_i, d_i), then we consider on (X, d) the product measure $\mu = \bigotimes \mu_i$.

Given a probability distribution σ on $(0, \infty)$ as above, we obtain an isotropic semigroup P^t on the product space X.

For example, consider $\mathbb{Q}_p^n$ that is the ultra-metric product of n copies of $\mathbb{Q}_p$, with the p-adic metric

$$d\left(x,y\right)=\left\|x-y\right\|_p=\max_{1\le i\le n}\left\|x_i-y_i\right\|_p.$$

The product of the normalized Haar measures μ on $\mathbb{Q}_p$ is the normalized Haar measure μ_n on $\mathbb{Q}_p^n$.

Hence, if $p^{-m}\le r<p^{-(m-1)}$ where $m\in\mathbb{Z}$ then, for any $x\in\mathbb{Q}_p^n$,

$$\mu_n\left(B_r(x)\right)=\prod_{i=1}^{n}\mu\left(B_r^{(i)}\left(x_i\right)\right)=p^{-nm}\simeq r^n.$$

Fix any $\beta>0$ and consider the distribution function

$$\sigma(r)=\exp\left(-\left(\frac{p}{r}\right)^{\beta}\right).$$

As in the one-dimensional case, computing $J(x,y)$ from (7), that is,

$$J(x,y)=\int_{d(x,y)}^{\infty}\frac{1}{\mu\left(B_r(x)\right)}d\ln\sigma(r),$$

and using the exact values of $\mu\left(B_r(x)\right)$, one obtains

$$J(x,y)=\frac{p^{\beta}-1}{1-p^{-(n+\beta)}}\,\|x-y\|_p^{-(n+\beta)}\,. \tag{19}$$

Similarly, (11) yields, in the case $n>\beta$, that

$$g(x,y)=\frac{1-p^{-\beta}}{1-p^{-(n-\beta)}}\;\|x-y\|_p^{-(n-\beta)}\,,$$

and (4) implies

$$p_t\left(x,y\right)\simeq\frac{t}{\left(t^{1/\beta}+\|x-y\|_p\right)^{n+\beta}}=\frac{1}{t^{n/\beta}}\left(1+\frac{\|x-y\|_p}{t^{1/\beta}}\right)^{-(n+\beta)}\,.$$

Hence, the jump kernel, Green function and the heat kernel for the isotropic Markov process in $\mathbb{Q}_p^n$ match the same quantities for the symmetric stable process of index β in $\mathbb{R}^n$ (apart from the values of constants and the range of β because $\beta\in(0,2)$ in $\mathbb{R}^n$ and $\beta\in(0,\infty)$ in $\mathbb{Q}_p^n$).

Summary of Lecture 2

Let (X, d) be a proper, separable ultra-metric space and μ be a Radon measure on X with full support. For any distribution function σ on $(0, \infty)$, we have constructed an isotropic heat semigroup $\{P^t\}_{t\geq 0}$ in $L^2(X)$, and obtained the following identity for its heat kernel:

$$p_t(x, y) = \int_{d(x,y)}^{\infty} \frac{d\sigma^t(r)}{\mu(B_r(x))}.$$

The associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular and has the jump kernel

$$J(x, y) = \int_{d(x,y)}^{\infty} \frac{1}{\mu(B_r(x))} d\ln\sigma(t).$$

Fix $\alpha, \beta > 0$, assume that (X, d, μ) α-regular, that is,

$$\mu(B_r(x)) \simeq r^{\alpha},$$

and choose σ as follows:

$$\sigma(r) = \exp\left(-\left(\frac{p}{r}\right)^{\beta}\right).$$

We have proved that

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{20}$$

and

$$J(x,y) \simeq d(x,y)^{-(\alpha+\beta)}. \tag{21}$$

Moreover, in $\mathbb{Q}_p^n$ the isotropic jump kernel satisfies the identity

$$J(x,y) = c_{n,\beta} d(x,y)^{-(n+\beta)}$$

and (20) holds with $\alpha = n$.

Suppose now that $(\mathcal{E}, \mathcal{F})$ is a general (not isotropic) regular Dirichlet form of jump type on the ultra-metric space.

How to characterize those Dirichlet forms whose heat kernels satisfy (21)?

It is true that if the jump kernel (not isotropic) satisfies (21) then the heat kernel satisfies (20)?

Even in $\mathbb{Q}_p^n$ these questions are highly non-trivial.

7 Heat kernels on more general spaces

Let (X, d) be a a separable, proper metric space and μ be a Radon measure on X with full support. Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(X, \mu)$ and $\{P_t\}_{t \geq 0}$ is the associated heat semigroup.

One of the most discussed problems is obtaining estimates of the corresponding heat kernel $p_t(x, y)$ (as well as its existence).

There are very few situations when the heat kernel can be computed exactly and explicitly. In $\mathbb{R}^n$ with the Lebesgue measure, the classical Dirichlet form

$$\mathcal{E}(f, f) = \int_{\mathbb{R}^n} |\nabla f|^2 \, dx,$$

has the generator $\mathcal{L} = -\Delta$ and the Gauss-Weierstrass heat kernel

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right),$$

that is the normal distribution at any time t.

For the symmetric stable process of index 1, generated by $\sqrt{-\Delta}$, the heat kernel is the Cauchy distribution with the parameter t, that is,

$$p_t(x,y) = \frac{c_n t}{\left(t^2 + |x-y|^2\right)^{\frac{n+1}{2}}},$$

with some $c_n > 0$.

In $\mathbb{R}^n$ with measure $d\mu = e^{|x|^2} dx$, the Dirichlet form

$$\mathcal{E}(f,f) = \int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu$$

has the generator $\mathcal{L} = -\Delta - 2x \cdot \nabla$ and the Mehler heat kernel

$$p_t(x,y) = \frac{1}{(2\pi \sinh 2t)^{n/2}} \exp\left(\frac{2x \cdot y \, e^{-2t} - |x|^2 - |y|^2}{1 - e^{-4t}} - nt\right).$$

In the hyperbolic space $\mathbb{H}^3$, the Laplace-Beltrami operator has the heat kernel

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \frac{r}{\sinh r} \exp\left(-\frac{r^2}{4t} - t\right) \qquad \text{where } r = d(x,y).$$

For many application quantitate properties of the heat kernels are important, so it becomes essential to have at least good estimates.

Let us recall some results about heat kernel bounds assuming that the space (X, d, μ) is α-regular, that is,

$$\mu\left(B_r\left(x\right)\right) \simeq r^{\alpha},$$

where necessarily $\alpha = \dim_H X$.

Let first X be a Riemannian manifold with the geodesic distance d and Riemannian measure μ. For the heat kernel of the local Dirichlet form

$$\mathcal{E}\left(f, f\right) = \int_X \left|\nabla f\right|^2 d\mu$$

the following is known: it satisfies the two-sides Gaussian estimates

$$p_t\left(x, y\right) \asymp \frac{c_1}{t^{\alpha/2}} \exp\left(-c_2 \frac{d^2\left(x, y\right)}{t}\right)$$

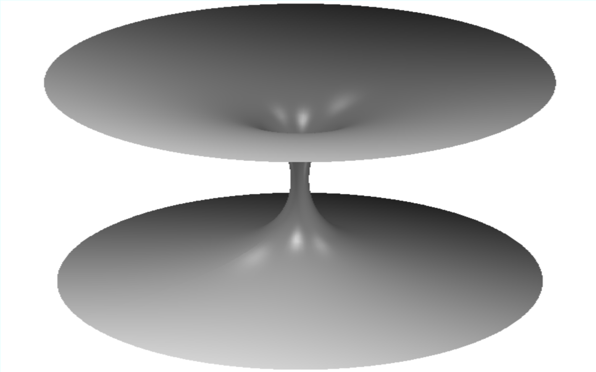
(where $c_1, c_2 > 0$) if and only if the following Poincaré inequality holds: for any ball $B = B_r\left(x_0\right)$ and any $f \in C^1\left(B\right)$,

$$\int_{\varepsilon B} \left(f - \overline{f}\right)^2 d\mu \leq C r^2 \int_B \left|\nabla f\right|^2 d\mu, \tag{22}$$

where $\overline{f} = \fint_{\varepsilon B} f d\mu$ and the constants C and $\varepsilon \in (0,1]$ are the same for all balls and functions.

For example, (22) holds in $\mathbb{R}^n$ and, moreover, on all manifolds of non-negative Ricci curvature.

However, it fails on the following manifold that is a connected sum of two copies of $\mathbb{R}^n$, because it has a “bottleneck”.



Development of Analysis on fractal spaces has brought into life *sub-Gaussian* estimates of heat kernels of local Dirichlet forms. This is the estimate of the form

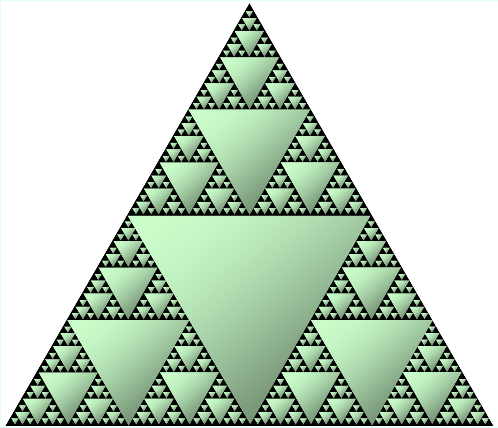
$$p_t(x,y) \asymp \frac{c_1}{t^{\alpha/\beta^*}} \exp\left(-c_2\left(\frac{d^{\beta^*}(x,y)}{t}\right)^{\frac{1}{\beta^*-1}}\right), \tag{23}$$

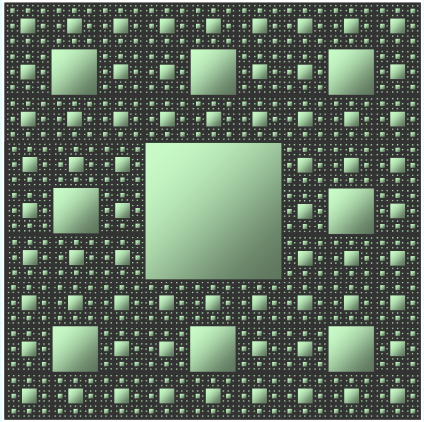
where β^* is a new parameter that is called the *walk dimension* of the corresponding diffusion process.

For example, the walk dimension of a diffusion process on a manifold, satisfying the Gaussian estimate, is clearly $\beta^* = 2$.

One can show that (23) implies $\beta^* \geq 2$.

It was proved in 1990s by M.Barlow, R.Bass et al. that, on a large class of fractals (like unbounded *Sierpinski gasket* and *carpet*), there is a diffusion process whose heat kernel satisfies the sub-Gaussian estimate (23) with $\beta^* > 2$.





For example, on the Sierpinski gasket $\alpha = \frac{\ln 3}{\ln 2}$ and $\beta^* = \frac{\ln 5}{\ln 2} \approx 2.32$, on the Sierpinski carpet $\alpha = \frac{\ln 8}{\ln 3}$ and $\beta^* \approx 2.09$.

Let us discuss a possibility of the heat kernel estimates (23) on a general metric measure space X. If (23) is true for some diffusion on X then X has to be α-regular and μ has to be comparable to the Hausdorff measure of dimension α. In particular, $\alpha = \dim_H X$ so that α is an invariant of the metric space (X, d).

To describe the nature of β^*, consider for any $\beta > 0$ the following quadratic form in $L^2(X, \mu)$:

$$\mathcal{E}_{\alpha,\beta}(f, f) = \int_X \int_X \frac{(f(x) - f(y))^2}{d(x, y)^{\alpha+\beta}} d\mu(x) \, d\mu(y).$$

It was proved by AG, Jiaxin Hu and K.S. Lau in 2003 that the walk dimension β^* admits the following characterization:

$$\boxed{\beta^* = \sup \{\beta > 0 : \mathcal{E}_{\alpha,\beta} \text{ extends to a regular Dirichlet form}\}}. \tag{24}$$

Consequently, β^* is also an invariant of the metric structure (X, d) alone!

The identity (24) holds under the hypothesis that a diffusion on X satisfies the sub-Gaussian estimate. However, the right hand side makes sense on an arbitrary α-regular metric space, so we can take now (24) as a *new definition* of the walk dimension β^*. It is valid for any α-regular metric space independently of heat kernels.

It follows from (24) that always $\beta^* \geq 2$ because, for any $\beta < 2$, Lipschitz functions with compact support are in the domain of $\mathcal{E}_{\alpha,\beta}$.

If X is a Riemannian manifold then one can deduce from (24) that $\beta^* = 2$. On fractals, as we know, typically $\beta^* > 2$.

Let us ask what is the walk dimension β^* of an ultra-metric space?

As we know, on an α-regular ultra-metric space, the isotropic Dirichlet form $\mathcal{E}$ with the distribution function $\sigma(r) = \exp(-(c/r)^\beta)$ with arbitrary $\beta > 0$ has the jump kernel

$$J(x, y) \simeq d(x, y)^{-(\alpha+\beta)}.$$

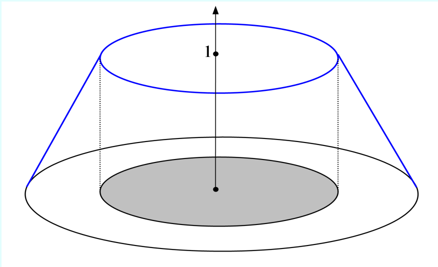
Since this jump kernel is comparable with the jump kernel of $\mathcal{E}_{\alpha,\beta}$, we have

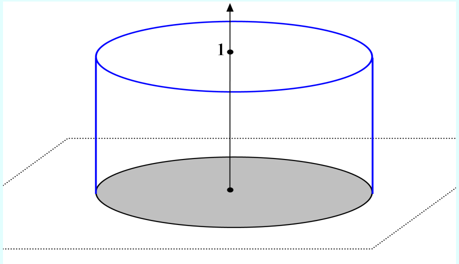
$$\mathcal{E}_{\alpha,\beta}(f, f) \simeq \mathcal{E}(f, f).$$

Since $\mathcal{E}$ is a regular Dirichlet form (Proposition 2), it follows that $\mathcal{E}_{\alpha,\beta}$ is also a regular Dirichlet form for any $\beta > 0$, which implies $\beta^* = \infty$!

Hence, in the family of all α-regular metric spaces, manifolds and ultra-metric spaces are extremal cases: for the manifolds we have $\beta^* = 2$, while for the ultra-metric spaces $\beta^* = \infty$.

However, these two extremal classes of metric spaces have something in common: both manifolds and ultra-metric spaces possess a priori rich classes of test functions with controlled energy: on manifolds these are usual bump or tent functions, while on ultra-metric spaces these are indicators of balls, as we have seen.





The presence of such test functions is very essential for the proofs of heat kernel estimates as all known techniques for obtaining off-diagonal upper bounds make use of such test functions.

In the setting of general metric spaces, one has to make an additional assumption about existence of “good” test functions.

To conclude the discussion about general metric spaces, let us mention the following result of AG and T.Kumagai 2008: if the heat kernel of a conservative Dirichlet form $(\mathcal{E},\mathcal{F})$ satisfies the estimate of the form

$$p_t(x,y) \asymp \frac{c_1}{t^{\alpha/\beta}} \Phi\left(c_2 \frac{d(x,y)}{t^{1/\beta}}\right)$$

for some positive α and β then either $\mathcal{E}$ is strongly local or

$$\Phi(s) \simeq (1+s)^{-(\alpha+\beta)}.$$

Since on ultra-metric spaces strongly local Dirichlet forms do not exist, we obtain that the only possible estimate of the above type is a stable-like estimate

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{25}$$

Our next purpose is to characterize those ultra-metric space and Dirichlet forms (not necessarily isotropic) when this estimate holds.

The following *necessary* conditions for (25) are known:

- the α-regularity: for any metric ball $B_r(x)$, we have

$$\mu\left(B_r(x)\right) \simeq r^\alpha \tag{V}$$

(consequently, $\alpha = \dim_H X$ and $\mu \simeq \mathcal{H}_\alpha$).

- the jump kernel estimate: for all $x, y \in X$,

$$J(x,y) \simeq d(x,y)^{-(\alpha+\beta)}. \tag{J}$$

Z.-Q.Chen and T.Kumagai proved in 2003 that, on general metric spaces (with a certain mild restriction on the metric), if $0 < \beta < 2$ then

$$(V)+(J) \Leftrightarrow (25).$$

However, if the walk dimension β^* of the space in question is larger than 2, then the value of β in (J) can be > 2. In this case, on top of (V) and (J) we need one more condition that ensures the existence of "good" test functions.

Such a condition was established in 2016 independently by

• Z.-Q. Chen, T. Kumagai, Jian Wang: condition CSJ (cutoff Sobolev inequality for jumps);

• AG, Jiaxin Hu, Eryan Hu: condition $Gcap$ (generalized capacity condition).

A common result of these works:

$$(V) + (J) + (Gcap) \Leftrightarrow (25).$$

We will show that, in the setting of *ultra-metric* spaces, the third condition is not needed.

8 α-regular ultra-metric spaces

Let (X, d) be a separable, proper ultra-metric space and let μ be an α-regular Radon measure on X. The results below were proved by A.Bendikov, AG, Eryan Hu 2017.

Theorem 6 *Let J be a symmetric non-negative function on $X \times X$ such that*

$$J(x,y) \simeq d(x,y)^{-(\alpha+\beta)} \tag{J}$$

for some $\beta > 0$. Then the quadratic form

$$\mathcal{E}(f,f) = \iint_{X\times X} (f(x) - f(y))^2 J(x,y) d\mu(x) d\mu(y)$$

determines a regular Dirichlet form in $L^2(X,\mu)$. Its heat kernel $p_t(x,y)$ exists, is continuous in (t,x,y), Hölder continuous in (x,y) and satisfies the stable-like estimate

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \tag{26}$$

for all $x, y \in X$ and $t > 0$. Consequently, $(V)+(J) \Leftrightarrow (26)$.

Next, let us relax pointwise upper and lower estimates of $J(x,y)$ in (J). We slightly change a setup and assume that we are given a symmetric Radon measure j on $X \times X$ of the form $dj = J(x,dy)d\mu(x)$. Both j and J are referred to as jump measures.

Definition. We say that J satisfies the β-*Poincaré inequality* if, for any ball $B = B_r(x_0)$ and any function $f \in L^2(B)$,

$$\int_{\varepsilon B} \left| f - \overline{f} \right|^2 d\mu \leq Cr^\beta \iint_{B\times B} (f(x) - f(y))^2 J(x,dy)d\mu(x) \qquad (PI)$$

where $\overline{f} = ⨍_{\varepsilon B} f d\mu$ and C and $\varepsilon \in (0,1]$ are constants.

Definition. We say that J satisfies the β-*tail condition* if, for any ball $B_r(x)$,

$$\int_{B_r(x)^c} J(x,dy) \leq Cr^{-\beta}. \qquad (TJ)$$

If $dj = J(x,y)d\mu(x)\,d\mu(y)$ and X is α-regular then the following implications hold:

$$J(x,y) \geq cd(x,y)^{-(\alpha+\beta)} \Rightarrow (PI)$$
$$J(x,y) \leq cd(x,y)^{-(\alpha+\beta)} \Rightarrow (TJ)$$

Theorem 7 (Main Theorem) *Let (X, d, μ) be α-regular ultra-metric space and let $J(x, dy)$ be a jump measure on $X \times X$ that satisfies (TJ). Then the quadratic form*

$$\mathcal{E}(f, f) = \iint_{X \times X} (f(x) - f(y))^2 J(x, dy) d\mu(x)$$

extends to a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ in $L^2(X, \mu)$. If in addition J satisfies (PI) then the heat kernel $p_t(x, y)$ of $(\mathcal{E}, \mathcal{F})$ exists, is continuous in (t, x, y), Hölder continuous in (x, y) and satisfies for all $x, y \in X$ and $t > 0$ the following "weak upper estimate"

$$p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-\beta}, \qquad (WUE)$$

and the "near-diagonal lower estimate"

$$p_t(x, y) \geq \frac{c}{t^{\alpha/\beta}} \quad \text{provided } d(x, y) \leq \delta t^{1/\beta}. \qquad (NLE)$$

Moreover, under the standing assumption (TJ), we have

$$(PI) \Leftrightarrow (WUE) + (NLE). \qquad (27)$$

Equivalence (27) is analogous to the aforementioned result that, on α-regular manifolds, the Poincaré inequality for the Dirichlet integral is equivalent to the two-sided Gaussian estimates of the heat kernel. An analogue of the condition (TJ) is in this case the locality of the Dirichlet form.

Note that the exponent $-\beta$ in (WUE) does not match the exponent $-(\alpha+\beta)$ in the optimal heat kernel bound (26). There are examples showing that, under (TJ) and (PI), one cannot guarantee any estimate of the form

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\gamma}$$

with $\gamma > \beta$.

In the same way, the lower bound (NLE) cannot be improved to any estimate of the form

$$p_t(x,y) \geq \frac{c}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\gamma}$$

with any, even very large, γ.

9 Example: jump measure on products

Here we give an example showing that the estimates (WUE) and (NLE) of Theorem 7 cannot be improved assuming only (TJ) and (PI).

Let $\{(X_i, d_i, \mu_i)\}_{i=1}^n$ be a sequence of ultra-metric measure spaces such that X_i is α_i-regular, where $\alpha_1, ..., \alpha_n$ is a prescribed sequence of positive reals. For example, we can take $X_i = \mathbb{Q}_p$ and

$$d_i\left(x, y\right) = \left\|x - y\right\|_p^{1/\alpha_i}.$$

Since $\mathbb{Q}_p$ with $\left\|x - y\right\|_p$ is 1-regular, it follows that (X_i, d_i) is α_i-regular.

Fix $\beta > 0$ and consider on each X_i the isotropic Dirichlet form $(\mathcal{E}_i, \mathcal{F}_i)$ associated with $\sigma\left(r\right) = \exp(-(c/r)^\beta)$, so that its jump kernel J_i satisfies

$$J_i(x, y) \simeq d_i(x, y)^{-(\alpha_i+\beta)}$$

and its heat kernel $p_t^{(i)}$ satisfies

$$p_t^{(i)}(x, y) \simeq \frac{1}{t^{\alpha_i/\beta}}\left(1 + \frac{d_i(x, y)}{t^{1/\beta}}\right)^{-(\alpha_i+\beta)}. \tag{28}$$

Consider now the product space $X = X_1 \times \ldots \times X_n$ with the ultra-metric

$$d(x, y) = \max_{1 \leq i \leq n} d_i(x_i, y_i)$$

and the product measure $\mu = \mu_1 \times \ldots \times \mu_n$. Then X is α-regular with

$$\alpha = \alpha_1 + \ldots + \alpha_n.$$

Let $\mathcal{L}_i$ be the generator of $\mathcal{E}_i$. We apply $\mathcal{L}_i$ to functions $f = f(x_1, ..., x_n)$ on X by considering f as a function of x_i only (like partial derivatives in $\mathbb{R}^n$). Consider the operator

$$\mathcal{L} = \mathcal{L}_1 + \ldots + \mathcal{L}_n$$

acting on functions on X.

Proposition 8 *The operator $\mathcal{L}$ is essentially self-adjoint, it generates a heat semigroup $\left\{e^{-t\mathcal{L}}\right\}_{t \geq 0}$ in $L^2(X)$ and its heat kernel satisfies the estimate*

$$p_t(x, y) \simeq \frac{1}{t^{\alpha/\beta}} \prod_{i=1}^{n} \left(1 + \frac{d_i(x_i, y_i)}{t^{1/\beta}}\right)^{-(\alpha_i + \beta)}. \tag{29}$$

Proof. Main idea: the operators $\mathcal{L}_i$ commute, whence

$$e^{-t\mathcal{L}} = e^{-t\mathcal{L}_1} e^{-t\mathcal{L}_2} \dots e^{-t\mathcal{L}_n}.$$

This implies that $e^{-t\mathcal{L}}$ has the heat kernel

$$p_t(x,y) = \prod_{i=1}^{n} p_t^{(i)}(x_i, y_i).$$

Substituting the estimates (28) for $p_t^{(i)}$, we obtain (29). ■

Let us verify that $p_t(x,y)$ satisfies both (WUE) and (NLE).

Indeed, for any pair x, y, choosing i so that $d(x,y) = d(x_i, y_i)$, we obtain

$$\begin{aligned} p_t(x,y) &\leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha_i+\beta)} \\ &\leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta}. \end{aligned}$$

If $d(x,y) \leq t^{1/\beta}$ then also $d_i(x_i, y_i) \leq t^{1/\beta}$ for all i whence

$$p_t(x,y) \geq \frac{c}{t^{\alpha/\beta}}.$$

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ generated by $\mathcal{L}$ has the form

$$\begin{aligned}\mathcal{E}\left(f,f\right) &= \left(\mathcal{L}f,f\right)_{L^2(X)} = \sum_{i=1}^{n} \left(\mathcal{L}_i f,f\right)_{L^2(X)} \\ &= \sum_{i=1}^{n} \int_{X_1} \ldots \overset{i}{\curlyvee} \ldots \int_{X_n} \mathcal{E}_i(f,f)\, d\mu_1 \ldots \overset{i}{\curlyvee} \ldots d\mu_n\end{aligned}$$

where

$$\mathcal{E}_i(f,f) = \int_{X_i} \left[f(x_1,...,y_i,...,x_n) - f(x_1,...,x_i,...,x_n)\right]^2 J_i(x_i,y_i) d\mu_i(x_i) d\mu_i(y_i).$$

It follows that $(\mathcal{E}, \mathcal{F})$ is a jump type Dirichlet form with the following *jump measure* (not jump kernel!)

$$J(x,dy) = \sum_{i=1}^{n} \delta_{x_1}(dy_1) \ldots \delta_{x_{i-1}}(dy_{i-1})\, J_i(x_i,y_i) d\mu_i(y_i) \delta_{x_{i+1}}(dy_{i+1}) \ldots \delta_{x_n}(dy_n),$$

where $\delta_{x_k}(dy_k)$ is a unit measure on X_k sitting at x_k.

It is easy to check that J satisfies (TJ):

$$\int_{B_r(x)^c} J(x,dy) = \sum_{i=1}^{n} \int_{B_r^{(i)}(x_i)^c} J_i(x_i,y_i)\, d\mu_i(y_i) \le Cr^{-\beta}.$$

Since the heat kernel on X satisfies (WUE) and (NLE), we conclude by Theorem 7, that the Poincaré inequality (PI) is also satisfied on X.

Consider the range of x, y, t such that

$$d_1(x_1,y_1) > t^{1/\beta} \text{ and } d_i(x_i,y_i) \le t^{1/\beta} \text{ for } i = 2,\ldots,n.$$

Then (29) yields

$$\begin{aligned} p_t(x,y) &\simeq \frac{1}{t^{\alpha/\beta}}\left(1+\frac{d_1(x_1,y_1)}{t^{1/\beta}}\right)^{-(\alpha_1+\beta)} \\ &= \frac{1}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha_1+\beta)}. \end{aligned}$$

Since α_1 can be chosen arbitrarily small, we see that (WUE) is optimal.

Similarly, consider the range of x, y such that

$$d_i\left(x_i, y_i\right) \simeq d_j\left(x_j, y_j\right) \quad \text{for all } i, j.$$

Then $d(x,y) \simeq d_i\left(x_i, y_i\right)$ and

$$\begin{aligned} p_t(x,y) &\simeq \frac{1}{t^{\alpha/\beta}} \prod_{i=1}^{n} \left(1 + \frac{d_i\left(x_i, y_i\right)}{t^{1/\beta}}\right)^{-(\alpha_i+\beta)} \\ &\simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+n\beta)}. \end{aligned}$$

Since n can be chosen arbitrarily large, while α and β are fixed, we see that one cannot ensure any lower bound of the form

$$p_t(x,y) \geq \frac{c}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-N}.$$

In this sense, (NLE) is optimal.

10 Operators of Vladimirov and Taibleson

Let us slightly modify the above example. For any $i = 1, ..., n$, let X_i be $\mathbb{Q}_p$ with the p-adic metric $d_i(x, y) = \|x - y\|_p$. That is, in the terminology of the previous example, all $\alpha_i = 1$.

Fix $\beta > 0$ and consider the fractional derivative $\mathfrak{D}_i^\beta$ acting in X_i. On the product space $\mathbb{Q}_p^n = X_1 \times ... \times X_n$ we have the operator

$$\mathcal{V}^\beta = \sum_{i=1}^{n} \mathfrak{D}_i^\beta,$$

that is called *Vladimirov operator*.

The operator $\mathcal{V}^\beta$ was introduced by Vladimirov and Volovich and was considered as a free Hamiltonian in their theory of p-adic Quantum Mechanics.

Since $\mathfrak{D}_i^\beta$ coincides with the isotropic Laplacian $\mathcal{L}_i$ on $X_i = \mathbb{Q}_p$, we obtain from Proposition 8 the following.

Corollary 9 *The operator $\mathcal{V}^\beta$ is essentially self-adjoint, and the heat semigroup $\exp\left(-t\mathcal{V}^\beta\right)$ has the heat kernel $p_t(x,y)$ that satisfies for all $t>0$ and $x,y\in\mathbb{Q}_p^n$ the estimate*

$$p_t(x,y)\simeq\frac{1}{t^{n/\beta}}\prod_{i=1}^{n}\left(1+\frac{\|x_i-y_i\|_p}{t^{1/\beta}}\right)^{-(1+\beta)}.$$

Corollary 10 *If $(n-1)/2<\beta<n$ then the Green function of $\mathcal{V}^\beta$ exists and satisfies the estimate*

$$g(x,y)\simeq\|x-y\|_p^{-(n-\beta)}. \tag{30}$$

The estimate (30) was known before only for a very special case $n=3$, $\beta=2$ and when all the components x_i-y_i are the same.

Another natural way of constructing a Markov operator on $\mathbb{Q}_p^n$ is to use the Fourier transform in $\mathbb{Q}_p^n$ that is defined for functions f on $\mathbb{Q}_p^n$ by

$$\widehat{f}(\xi)=\int_{\mathbb{Q}_p^n}e^{2\pi i\langle x,\xi\rangle}f(x)\,d\mu(x),$$

where $\xi\in\mathbb{Q}_p^n$ and $\langle x,\xi\rangle=\sum_{k=1}^n\{x_k\xi_k\}$.

Definition. For any $\beta > 0$ the *Taibleson operator* $\mathcal{T}^\beta$ is defined on functions $f \in \mathcal{C}_0(\mathbb{Q}_p^n)$ by $\widehat{\mathcal{T}^\beta f}(\xi) = \|\xi\|_p^\beta \widehat{f}(\xi),\ \xi \in \mathbb{Q}_p^n$.

Clearly, in the case $n = 1$ the operator $\mathcal{T}^\beta$ coincides with $\mathfrak{D}^\beta$ and $\mathcal{V}^\beta$. In the case $n > 1$, the operators $\mathcal{T}^\beta$ and $\mathcal{V}^\beta$ are different! In particular, this can be seen from the following result.

Theorem 11 *The operator $\mathcal{T}^\beta$ is essentially self-adjoint, it generates a heat semigroup in $L^2(\mathbb{Q}_p^n)$ that admits a continuous heat kernel $p_t(x, y)$ satisfying the estimate*

$$p_t(x, y) \simeq \frac{1}{t^{n/\beta}} \left(1 + \frac{\|x - y\|_p}{t^{1/\beta}}\right)^{-(n+\beta)}.$$

The Green function of $\mathcal{T}^\beta$ is finite if and only if $\beta < n$, and in this case it satisfies the identity $g(x, y) = c_{n,p} \|x - y\|_p^{-(n-\beta)}$.

Proof. Everything follows from the observation that $\mathcal{T}^\beta$ coincides on $\mathcal{C}_0$ with the isotropic Laplacian $\mathcal{L}$ associated with the distribution function $\sigma(r) = \exp(-(p/r)^\beta)$. For the proof of $\mathcal{T}^\beta = \mathcal{L}$, we compare eigenfunctions and eigenvalues of $\mathcal{T}^\beta$ and $\mathcal{L}$ and show that they coincide. ■

11 Semi-bounded jump kernels

Let (X, d, μ) be α-regular ultra-metric space and $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form with a jump kernel $J(x, y)$. Consider two conditions:

$$J(x, y) \leq C d(x, y)^{-(\alpha+\beta)} \tag{$J_\leq$}$$

and

$$J(x, y) \geq c d(x, y)^{-(\alpha+\beta)}. \tag{$J_\geq$}$$

Theorem 12 *If ($J_\leq$) and (PI) are satisfied then the heat kernel satisfies for all $x, y \in X$ and $t > 0$ the optimal upper bound*

$$p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \tag{UE}$$

and the near-diagonal lower bound

$$p_t(x, y) \geq \frac{c}{t^{\alpha/\beta}} \quad \text{provided } d(x, y) \leq \delta t^{1/\beta}. \tag{NLE}$$

In fact, we have

$$(J_\leq) + (PI) \Leftrightarrow (UE) + (NLE).$$

Theorem 13 *If* ($J_{\geq}$) *and* (TJ) *are satisfied then the heat kernel satisfies for all* $x, y \in X$ *and* $t > 0$ *the optimal lower bound*

$$p_t(x,y) \geq \frac{c}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \tag{LE}$$

and the weak upper bound

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta} . \tag{WUE}$$

Moreover, under the standing assumption (TJ)*, we have*

$$(J_{\geq}) \Leftrightarrow (WUE) + (LE).$$

Clearly, Theorems 12 and 13 imply that

$$(J_{\leq}) + (J_{\geq}) \Leftrightarrow (UE) + (LE),$$

which is equivalent to Theorem 6.

12 Example: degenerated jump kernel

Here we construct an example of a jump kernel $J(x,y)$ on $X=\mathbb{Q}_p$ that satisfies $(J_{\leq})$ and (PI) but not $(J_{\geq})$. In fact, J vanishes on large subsets.

Let J be a symmetric kernel on $X \times X$ and let Φ be an increasing positive function on $(0,\infty)$. We say that J satisfies Φ-*Poincaré inequality* if, for any ball $B \subset X$ of radius r and for any $f \in L^2(B)$,

$$\int_{B\times B} (f(x)-f(y))^2 \, d\mu(x)d\mu(y) \leq \Phi(r) \int_{B\times B} (f(x)-f(y))^2 J(x,y) d\mu(x)d\mu(y).$$

Lemma 14 *The above inequality is equivalent to*

$$\int_B \left(f-\overline{f}\right)^2 d\mu \leq \frac{\Phi(r)}{2\mu(B)} \int_{B\times B} (f(x)-f(y))^2 J(x,y) d\mu(x)d\mu(y), \quad (31)$$

where $\overline{f} = \fint_B f d\mu$.

Note that if $\mu(B) \simeq r^\alpha$ and $\Phi(r) = r^{\alpha+\beta}$ then (31) coincides with the β-Poincaré inequality.

Proof. We have

$$\begin{aligned}\int_B \int_B (f(x) - f(y))^2 \, d\mu(x) d\mu(y) &= \int_B \int_B \left(f(x)^2 - 2f(x)f(y) + f(y)^2\right) d\mu(x) d\mu(y) \\ &= 2\mu(B) \int_B f^2 d\mu - 2 \left(\int_B f d\mu \right)^2 \\ &= 2\mu(B) \left(\int_B f^2 d\mu - \overline{f}^2 \mu(B) \right)\end{aligned}$$

and

$$\int_B \left(f - \overline{f}\right)^2 d\mu = \int_B f^2 d\mu - 2\overline{f} \int_B f d\mu + \overline{f}^2 \mu(B) = \int_B f^2 d\mu - \overline{f}^2 \mu(B).$$

Hence, we obtain

$$\int_{B \times B} (f(x) - f(y))^2 \, d\mu(x) d\mu(y) = 2\mu(B) \int_B \left(f - \overline{f}\right)^2 d\mu,$$

whence the claim follows. ■

Set $\Phi(r) = r^{\alpha+\beta}$ with $\alpha = 1$. We need to construct on $\mathbb{Q}_p$ a jump kernel that satisfies const Φ-Poincaré inequality, vanishes on large subsets and such that

$$J(x,y) \leq \frac{1}{\Phi(d(x,y))}.$$

For simplicity, we construct J not on $\mathbb{Q}_p$ but on a discrete subset of $\mathbb{Q}_p$.

Let $M \subset \mathbb{Q}_p$ be the set of p-adic fractions $.x_1x_2...$, that is, M is the set of sequences $x = \{x_i\}_{i=1}^{\infty}$, where $x_i \in \mathbb{F}_p$ and $x_i = 0$ for large enough i. The set M has the additive group structure as follows:

$$x + y = \{x_i + y_i\}_{i=1}^{\infty},$$

where the sum $x_i + y_i$ is understood in $\mathbb{F}_p$.

Recall that $\|x\|_p = p^n$ if $x_n \neq 0$ and $x_i = 0$ for all $i > n$. The distance function on M is $d(x,y) = \|x-y\|_p$, and balls are defined by

$$B_r(x) = \{y \in M : d(x,y) \leq r\}.$$

Define a function S on M by

$$S(x) = \sum_{i=1}^{\infty} x_i \in \mathbb{F}_p,$$

and consider the following subset N of $M \times M$:

$$N = \{(x, y) \in M \times M : S(x) = 0 \text{ and } S(y) = 1 \text{ or } S(x) = 1 \text{ and } S(y) = 0\}.$$

Proposition 15 *Let $p \geq 3$. For the jump kernel*

$$J(x, y) = \frac{\mathbf{1}_{N^c}(x, y)}{\Phi(d(x, y))},$$

the following inequality holds for any ball B of radius r and any function f on B:

$$\sum_{(x,y)\in B\times B} (f(x) - f(y))^2 \leq 5\Phi(r) \sum_{(x,y)\in B\times B} (f(x) - f(y))^2 J(x, y) \quad (32)$$

Proof. We have

$$\begin{aligned}\sum_{(x,y)\in(B\times B)\cap N^c}(f(x)-f(y))^2 &\leq \sum_{(x,y)\in(B\times B)\cap N^c}(f(x)-f(y))^2\frac{\Phi(r)}{\Phi(d(x,y))} \\ &= \Phi(r)\sum_{(x,y)\in B\times B}(f(x)-f(y))^2 J(x,y)\end{aligned}$$

We will prove that

$$\sum_{(x,y)\in(B\times B)\cap N}(f(x)-f(y))^2 \leq 4\sum_{(x,y)\in(B\times B)\cap N^c}(f(x)-f(y))^2, \quad (33)$$

which will then imply (32).

For simplicity, let $p=3$. Observe first the following: any two points $x,y\in M$ form with the point

$$z=-(x+y)$$

an equilateral triangle. Indeed, we have $z-x=-2x-y=x-y$ (since $-2=1 \bmod 3$), whence $\|z-x\|_3=\|x-y\|_3$ and in the same way $\|z-y\|_3=\|x-y\|_3$.

Consequently, if $x, y \in B$ then also $z \in B$ since x is a center of B.

The second observation is that if $(x, y) \in N$ then both (x, z) and (y, z) belong to N^c. Indeed, by the definition of z we have

$$S(z) = -(S(x) + S(y)).$$

Since $(x, y) \in N$, we have $S(x) + S(y) = 1$ whence $S(z) = -1 = 2$. Consequently, any pair $(\cdot, z)$ belongs to N^c.

Combining the above observations, we conclude that

$$\text{if } (x, y) \in (B \times B) \cap N \ \text{ then } \ (x, z) \in (B \times B) \cap N^c,$$

and the same is true for (y, z).

Next, we have

$$(f(x) - f(y))^2 \leq 2(f(x) - f(z))^2 + 2(f(y) - f(z))^2$$

and

$$\sum_{(x,y)\in(B\times B)\cap N} \left(f\left(x\right)-f\left(y\right)\right)^2 \leq 2\sum_{(x,y)\in(B\times B)\cap N} \left(f\left(x\right)-f\left(z\right)\right)^2 +2\sum_{(x,y)\in(B\times B)\cap N} \left(f\left(y\right)-f\left(z\right)\right)^2 .$$

Observe that the mapping

$$(x,y)\mapsto(x,z)=\left(x,-\left(x+y\right)\right),$$

is injective because the pair (x,z) allows to recover the pair (x,y) uniquely by $y=-\left(x+z\right)$. Therefore,

$$\sum_{(x,y)\in(B\times B)\cap N} \left(f\left(x\right)-f\left(z\right)\right)^2 \leq \sum_{(x,z)\in(B\times B)\cap N^c} \left(f\left(x\right)-f\left(z\right)\right)^2 ,$$

The same applies to the sum of $\left(f\left(y\right)-f\left(z\right)\right)^2$, and we obtain

$$\sum_{(x,y)\in(B\times B)\cap N} \left(f\left(x\right)-f\left(y\right)\right)^2 \leq 4\sum_{(x,z)\in(B\times B)\cap N^c} \left(f\left(x\right)-f\left(z\right)\right)^2 ,$$

thus proving (33). ■

13 Approach to the proof

We outline most essential parts of the proofs of Theorems 6, 7, 12, 13. Let (X, d, μ) be an α-regular ultra-metric space and $(\mathcal{E}, \mathcal{F})$ be a jump type Dirichlet form with the jump kernel $J(x, y)$. We write

$$dj = J(x, y)d\mu(x)d\mu(y) = J(x, dy)d\mu(x).$$

Assuming that J satisfies the β-tail condition

$$\int_{B_r(x)^c} J(x, dy) \leq Cr^{-\beta} \tag{TJ}$$

and the β-Poincaré inequality

$$\int_{B_r} \left|f - \overline{f}\right|^2 d\mu \leq Cr^{\beta} \int_{B_r} \int_{B_r} (f(x) - f(y))^2 J(x, dy)d\mu(x), \tag{PI}$$

we need to prove the weak upper estimate

$$p_t(x, y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-\beta} \tag{WUE}$$

and the near-diagonal lower estimate

$$p_t\left(x,y\right)\geq\frac{c}{t^{\alpha/\beta}}\ \ \text{provided } d\left(x,y\right)\leq\delta t^{1/\beta},\tag{NLE}$$

for some $\delta>0$. If in addition J satisfies

$$J\left(x,y\right)\leq Cd\left(x,y\right)^{-(\alpha+\beta)}\tag{$J_\leq$}$$

then heat kernel should satisfy the optimal upper estimate

$$p_t(x,y)\leq\frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)},\tag{UE}$$

and if in addition

$$J\left(x,y\right)\geq cd\left(x,y\right)^{-(\alpha+\beta)}\tag{$J_\geq$}$$

then heat kernel should satisfy the optimal lower estimate

$$p_t(x,y)\geq\frac{c}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}.\tag{LE}$$

There are also issues with the existence of the heat kernel and its Hölder contnuity, as well as the opposite implications.

The proof is very long and consists of many steps. We outline the structure of the proof and some most essential moments.

Overall, the proof uses the same techniques as in general metric spaces but the presense of an ultra-metric brings some simplifications.

For any open set $\Omega \subset X$, consider the function space $\mathcal{F}(\Omega)$ that is the closure of $\mathcal{F} \cap \mathcal{C}_0(\Omega)$ in $\mathcal{F}$. Then $(\mathcal{E}, \mathcal{F}(\Omega))$ is a regular Dirichlet form in $L^2(\Omega)$ that corresponds to a Markov process killed outside Ω.

It is important, that in ultra-metric space satisfying (TJ), for any ball $B = B_r(x)$, $\boxed{1_B \in \mathcal{F}(B)}$ because $1_B \in \mathcal{C}_0(B)$ and $\mathcal{E}(1_B, 1_B) \leq C\mu(B) r^{-\beta}$.

Denote by P_t^Ω the heat semigroup of $(\mathcal{E}, \mathcal{F}(\Omega))$ and by

$$G^\Omega = \int_0^\infty P_t^\Omega \, dt$$

the Green operator. It is known that P_t^Ω and G^Ω are increasing in Ω.

We say that a function $u \in \mathcal{F}$ is *superharmonic* in Ω if $\mathcal{E}(u, \varphi) \geq 0$ for any non-negative $\varphi \in \mathcal{F}(\Omega)$. A function u is *subharmonic* if $-u$ is superharmonic. Finally, u is *harmonic* if u is super- and subharonic.

Step 1. (PI) implies the Nash inequality: for any $f\in\mathcal{F}\cap L^1(X)$,

$$\|f\|_{L^2}^{2(1+\nu)}\leq C\mathcal{E}(f,f)\|f\|_{L^1}^{2\nu}, \tag{34}$$

where $\nu=\beta/\alpha$. The latter implies the existence of the heat kernel and the diagonal upper estimate, for all $t>0$ and almost all $x,y\in X$,

$$p_t(x,y)\leq Ct^{-\alpha/\beta}. \tag{DUE}$$

One of the consequences of (DUE) is the following estimate of the meat exit time from balls: for any ball B of radius r,

$$G^B1\leq Cr^\beta. \tag{35}$$

In the case $\alpha>\beta$ it is simple (while the case $\alpha\leq\beta$ requires more care):

$$\begin{aligned}
G^B1 &\leq G1_B=\int_0^\infty P_t1_B\,dt\\
&\leq \int_0^{r^\beta}P_t1\,dt+\int_{r^\beta}^\infty\int_B p_t(x,y)\,d\mu(y)\,dt\\
&\leq r^\beta+C\int_B\left(\int_{r^\beta}^\infty t^{-\alpha/\beta}dt\right)d\mu\leq r^\beta+Cr^\alpha\left(r^\beta\right)^{1-\alpha/\beta}=Cr^\beta.
\end{aligned}$$

One more consequence of the Nash inequality (34) is the Faber-Krahn inequality: for any measurable set $E \subset X$ of finite measure and any $f \in \mathcal{F}$ such that $f = 0$ a.e. outside E, we have

$$\mathcal{E}(f,f) \geq c\mu(E)^{-\nu} \|f\|_{L^2}^2 . \tag{36}$$

Indeed, by Cauchy-Schwarz inequality,

$$\|f\|_{L^1}^2 \leq \mu(E) \|f\|_{L^2}^2$$

so by (34)

$$\mathcal{E}(f,f) \geq c\|f\|_{L^2}^{2(1+\nu)} \|f\|_{L^1}^{-2\nu} \geq c \|f\|_{L^2}^2 \mu(E)^{-\nu} .$$

Step 2. This is the largest and most technical part of the proof. One obtains a weak Harnack inequality for harmonic functions of $(\mathcal{E}, \mathcal{F})$, where the main ingrdient of the proof is Lemma of growth. We give some details below. The weak Harnack inequality implies an oscillation inequality for harmonic functions and, consequently, the Hölder continuity of harmonic functions.

The mean exit time estimate (35) implies $\left\|G^B f\right\|_{L^\infty} \leq Cr^\beta \left\|f\right\|_{L^\infty}$, which allows to extends oscillation inequality to solutions u of $\mathcal{L}u = f$ with bounded functions f.

Considering a function $u\left(t,\cdot\right) = P_t\varphi$ as solution to $\mathcal{L}u = -\partial_t u$ and estimating $\left\|\partial_t u\right\|_{L^\infty}$ by means of (DUE), we obtain the oscillation inequality and the Höder continuity for $P_t f$ and, hence, also for the heat kernel.

Step 3. Here one obtains the lower bound for mean exit time:

$$G^B 1 \geq cr^\beta \ \text{ in } B \tag{37}$$

that is, in fact, a consequence of the Lemma of growth. The function $u = G^B 1$ is superharmonic in B; hence, by a corollary of a Lemma of growth, it satisfies

$$\inf_B u \geq c\left(⨍_B \frac{1}{u} d\mu\right)^{-1}.$$

On the other hand, using $\phi = 1_B \in \mathcal{F}\left(B\right)$, we obtain

$$\int_B \frac{1}{u} d\mu = (\phi, \frac{\phi^2}{u}) = \mathcal{E}(G^B\phi, \frac{\phi^2}{u}) = \mathcal{E}(u, \frac{\phi^2}{u}).$$

Next one uses the following general inequality (Lemma 19 below):

$$\mathcal{E}(u,\frac{\phi^2}{u}) \leq 3\mathcal{E}(\phi,\phi).$$

Since by (TJ) $\mathcal{E}(\phi,\phi) \leq Cr^{\alpha-\beta}$, we obtain

$$\int_B \frac{1}{u}d\mu \leq Cr^{-\beta},$$

whence (37) follows.

The estimates (35) and (37) yield

$$G^B 1 \simeq r^\beta \text{ in } B.$$

This implies the following *survival estimate*:

$$\boxed{P_t^B 1 \geq \varepsilon \quad \text{in } B, \quad \text{provided} \quad t^{1/\beta} \leq \delta r,} \tag{S}$$

with some $\varepsilon, \delta > 0$. Indeed, (S) follows from a general inequality

$$P_t^B 1 \geq \frac{G^B 1 - t}{\|G^B 1\|_{L^\infty}}.$$

Step 4. Here we prove (NLE). For any ball $B = B_r(x)$, assuming $t^{1/\beta} \leq \delta r$, we have, using the semigroup identity and (S),

$$\begin{aligned} p_{2t}(x,x) &= \int_X p_t(x,y)^2 \, d\mu(y) \geq \int_B p_t(x,y)^2 \, d\mu(x) \\ &\geq \frac{1}{\mu(B)} \left(\int_B p_t(x,y) \, d\mu(x) \right)^2 \geq \frac{\left(P_t^B 1\right)^2}{\mu(B)} \geq \frac{\varepsilon^2}{\mu(B)} \simeq r^{-\alpha}. \end{aligned}$$

Choosing $r = \delta^{-1} r^{1/\beta}$, we obtain

$$p_t(x,x) \geq c t^{-\alpha/\beta}.$$

By the oscillation inequality from the second step,

$$|p_t(x,x) - p_t(x,y)| \leq C t^{-\alpha/\beta} \left(\frac{d(x,y)}{t^{1/\beta}} \right)^\theta.$$

Hencee, if $d(x,y) \leq \delta t^{1/\beta}$ with small enough δ, then

$$|p_t(x,x) - p_t(x,y)| \leq \frac{c}{2} t^{-a/\beta},$$

whence (NLE) follows.

Step 5. Here we prove (WUE). The main difficulty is in obtaining the following estimate: for any ball B of radius r and any $t > 0$,

$$\boxed{P_t 1_{B^c} \leq C\frac{t}{r^\beta}}. \tag{TP}$$

If this is already known then we have, by setting $r = d(x, y)/2$,

$$\begin{aligned} p_{2t}(x, y) &= \int_X p_t(x, z)\, p_t(z, y)\, d\mu(z) \\ &\leq \left(\int_{B_r(x)^c} + \int_{B_r(y)^c}\right) p_t(x, z)\, p_t(z, y)\, d\mu(z) \\ &\leq (\sup p_t)\, P_t 1_{B_r(x)^c} + (\sup p_t)\, P_t 1_{B_r(y)^c} \\ &\leq C t^{-\alpha/\beta} \frac{t}{r^\beta}. \end{aligned}$$

Since by (DUE) also $p_t(x, y) \leq C t^{-\alpha/\beta}$, it follows

$$p_{2t}(x, y) \leq C t^{-\alpha/\beta} \min\left(1, \frac{t}{r^\beta}\right) \simeq t^{-\alpha/\beta}\left(1 + \frac{r}{t^{1/\beta}}\right)^{-\beta}.$$

However, the main difficulty here lies in proving (TP) which itself a multi-step procedure that is based on reiterating of the suvival estimate (S). Indeed, (S) implies, for $t^{1/\beta} \leq \delta r$, that

$$P_t 1_{B^c} \leq 1 - P_t 1_B \leq 1 - P_t^B 1 \leq 1 - \varepsilon,$$

which gives (TP) provided $t^{1/\beta} = \delta r$. A certain bootstrapping argument allows to extend this to all t.

Step 6. In the case when J satisfies $(J_{\leq})$, one can extend the argument of Step 5 to prove the optimal upper estimate (UE), which requires additional techniques. One uses the truncated jump kernel

$$J^{(\rho)} = \min(J, \rho),$$

the heat kernel $q_t^{(\rho)}(x, y)$ associated with $J^{(\rho)}$, and the following general estimate

$$p_t(x, y) \leq q_t^{(\rho)}(x, y) + 2t \sup_{\{x', y' \in X : d(x', y') \geq \rho\}} J(x', y').$$

For the truncated heat kernel one obtains the estimate

$$q_t^{(\rho)}(x,y) \leq Ct^{-\alpha/\beta} \exp\left(-4\rho^{-\beta}t - c\min(\frac{d(x,y)}{\rho}, \frac{\rho}{t^{1/\beta}})\right),$$

which together with ($J_{\leq}$) allows to obtain (UE).

Step 7. In the case when J satisfies ($J_{\geq}$), one uses the following general result: assuming that conditions (S) and (NLE) are satisfied, the following estimate holds for all $t > 0$, $x, y \in X$:

$$p_t(x,y) \geq \frac{c}{t^{\alpha/\beta}} \min(1,\ \{t\mu(B_{t^{1/\beta}}(y)) \operatorname*{essinf}_{\substack{x' \in B_{t^{1/\beta}}(x) \\ y' \in B_{t^{1.\beta}}(y)}} J(x',y')\}). \tag{38}$$

Hence, if $r := d(x,y) \geq \delta t^{1/\beta}$ then $d(x',y') \leq Cr$ and, hence, $J(x',y') \geq cr^{-(\alpha+\beta)}$ which implies

$$p_t(x,y) \geq \frac{c}{t^{\alpha/\beta}} \min\left(1, \frac{t^{1+\alpha/\beta}}{r^{\alpha+\beta}}\right) \simeq \frac{1}{t^{\alpha/\beta}}\left(1 + \frac{r}{t^{1/\beta}}\right)^{-(\alpha+\beta)}.$$

14 Lemma of growth

For any measurable function v on X and for any ball B on X, define the *tail* of v outside B by

$$T_B(v) := \sup_{x \in B} \int_{B^c} |v(y)| J(x, dy).$$

Lemma 16 *Let B be a ball. For any $u \in \mathcal{F} \cap L^\infty$ that non-negative and subharmonic in B, and for $\phi = 1_B$, we have*

$$\mathcal{E}(u\phi, u\phi) \leq 2T_B(u) \int_B u d\mu. \tag{39}$$

Proof. Since $\phi \in \mathcal{F}(B)$, both $u\phi$ and $u\phi^2$ belong to $\mathcal{F}(B)$. We have:

$$\mathcal{E}(u\phi, u\phi) = \mathcal{E}(u, u\phi^2) + \int_{X \times X} u(x)u(y) \left(\phi(x) - \phi(y)\right)^2 dj.$$

By subharmonicity of u, we have $\mathcal{E}(u, u\phi^2) \leq 0$.

It follows that

$$\begin{aligned}
\mathcal{E}(u\phi, u\phi) &\leq \left(\int_{B\times B} + \int_{B^c\times B} + \int_{B\times B^c} + \int_{B^c\times B^c}\right) u(x)u(y)\left(\phi(x)-\phi(y)\right)^2 dj \\
&= 2\int_{B\times B^c} u(x)u(y)\left(\phi(x)-\phi(y)\right)^2 dj \quad \text{(by symmetrization)} \\
&\leq 2\int_B u(x)d\mu(x)\cdot\sup_{x\in B}\int_{B^c}|u(y)|J(x,dy),
\end{aligned}$$

which is equivalent to (39).

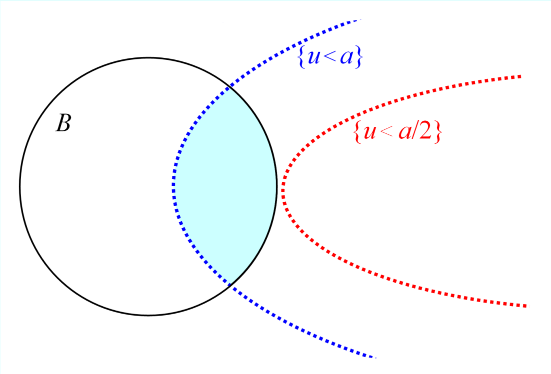
Lemma 17 (Lemma of growth) *If $u \in \mathcal{F} \cap L^\infty$ is superharmonic and non-negative in a ball B of radius R and if, for some $a > 0$,*

$$\frac{\mu(B \cap \{u < a\})}{\mu(B)} \leq \varepsilon_0 \left(1 + \frac{R^\beta T_B(u_-)}{a}\right)^{-\alpha/\beta}, \tag{40}$$

then

$$\operatorname*{essinf}_B u \geq \frac{a}{2},$$

where ε_0 is a positive constant depending on the main hypotheses.



Proof. For any $s > 0$, set

$$m_s = \frac{\mu(B \cap \{u < s\})}{\mu(B)} \quad \text{and} \quad \widetilde{m}_s = \mu(B \cap \{u < s\})$$

In the first part of the proof, we show that, for all $b > a > 0$,

$$\boxed{m_a \le CL \left(\frac{b}{b-a} \right)^2 m_b^{1+\beta/\alpha}}, \tag{41}$$

where

$$L := 1 + \frac{R^\beta T_B(u_-)}{b}. \tag{42}$$

Set $v = (b-u)_+$ and $\phi = 1_B$. Then we have

$$\widetilde{m}_a = \int_{B \cap \{u<a\}} \phi^2 d\mu \le \int_B \phi^2 \underbrace{\left(\frac{(b-u)_+}{b-a} \right)^2}_{\ge 1 \text{ on } \{u<a\}} d\mu = \frac{1}{(b-a)^2} \int_B (\phi v)^2 d\mu. \tag{43}$$

Note that $\phi v = 0$ outside the set $E = B \cap \{u < b\} = B \cap \{v > 0\}$ because either $\phi = 0$ or $v = 0$.

By the Faber-Krahn inequality (36), we obtain

$$\int_B (\phi v)^2 d\mu = \int_E (\phi v)^2 d\mu \le C\mathcal{E}(\phi v, \phi v)\mu(E)^\nu = C\mathcal{E}(\phi v, \phi v)\widetilde{m}_b^\nu.$$

Combining this inequality with (43), we obtain

$$\widetilde{m}_a \le \frac{1}{(b-a)^2}\int_B (\phi v)^2 d\mu \le C\frac{\mathcal{E}(\phi v, \phi v)}{(b-a)^2}\widetilde{m}_b^\nu. \tag{44}$$

Since u is superharmonic in B, the function $v = (b-u)_+$ is subharmonic in B, and we obtain by Lemma 16 and (TJ) that

$$\begin{aligned}\mathcal{E}(\phi v, \phi v) &\le 2T_B(v)\int_B v d\mu \le 2T_B(v)\int_B b1_{\{u<b\}} d\mu \\ &\le 2\left(T_B(b) + T_B(u_-)\right) b\widetilde{m}_b \le C\left(bR^{-\beta} + T_B(u_-)\right) b\widetilde{m}_b \le CLb^2R^{-\beta}\widetilde{m}_b.\end{aligned}$$

Combining this with (44) yields

$$\widetilde{m}_a \le C\frac{Lb^2R^{-\beta}}{(b-a)^2}\widetilde{m}_b^{1+\nu} \le C\frac{Lb^2}{(b-a)^2}m_b^{1+\nu}R^{-\beta}(R^\alpha)^{1+\beta/\alpha} = C\frac{Lb^2}{(b-a)^2}m_b^{1+\nu}R^\alpha.$$

Dividing by R^α and using $\widetilde{m}_a/R^\alpha \simeq m_a$, we obtain (41).

In the second part of the proof, consider the following sequence

$$a_k := \frac{1}{2}(1 + 2^{-k})a, \quad k = 0, 1, 2, ...,$$

so that $a_k \searrow \frac{1}{2}a$ as $k \to \infty$. Set also

$$m_k := m_{a_k} = \frac{\mu(B \cap \{u < a_k\})}{\mu(B)}.$$

Applying the inequality (41) with $a = a_k$ and $b = a_{k-1}$, we obtain, for any $k \geq 1$,

$$m_k \leq C \left(1 + \frac{R^\beta T_B(u_-)}{a_{k-1}}\right) \left(\frac{a_{k-1}}{a_{k-1} - a_k}\right)^2 m_{k-1}^q$$

where $q = 1 + \beta/\alpha$. Since $a_{k-1} \geq \frac{1}{2}a$ and

$$\frac{a_{k-1}}{a_{k-1} - a_k} = \frac{1 + 2^{-(k-1)}}{2^{-(k-1)} - 2^{-k}} \leq 2^{k+1},$$

it follows that

$$m_k \leq CL \cdot 4^k \cdot m_{k-1}^q, \tag{45}$$

where

$$L = 1 + \frac{R^\beta T_B(u_-)}{a}.$$

Iterating (45), we obtain

$$\begin{aligned} m_k &\leq (CL)^{1+q+\cdots+q^{k-1}} \cdot 4^{k+q(k-1)+\cdots+q^{k-1}} \cdot m_0^{q^k} \\ &\leq \left((CL)^{\frac{1}{q-1}} \cdot 4^{\frac{q}{(q-1)^2}} \cdot m_0\right)^{q^k}, \end{aligned} \tag{46}$$

where in the second line we have used that

$$k + q(k-1) + \cdots + q^{k-1} = \frac{q^{k+1} - (k+1)q + k}{(q-1)^2} \leq \frac{q}{(q-1)^2} q^k,$$

and $C > 1$. It follows from (46) and $q > 1$ that if

$$(CL)^{\frac{1}{q-1}} \cdot 4^{\frac{q}{(q-1)^2}} \cdot m_0 \leq \frac{1}{2}, \tag{47}$$

then

$$\lim_{k\to\infty} m_k = 0. \tag{48}$$

Clearly, (47) is equivalent to

$$m_0 \leq 2^{-\frac{2q}{(q-1)^2}-1} \cdot (CL)^{-\frac{1}{q-1}}.$$

Since $\frac{1}{q-1} = \frac{\alpha}{\beta}$, we see that this condition is equivalent to the hypothesis (40) with

$$\varepsilon_0 := 2^{-\frac{2q}{(q-1)^2}-1} C^{-\frac{1}{q-1}}.$$

Assuming that ε_0 is defined so, we see that (47) is satisfied and, hence, we have (48). It follows that

$$\mu(B \cap \{u \leq \frac{a}{2}\}) = 0,$$

which implies $\operatorname{essinf}_B u \geq a/2$. ■

Lemma 18 *Let a non-negative function $u \in \mathcal{F} \cap L^\infty$ be superharmonic in a ball B. Then*

$$\operatorname*{essinf}_B u \geq \frac{\varepsilon_0}{2} \left(\fint_B \frac{1}{u} d\mu \right)^{-1},$$

where ε_0 is the same as in Lemma 17.

Proof. We will apply Lemma 17 with a suitable value of a. Indeed, for any $a > 0$, we have

$$\mu(B \cap \{u < a\}) = \mu(B \cap \{\frac{1}{u} > \frac{1}{a}\}) \leq a \int_B \frac{1}{u} d\mu = a\mu(B) \fint_B \frac{1}{u} d\mu.$$

Since u is non-negative on X, we have that $R^\beta T_B(u_-) = 0$. Setting

$$a := \varepsilon_0 \left(\fint_B \frac{1}{u} d\mu \right)^{-1},$$

we obtain that

$$\mu(B \cap \{u < a\}) \leq \varepsilon_0 \mu(B).$$

Hence, by Lemma 17, we conclude that

$$\operatorname*{essinf}_B u \geq \frac{a}{2} = \frac{\varepsilon_0}{2} \left(\fint_B \frac{1}{u} d\mu \right)^{-1},$$

which was to be proved. ■

15 Weak Harnack inequality

Lemma 19 *Let* $u \in \mathcal{F} \cap L^\infty$ *and assume that* $\operatorname{essinf}_B u > 0$ *for some ball* B*. Then, for* $\phi = 1_B$,

$$\mathcal{E}(u, \frac{\phi^2}{u}) + \frac{1}{2}\int_B \int_B \left|\ln \frac{u(y)}{u(x)}\right|^2 dj(x,y) \leq 3\mathcal{E}(\phi,\phi) - 2\int_B \int_{B^c} \frac{u\,(y)}{u(x)} dj(x,y).$$

Lemma 20 *Let* $u \in \mathcal{F} \cap L^\infty$ *be superharmonic in a ball* B *of radius* R *and let* $u \geq \lambda > 0$ *in* B*. Fix positive numbers* a, b *and consider in* B *the function:*

$$v := \left(\ln \frac{a}{u}\right)_+ \wedge b.$$

Then

$$\fint_B \fint_B (v(x) - v(y))^2 d\mu(x) d\mu(y) \leq C\left(1 + \frac{R^\beta T_B\,(u_-)}{\lambda}\right). \tag{49}$$

Proof. Note first that

$$|v(x) - v(y)| \leq \left| \ln \frac{u(y)}{u(x)} \right|.$$

By (PI) as in Lemma 14 and by Lemma 19, we obtain

$$\begin{aligned}
& \fint_B \fint_B (v(x) - v(y))^2 d\mu(x) d\mu(y) \\
\leq \;& CR^{\beta-\alpha} \int_B \int_B (v(x) - v(y))^2 dj(x,y) \\
\leq \;& CR^{\beta-\alpha} \int_B \int_B \left| \ln \frac{u(y)}{u(x)} \right|^2 dj(x,y) \\
\leq \;& CR^{\beta-\alpha} \left(6\mathcal{E}\left(\phi,\phi\right) + 4 \int_B \int_{B^c} \frac{u(y)_-}{u(x)} d\mu\left(x\right) J(x,dy) \right) \\
\leq \;& CR^{\beta-\alpha} \left(R^{\alpha-\beta} + R^{\alpha} \sup_{x \in B} \int_{B^c} \frac{u(y)_-}{\lambda} J(x,dy) \right) \\
= \;& C \left(1 + \frac{R^{\beta} T_B(u_-)}{\lambda} \right).
\end{aligned}$$

■

Lemma 21 (Weak Harnack inequality) *Let B be a ball of radius R and let $u \in \mathcal{F} \cap L^\infty$ be superharmonic and non-negative in B. Then, for any $a > 0$, such that*

$$\frac{\mu(B \cap \{u \geq a\})}{\mu(B)} \geq \frac{1}{2} \tag{50}$$

and

$$R^\beta T_B(u_-) \leq \varepsilon a, \tag{51}$$

we have

$$\operatorname*{essinf}_B u \geq \varepsilon a, \tag{52}$$

where $\varepsilon > 0$ is a constant that depends only on the main hypotheses.

If $u \geq 0$ on X then the condition (51) is trivially satisfied. A (strong) Harnack inequality for non-negative harmonic functions would say that

$$\operatorname*{essinf}_B u \geq \varepsilon \operatorname*{esssup}_B u.$$

In particular, for any $a < \operatorname{esssup}_B u$, we would have (52). That is, the hypothesis (50) could be relaxed in this case to $\mu(B \cap \{u \geq a\}) > 0$. Hence, Lemma 21 is a weak version of the Harnack inequality.

Proof. Let λ, b be two positive parameters to be determined later. Consider the functions $u_\lambda := u + \lambda$ and

$$v := \left(\ln \frac{a+\lambda}{u_\lambda}\right)_+ \wedge b.$$

Note that $0 \le v \le b$ and in B

$$\begin{aligned} v = 0 \quad &\Leftrightarrow \quad \frac{a+\lambda}{u_\lambda} \le 1 \quad &&\Leftrightarrow \quad u \ge a \\ v = b \quad &\Leftrightarrow \quad \frac{a+\lambda}{u_\lambda} \ge e^b \quad &&\Leftrightarrow \quad u_\lambda \le (a+\lambda)e^{-b} =: q. \end{aligned}$$

We will apply Lemma 17 to u_λ instead of u. Set

$$\omega := \frac{\mu(B \cap \{u \ge a\})}{\mu(B)} = \frac{\mu(B \cap \{v = 0\})}{\mu(B)} \tag{53}$$

and

$$m := \frac{\mu(B \cap \{u_\lambda \le q\})}{\mu(B)} = \frac{\mu(B \cap \{v = b\})}{\mu(B)}. \tag{54}$$

By Lemma 17, if

$$m \leq \varepsilon_0 \left(1 + \frac{R^\beta T_B((u_\lambda)_-)}{q}\right)^{-\alpha/\beta}, \tag{55}$$

then

$$\operatorname*{essinf}_B u_\lambda \geq \frac{q}{2}. \tag{56}$$

Since $u \geq 0$ in B, we have

$$L := R^\beta T_B(u_-) \geq R^\beta T_B\left((u_\lambda)_-\right).$$

Hence, in order to have (55), it suffices to ensure that

$$m \leq \varepsilon_0 \left(1 + \frac{L}{q}\right)^{-\alpha/\beta}. \tag{57}$$

Let us estimate m from above using the definition (53) and (54) of ω and m, as well as Lemma 20.

We obtain

$$
\begin{aligned}
b^2 m\omega &= \frac{1}{\mu(B)^2}\int_{B\cap\{v=0\}}\int_{B\cap\{v=b\}} b^2 d\mu(x)d\mu(y) \\
&= \frac{1}{\mu(B)^2}\int_{B\cap\{v=0\}}\int_{B\cap\{v=b\}} (v(x)-v(y))^2 d\mu(x)d\mu(y) \\
&\le \fint_B\fint_B (v(x)-v(y))^2 d\mu(x)d\mu(y) \\
&\le C\left(1+\frac{R^\beta T_B((u_\lambda)_-)}{\lambda}\right) \le C\left(1+\frac{L}{\lambda}\right).
\end{aligned}
$$

It follows that

$$
m \le \frac{C}{b^2\omega}\left(1+\frac{L}{\lambda}\right) \le \frac{2C}{b^2}\left(1+\frac{L}{\lambda}\right),
$$

where we have used that $\omega \ge 1/2$, which is true by (50). Hence, the condition (57) will be satisfied provided

$$
\frac{2C}{b^2}\left(1+\frac{L}{\lambda}\right) \le \varepsilon_0\left(1+\frac{L}{q}\right)^{-\alpha/\beta},
$$

which is equivalent to

$$b^2 \geq \frac{2C}{\varepsilon_0}\left(1+\frac{L}{\lambda}\right)\left(1+\frac{L}{q}\right)^{\alpha/\beta}. \tag{58}$$

Fix $\varepsilon > 0$ to be determined later, and specify the parameters λ, b as follows:

$$\lambda := \varepsilon a, \quad b := \ln\frac{1+\varepsilon}{4\varepsilon}.$$

Then we have

$$q = (a+\lambda)e^{-b} = 4\varepsilon a,$$

and the inequality (58) is equivalent to

$$\left(\ln\frac{1+\varepsilon}{4\varepsilon}\right)^2 \geq \frac{2C}{\varepsilon_0}\left(1+\frac{L}{\varepsilon a}\right)\left(1+\frac{L}{4\varepsilon a}\right)^{\alpha/\beta}. \tag{59}$$

Since by (51) we have $L \leq \varepsilon a$, the inequality (59) will follow from

$$\left(\ln\frac{1+\varepsilon}{4\varepsilon}\right)^2 \geq \frac{4C}{\varepsilon_0}\left(\frac{5}{4}\right)^{\alpha/\beta}.$$

The latter can be achieved by choosing ε small enough. With this choice of ε we conclude that (56) holds, which implies

$$\operatorname*{essinf}_{B} u \geq \frac{q}{2} - \lambda = 2\varepsilon a - \varepsilon a = \varepsilon a,$$

thus finishing the proof. ■