HEAT KERNELS AND NON-LOCAL DIRICHLET FORMS ON ULTRAMETRIC SPACES

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Abstract. We consider a class of jump measures on ultrametric spaces and the associated non-local regular Dirichlet forms. We obtain equivalent conditions for certain heat kernel upper and lower estimates in terms of the properties of the jump measure. In particular, heat kernel estimates are given for quite degenerate/singular jump measures as shown in a number of examples.

Contents

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1. Introduction and motivation

The purpose of this paper is to obtain upper and lower estimates of heat kernels of certain jump type Dirichlet forms on ultrametric spaces. In particular, our results apply on such well-known examples of ultrametric spaces as the fields $\mathbb{Q}_p$ of p-adic numbers and their self-products $\mathbb{Q}_p^n$. For general metric spaces most of these results are not known yet.

1.1. **Jump type Dirichlet forms.**

Let (M, d) be a locally compact separable metric space and μ be a Radon measure on M with full support. Let $(\mathcal{E}, \mathcal{F})$ be a regular jump type Dirichlet form in $L^2(M, \mu)$ with the jump kernel $J(x, y)$, that is, for all $f, g \in \mathcal{F} \subset L^2$ we have

$$\mathcal{E}(f, g) = \iint_{M\times M} (f(x) - f(y))(g(x) - g(y)) J(x, y) d\mu(x) d\mu(y). \tag{1.1}$$

(see [14] for the theory of Dirichlet forms). The Dirichlet form has a generator $\mathcal{L}$ that is a non-negative definite self-adjoint operator in L^2 and the associated heat semigroup $\{e^{-t\mathcal{L}}\}_{t\geq 0}$. The *heat kernel* $p_t(x, y)$ of $(\mathcal{E}, \mathcal{F})$ (or of $\mathcal{L}$) is the integral kernel of the heat operator $P_t = e^{-t\mathcal{L}}$ (should the former exist). Equivalently, $p_t(x, y)$ is the transition density of the associated jump process.

Given a symmetric, non-negative, measurable function J on $M \times M$, one may ask if the bilinear form (1.1) becomes a regular Dirichlet form with an appropriate domain $\mathcal{F}$, whether it admits the heat kernel and how to estimate the latter quantitatively.

For example, consider in $\mathbb{R}^n$ the jump kernel

$$J(x, y) = |x - y|^{-(n+\beta)},$$

where β is a real parameter. If $0 < \beta < 2$ then $\mathcal{E}$ is a regular Dirichlet form with the generator const $(-\Delta)^{\beta/2}$ (where Δ is the Laplace operator), and the associated jump process is the symmetric stable Levy process of the index β. In the case $\beta = 1$ we have

$$p_t(x, y) = \frac{c_n t}{\left(t^2 + |x - y|^2\right)^{\frac{n+1}{2}}}$$

with some $c_n > 0$, which is the Cauchy distribution with the parameter t. For an arbitrary $0 < \beta < 2$, the heat kernel of the symmetric stable process of index β admits the following estimate:

$$p_t(x, y) \simeq \frac{t}{\left(t^{1/\beta} + |x - y|\right)^{n+\beta}} = \frac{1}{t^{n/\beta}}\left(1 + \frac{|x - y|}{t^{1/\beta}}\right)^{-(n+\beta)}, \tag{1.2}$$

where the sign $\simeq$ means that the ratio of the two sides is bounded between two positive constants. The estimate (1.2) is obtained by the subordination techniques from the heat kernel of Δ.

Assume now that

$$J(x, y) \simeq |x - y|^{-(n+\beta)}.$$

Then (1.2) is still true, which follows from a result of Chen and Kumagai [6]. One can ask, under what conditions on an arbitrary metric measure space (M, d, μ) and a jump kernel J, the heat kernel of the associated Dirichlet form exists and satisfies the *stable-like estimate*

$$p_t(x, y) \simeq \frac{1}{t^{\alpha/\beta}}\left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \tag{1.3}$$

with some positive parameters α, β. If the Dirichlet form is conservative then the following two conditions are *necessary* for (1.3) (see [6], [21], and [23]):

- the α-regularity: for any metric ball $B(x, r)$, we have

$$\mu(B(x, r)) \simeq r^\alpha. \tag{V}$$

- the jump kernel estimate:

$$J(x,y) \simeq d(x,y)^{-(\alpha+\beta)}. \tag{J}$$

It follows from (V) that α is necessarily the Hausdorff dimension of M and $\mu \simeq \mathcal{H}_\alpha$. The value of the parameter β in (1.3) or (J) is called the *index* of the Dirichlet form or that of the associated jump process.

Chen and Kumagai [5], [6] proved that if $0 < \beta < 2$ then (V) and (J) are also sufficient for (1.3), that is,

$$(V)+(J) \Leftrightarrow (1.3).$$

There are many examples of fractal spaces where a jump kernel (J) generates a regular Dirichlet form even with $\beta > 2$. Indeed, on large families of fractals there are diffusion processes with the walk dimension $\beta^* > 2$. By using the subordination techniques, one obtains a jump process with any index $\beta \in (0, \beta^*)$, in particular, β can be larger than 2 (see [2], [17], [21]).

In the case of $\beta > 2$, in order to ensure the estimate (1.3), one needs on top of (V) and (J) one more quite complicated condition, which was established independently in [7], [8], [9] (cutoff Sobolev inequality) and [19] (generalized capacity condition). One of the purposes of this paper is to show that in the setting of *ultrametric* spaces one can manage without the third condition for *any* $\beta \in (0, +\infty)$.

1.2. **Ultrametric spaces.** Let (M, d) be a metric space. The metric d is called an *ultrametric* if it satisfies the ultrametric inequality

$$d(x,y) \le \max\{d(x,z), d(z,y)\} \quad \text{for every } x, y, z \in M. \tag{1.4}$$

In this case (M, d) is called an ultrametric space.

Consider for any $x \in M$ and $r > 0$ the metric ball

$$B(x,r) = \{y \in M : d(x,y) \le r\}.$$

It is easy to deduce from the ultrametric inequality (1.4) the following properties of the space in question. These properties will be frequently used in what follows.

- Any two balls of the same radius are either disjoint or coincide. More generally, any two balls are either disjoint or one contains the other. Consequently, the collection of all distinct balls of the same radius r forms a partition of M.
- Every point inside a ball is its center. This implies that balls are not only closed sets but also open. Consequently, the topological boundary of any ball is empty. One more consequence is that any ultrametric space is totally disconnected.
- From any covering of a compact set by a family of balls there is a finite subcover that consists of mutually disjoint balls.

A well-known example of an ultrametric space is the field $\mathbb{Q}_p$ of p-adic numbers, where p is a prime. Recall that $\mathbb{Q}_p$ is defined as the closure of $\mathbb{Q}$ with respect to the p-adic norm $\|x\|_p$ that satisfies the ultrametric inequality

$$\|x+y\|_p \le \max\left\{\|x\|_p, \|y\|_p\right\}.$$

Hence, $\mathbb{Q}_p$ with the metric $\|x-y\|_p$ is an ultrametric space. Analysis on $\mathbb{Q}_p$ and $\mathbb{Q}_p^n$ was developed, in particular, in [1], [3], [31], [32], [33].

1.3. **Isotropic Dirichlet forms.** Let (M, d) be an ultrametric space where all balls are compact. Let μ be a Radon measure on M with full support. In [3], the authors introduced an *isotropic* jump kernel on M given by

$$J(x,y) = \int_{d(x,y)}^{\infty} \frac{1}{\sigma(r)} \frac{d\sigma(r)}{\mu(B(x,r))}, \tag{1.5}$$

where σ is any cumulative probability distribution function on $(0, \infty)$ that is strictly monotone increasing and left-continuous. This jump kernel determines a regular Dirichlet form that is referred to as *isotropic Dirichlet form*, and its heat kernel admits the following explicit formula

$$p_t(x,y)=\int_{d(x,y)}^{\infty}\frac{t\sigma^{t-1}(r)d\sigma(r)}{\mu(B(x,r))}. \tag{1.6}$$

Assume that (M,d,μ) is Ahlfors α-regular, that is, it satisfies (V). Choosing σ to be the Fréchet distribution:

$$\sigma(r)=\exp\left(-\left(\frac{c}{r}\right)^{\beta}\right), \tag{1.7}$$

where $c,\beta>0$ are arbitrary, we obtain that the jump kernel (1.5) satisfies (J) and the heat kernel (1.6) satisfies the stable-like estimate (1.3).

For example, let $M=\mathbb{Q}_p^n$ where p is a prime and $n\in\mathbb{Z}_+$, and let μ be the Haar measure on $\mathbb{Q}_p^n$. The space $(\mathbb{Q}_p^n,\|\cdot\|_p,\mu)$ is n-regular so that the isotropic heat kernel with the function σ from (1.7) admits the estimate

$$p_t(x,y)\simeq\frac{1}{t^{n/\beta}}\left(1+\frac{\|x-y\|_p}{t^{1/\beta}}\right)^{-(n+\beta)}. \tag{1.8}$$

In this case the jump kernel in $\mathbb{Q}_p^n$ can be computed exactly as follows

$$J(x,y)=\frac{c_{n,p,\beta}}{\|x-y\|_p^{n+\beta}}. \tag{1.9}$$

The generator of the Dirichlet form associated with the jump kernel (1.9) coincides with Taibleson operator introduced in [31] (see also [32], [3] and references therein).

Let (M,d) be a general metric space and μ be an α-regular measure on M. For any $\beta>0$, consider the following quadratic form in $L^2(M,\mu)$:

$$\mathcal{E}_{\alpha,\beta}(f,f)=\iint_{M\times M}\frac{(f(x)-f(y))^2}{d(x,y)^{\alpha+\beta}}d\mu(x)d\mu(y).$$

Define the *walk dimension* β^* of M by

$$\beta^*=\sup\{\beta>0:\mathcal{E}_{\alpha,\beta}\text{ extends to a regular Dirichlet form}\}. \tag{1.10}$$

Note that β^* is an invariant of the metric space (M,d) alone because $\alpha=\dim_H M$ and $\mu\simeq\mathcal{H}_\alpha$, where $\dim_H M$ is the Hausdorff dimension of M. It is known that if M carries a diffusion process $\{X_t\}$ whose transition density satisfies a sub-Gaussian upper and lower estimates, then the walk dimension of $\{X_t\}$ (a parameter in sub-Gaussian estimates) must be equal to β^* (see [21]).

It follows from (1.10) that always $\beta^*\geq 2$ because for any $\beta<2$, Lipschitz functions with compact supports are in the domain of $\mathcal{E}_{\alpha,\beta}$.

If M is a Riemannian manifold then $\beta^*=2$, while on fractals typically $\beta^*>2$. On α-regular ultrametric spaces, as it follows from the above construction of isotropic Dirichlet forms, we have always $\beta^*=\infty$. Hence, in the family of all α-regular metric spaces, manifolds and ultrametric spaces are extremal opposite cases as far as the walk dimension is concerned.

However, these two extremal classes of metric spaces have something in common: both manifolds and ultrametric spaces possess a priori rich classes of test functions with controlled energy: on manifolds these are usual bump or tent functions, while on ultrametric spaces these are indicators of balls, as we will see below. The existence of such classes of test functions is vital for obtaining upper bounds of the heat kernel.

2. Statement of the main results

In this section we state the main results of this paper, while the proofs will be given in the rest of the paper.

Throughout the paper, (M,d) is a locally compact separable *ultrametric* space and μ is a Radon measure on M with full support. Denote by $\mathcal{B}(M)$ the set of all Borel functions on

M. For any open set $U \subset M$, denote by $C_0(U)$ the space of all continuous functions f on M with compact supports $\operatorname{supp} f \subset U$.

2.1. **Jump kernel and Dirichlet form.** Throughout the paper we fix some parameter $\overline{R} \in (0, \operatorname{diam} M]$ and a kernel $J(x,E)$ on $M \times \mathcal{B}(M)$ that satisfies the following two conditions: for any $r \in \left(0, \overline{R}\right)$,

$$J(x, B(x,r)^c) \text{ is a } \mu\text{-locally integrable function of } x \in M, \tag{j.1}$$

and J is symmetric, that is, for all $u, v \in \mathcal{B}_+(M)$,

$$\int_M \int_M u(x)v(y)J(x,dy)d\mu(x) = \int_M \int_M u(y)v\left(x\right)J(x,dy)d\mu(x). \tag{j.2}$$

For example, the kernel

$$J\left(x,E\right) = \int_E J(x,y)d\mu(y)$$

satisfies (j.1) and (j.2) provided $J\left(x,y\right)$ is a non-negative symmetric measurable function of $(x,y) \in M \times M$ such that

$$\int_K \int_{(K_r)^c} J(x,y)d\mu(x)d\mu(y) < \infty \tag{2.1}$$

for any compact set $K \subset M$ and any $r \in \left(0, \overline{R}\right)$, where K_r is the r-neighborhood of K.

Any kernel J satisfying (j.1) and (j.2) determines a positive symmetric Radon measure j on $M \times M \setminus \operatorname{diag}$ that is defined by

$$\int_{M\times M\setminus \operatorname{diag}} f(x,y)dj(x,y) = \int_M \left(\int_M f(x,y)J(x,dy)\right)d\mu(x),$$

for any $f \in C_0(M \times M \setminus \operatorname{diag})$.

Consider the following bilinear form $(\mathcal{E}, \mathcal{F}_{\max})$ on $L^2(M,\mu)$:

$$\begin{cases} \mathcal{E}(u,v) = \int\int_{M\times M\setminus \operatorname{diag}} \left(u(x)-u(y)\right)\left(v(x)-v(y)\right)dj(x,y) \\ \mathcal{F}_{\max} = \left\{u \in L^2(M) : u \text{ is Borel measurable and } \mathcal{E}(u,u) < \infty\right\} \end{cases}. \tag{2.2}$$

The argument in [14, Example 1.2.4, p. 14] shows that $\mathcal{E}$ is well-defined, that is, for any $u \in \mathcal{B}\left(M\right)$,

$$u = 0 \ \mu\text{-a.e.} \Rightarrow \mathcal{E}(u,u) = 0.$$

We will prove below that, under conditions (j.1) and (j.2), $(\mathcal{E}, \mathcal{F}_{\max})$ is a Dirichlet form, and construct further a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ with the domain $\mathcal{F} \subset \mathcal{F}_{\max}$.

Definition 2.1. A function f on M is said to be *locally constant* if, for any $x \in M$, there is $\varepsilon > 0$ such that $f \equiv \text{const}$ in $B(x,\varepsilon)$.

Denote by $\mathcal{D}$ the space of all locally constant functions on M with compact supports. Clearly, we have $\mathcal{D} \subset C_0\left(M\right)$.

Since any ball is closed and open, the indicator function $\mathbf{1}_B$ of any compact ball B belongs to $\mathcal{D}$. Moreover, using properties of ultrametric balls, it is easy to verify that $\mathcal{D}$ consists of finite linear combinations of indicator functions of compact balls:

$$\mathcal{D} = \left\{\sum_{i=0}^n c_i \mathbf{1}_{B_i} : n \in \mathbb{N},\ c_i \in \mathbb{R},\ B_i \text{ is a compact ball}\right\}, \tag{2.3}$$

where the balls $\{B_i\}_{i=0}^n$ can be chosen to be mutually disjoint (see the proof of Lemma 4.1).

Theorem 2.2. *Assume (j.1) and (j.2) are satisfied.*

(I) *Then $(\mathcal{E}, \mathcal{F}_{\max})$ is a Dirichlet form on $L^2(M)$ and $\mathcal{D} \subset \mathcal{F}_{\max}$.*

(II) *Set*

$$\mathcal{F} := \overline{\mathcal{D}}^{\mathcal{E}_1}, \tag{2.4}$$

where the closure is taken with respect to the inner product $\mathcal{E}_1 = \mathcal{E} + (\cdot,\cdot)_{L^2}$ *in* $\mathcal{F}$*. Then* $(\mathcal{E},\mathcal{F})$ *is a regular Dirichlet form on* $L^2(M)$*.*

In particular, indicator functions $\mathbf{1}_B$ of compact balls B belong to $\mathcal{F}$.

The proof of Theorem 2.2 is given in Section 4. Unless otherwise stated, in the rest of this paper, $(\mathcal{E},\mathcal{F})$ is always referred to the regular Dirichlet form constructed in Theorem 2.2(II). Let us emphasize that so far we have not made any additional assumption about μ.

2.2. **Heat kernel estimates.** Let us now state our main results about the heat kernel estimates. For these results, we always assume the space M is proper, that is, all balls $B(x,r)$ in M are compact. In particular, we have $\mu(B(x,r)) < \infty$.

Throughout the paper we fix positive reals α,β and $\overline{R} \in (0, \operatorname{diam} M]$. Note that $\overline{R}$ could be ∞ if $\operatorname{diam} M = \infty$.

Definition 2.3. We say that the condition ($V_\le$) is satisfied if, for all $x \in M$ and $r \in (0,\infty)$,

$$\mu(B(x,r)) \le Cr^\alpha, \tag{$V_\le$}$$

for some constant $C > 0$. We say that the condition ($V_\ge$) is satisfied if, for all $x \in M$ and all $r \in (0,\overline{R})$,

$$\mu(B(x,r)) \ge C^{-1} r^\alpha. \tag{$V_\ge$}$$

We say that (V) is satisfied if both ($V_\le$) and ($V_\ge$) are satisfied.

Let us emphasize that ($V_\le$) is assumed to be true for all $r > 0$ while ($V_\ge$) should be satisfied only for $r \in (0,\overline{R})$. This convention allows us to cover compact ultrametric spaces M.

Definition 2.4. We say that the *tail condition* (TJ) is satisfied if there exists $C > 0$ such that, for any ball $B = B(x,r)$ with $x \in M$ and $r \in (0,\overline{R})$,

$$J(x, B^c) \le Cr^{-\beta}. \tag{TJ}$$

Clearly, if (TJ) is satisfied then J satisfies ($j.1$) so that Theorem 2.2 applies.

For any measurable set $A \subset M$ and any integrable function f on A, set

$$f_A := \fint_A f d\mu := \frac{1}{\mu(A)} \int_A f d\mu.$$

For any ball $B = B(x_0, r)$ and any $\lambda > 0$, set

$$\lambda B = B(x_0, \lambda r).$$

Since any point in the ball B can be used as its center, the notation λB is sensitive to the choice of the center of B if $\lambda < 1$.

Definition 2.5. We say that the *Poincaré inequality* (PI) is satisfied if there exist $\kappa \in (0,1]$ and $C > 0$ such that, for any ball $B := B(x_0,r)$ with $x_0 \in M$ and $r \in (0,\overline{R})$, and for any $f \in \mathcal{F}$,

$$\int_{\kappa B} |f - f_{\kappa B}|^2 d\mu \le Cr^\beta \int_B \int_B (f(x) - f(y))^2 dj(x,y). \tag{PI}$$

Definition 2.6. We say that the *weak upper estimate* (wUE) is satisfied, if the heat kernel $p_t(x,y)$ exists and satisfies the following estimate

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y) \wedge \overline{R}}{t^{1/\beta}}\right)^{-\beta}, \tag{wUE}$$

for some $C > 0$, for all $t \in (0, \overline{R}^\beta)$ and for μ-almost all $x, y \in M$.

In particular, if $\overline{R} \geq \operatorname{diam} M$ then (wUE) is equivalent to

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta}$$

for all $t \in (0, \overline{R}^{\beta})$ and for μ-almost all $x, y \in M$.

Definition 2.7. We say that *near diagonal lower estimate* (nLE) is satisfied if the heat kernel $p_t(x,y)$ exists and satisfies the following estimate:

$$p_t(x,y) \geq c t^{-\alpha/\beta}, \tag{nLE}$$

for some $c, \delta > 0$, for all $t \in (0, \overline{R}^{\beta})$ and μ-almost all $x, y \in M$ such that

$$d(x,y) \leq \delta t^{1/\beta}.$$

Our main result is the following theorem.

Theorem 2.8. *Let M be a proper ultrametric space. If (V), (TJ) and (PI) are satisfied then the heat kernel $p_t(x,y)$ of $(\mathcal{E}, \mathcal{F})$ exists, is continuous jointly in t, x, y, Hölder continuous jointly in x, y and satisfies (wUE) and (nLE).*

Moreover, under the standing assumptions (V) and (TJ), the following equivalence takes place:

$$(PI) \Leftrightarrow (wUE) + (nLE). \tag{2.5}$$

Note also that the conditions (TJ) and (PI) can be satisfied for quite degenerate/singular jump measures as will be shown by examples in Section 15.

Let us emphasize that similar results for general metric spaces are not known and, most probably, they are not true without additional conditions. It would be interesting to obtain a version of Theorem 2.8 for general metric spaces.

Remark 2.9. Let M be a Riemannian manifold with the geodesic distance d and the Riemannian measure μ. Let $(\mathcal{E}, \mathcal{F})$ be the classical local Dirichlet form

$$\mathcal{E}(u,u) = \int_M |\nabla u|^2 \, d\mu.$$

Then, under the standing assumption (V), the corresponding Poincaré inequality

$$\int_B |f - f_B|^2 d\mu \leq C r^2 \int_B |\nabla f|^2 \, d\mu,$$

is equivalent to the following Gaussian estimate of the heat kernel $p_t(x,y)$:

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/2}} \exp\left(-c\frac{d(x,y)^2}{t}\right)$$

(see [16], [28], [29]). The equivalence (2.5) of Theorem 2.8 can be regarded as a version of this result for ultrametric spaces.

The following stability result is an easy consequence of Theorem 2.8.

Corollary 2.10. *Assume that (V) is satisfied. Let $J^{(1)}$ and $J^{(2)}$ be two kernels both satisfying ($j.1$), ($j.2$). Let $(\mathcal{E}^{(1)}, \mathcal{F}^{(1)})$, $(\mathcal{E}^{(2)}, \mathcal{F}^{(2)})$ be two regular Dirichlet forms determined by $J^{(1)}$ and $J^{(2)}$ respectively (as in Theorem 2.2). Assume that, for some $C > 0$,*

$$C^{-1} J^{(2)} \leq J^{(1)} \leq C J^{(2)}.$$

If $(\mathcal{E}^{(1)}, \mathcal{F}^{(1)})$ satisfies (TJ), (wUE) and (nLE), then $(\mathcal{E}^{(2)}, \mathcal{F}^{(2)})$ also satisfies (TJ), (wUE) and (nLE).

In Section 15, we give examples with $\overline{R}=\infty$ showing that, under the hypotheses (V), (TJ) and (PI) of Theorem 2.8, the estimate (wUE) cannot be improved to

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\beta+\varepsilon)}$$

for any $\varepsilon>0$. Similarly, the lower bound (nLE) cannot be improved to

$$p_t(x,y) \ge \frac{c}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-N}$$

for any $N>0$. Hence, in an ultrametric space, the Poincaré inequality does not yield matching upper and lower bounds. For the latter one needs stronger assumptions as below.

Definition 2.11. We say that the condition ($J_\le$) is satisfied if the jump kernel J has the form:

$$J(x,dy)=J(x,y)d\mu(y), \tag{2.6}$$

where $J(x,y)$ is a symmetric function of $x,y\in M$ such that, for all distinct $x,y\in M$,

$$J(x,y)\le Cd(x,y)^{-(\alpha+\beta)}. \tag{$J_\le$}$$

Similarly, we say that the condition ($J_\ge$) is satisfied if, for all distinct $x,y\in M$,

$$J(x,y)\ge C^{-1}d(x,y)^{-(\alpha+\beta)}. \tag{$J_\ge$}$$

We say that the condition (J) is satisfied if both ($J_\le$) and ($J_\ge$) are satisfied, that is, if, for all $x,y\in M$,

$$J(x,y)\simeq d(x,y)^{-(\alpha+\beta)}. \tag{J}$$

It is easy to see that

$$(V_\le)+(J_\le)\Rightarrow(TJ) \tag{2.7}$$

(see [19, Prop. 6.4]) and

$$(V_\ge)+(J_\ge)\Rightarrow(PI) \tag{2.8}$$

(see Lemma 3.1 and the argument after that).

Hence, (TJ) can be regarded as a weak version of the upper bound ($J_\le$), and (PI) can be regarded as a weak version of the lower bound ($J_\ge$).

Definition 2.12. We say that the optimal *upper estimate* (UE) is satisfied if the heat kernel $p_t(x,y)$ exists and satisfies the following upper bound:

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}, \tag{UE}$$

for some $C>0$, for all $t\in(0,\overline{R}^\beta)$ and for μ-almost all $x,y\in M$.

We say that the optimal *lower estimate* (LE) is satisfied if the heat kernel $p_t(x,y)$ exists and satisfies the following lower bound:

$$p_t(x,y) \ge \frac{c}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}, \tag{LE}$$

for some $c>0$, for all $t\in(0,\overline{R}^\beta)$ and for μ-almost all $x,y\in M$.

We say that the heat kernel satisfies two-sided stable-like estimate if both (UE) and (LE) are satisfied, that is, if

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}, \tag{2.9}$$

for all $t\in(0,\overline{R}^\beta)$ and μ-almost all $x,y\in M$.

Corollary 2.13. *Let* (V) *be satisfied and* J *have the form* (2.6).

(a) *We have*

$$(J_{\leq}) + (PI) \Leftrightarrow (UE) + (nLE).$$

(b) *If in addition* (TJ) *is satisfied then*

$$(J_{\geq}) \Leftrightarrow (wUE) + (LE).$$

(c) *We have*

$$(J) \Leftrightarrow (2.9).$$

In all the cases (a), (b), (c), *the heat kernel* $p_t(x,y)$ *exists, is continuous jointly in* t,x,y *and Hölder continuous jointly in* x,y.

Corollary 2.13(c) recovers the estimate (1.8) of [3] that was proved for the jump kernel (1.9) in $\mathbb{Q}_p^n$. Corollary 2.13(c) can be deduced from the previously known results for general metric spaces. Indeed, the case of an arbitrary $\beta > 0$ can be reduced to the case $\beta = 1$ by a simple change of distance function $\widetilde{d}(x,y) = d(x,y)^\beta$ (that is again a metric by the ultrametric property), and then one can apply the results of [6] or [19] to obtain (2.9). However, Theorem 2.8 and parts (a), (b) of Corollary 2.13 cannot be obtained in this way.

2.3. **Structure of the paper.** In Section 3 we give examples of ultrametric spaces and jump kernels satisfying (TJ) and (PI). These examples show, in particular, that our results work for highly anisotropic cases, in particular, the jump measure can vanish on very large area of $(M \times M) \setminus \mathrm{diag}$. In Section 4 we prove Theorem 2.2 about construction of a regular Dirichlet form.

The major part of the paper is devoted to the proof of Theorem 2.8. The proof of the key implication

$$(V) + (TJ) + (PI) \Rightarrow (wUE) + (nLE), \tag{2.10}$$

is fulfilled in Sections 5–12.

In Section 13 we deduce (PI) from the heat kernel bounds (wUE) and (nLE), and in Section 14 we conclude the proofs of Theorem 2.8 and Corollary 2.13 by combining the results of the previous sections.

In Section 15 we give more examples to show that the heat kernel bounds (wUE) and (nLE) of Theorem 2.8 are sharp in certain sense.

Let us now describe the main steps in the proof of the implication (2.10).

Step 1. We show that the Poincaré inequality (PI) implies the Nash inequality (Lemma 5.2). The latter yields by the well-known argument the existence of the heat kernel and on-diagonal upper bound

$$p_t(x,y) \leq Ct^{-\alpha/\beta}, \tag{2.11}$$

for all $t \in (0, \overline{R}^\beta)$ and μ-a.a. $x,y \in M$ (Lemma 5.5). One more consequence of the Nash inequality is the Faber-Krahn inequality (Lemma 5.3).

The on-diagonal upper estimate (2.11) implies the upper bounds of the meat exit time from balls: for any ball B of radius $r \in (0, \sigma\overline{R})$ (where $\sigma \in (0,1)$ is the same as in Lemma 5.3),

$$\operatorname*{ess\,sup}_B G^B 1 \leq Cr^\beta, \tag{2.12}$$

where G^B is the Green operator in B (Lemma 10.2).

Step 2. This is the largest and most technical part of the proof. We first prove Lemma of growth (Lemma 6.4) that is based on the Faber-Krahn inequality, and Lemma 7.2 where the Poincaré inequality is used at full strength. These lemmas imply a weak Harnack inequality for harmonic functions of $(\mathcal{E}, \mathcal{F})$ (Lemma 8.1), that in turn yields the oscillation inequalities for harmonic functions (Lemmas 9.1, 9.3, 9.4) and, consequently, the Hölder continuity of harmonic functions.

Step 3. The mean exit time estimate (2.12) implies that

$$\left\|G^B f\right\|_{L^\infty} \leq C r^\beta \left\|f\right\|_{L^\infty},$$

which allows to extend the oscillation inequality to solutions u of $\mathcal{L}u = f$ (Lemma 11.2).

Considering a function $u(t,\cdot) = P_t f$ as the solution to $\mathcal{L}u = -\partial_t u$ and estimating $\|\partial_t u\|_{L^\infty}$ by means of (2.11), we obtain the oscillation inequality and the Hölder continuity for $P_t f$ and, hence, also for the heat kernel (Lemma 12.1 and [19, Lemmas 5.10, 5.11, 5.12, 5.13]).

Step 4. Using one of the consequences of Lemma of growth, we obtain the lower bound for the mean exit time in any ball B of radius $r \in (0, \overline{R})$:

$$\operatorname*{ess\,inf}_B G^B 1 \geq c r^\beta \quad \text{in } B, \tag{2.13}$$

(Lemma 10.4). The estimates (2.12) and (2.13) imply the following *survival estimate*: for any ball B of radius $r \in (0, \overline{R})$,

$$\operatorname*{ess\,inf}_B P_t^B 1 \geq \varepsilon \quad \text{in } B, \quad \text{provided} \quad t^{1/\beta} \leq \delta r \tag{2.14}$$

(Lemma 10.6).

Step 5. The survival estimate (2.14) implies the on-diagonal lower bound

$$p_t(x,x) \geq c t^{-\alpha/\beta},$$

which together with the oscillation inequality yields the near diagonal lower estimate (nLE) (Lemma 12.2).

Step 6. Here we prove the off-diagonal upper estimate (wUE). The main difficulty is in obtaining the following estimate: for any ball B of radius $r < \overline{R}$ and any $t > 0$,

$$P_t \mathbf{1}_{B^c} \leq C \frac{t}{r^\beta}. \tag{2.15}$$

It is done by comparing P_t to a semigroup Q_t with a truncated jump kernel $dj^{(\rho)} = \mathbf{1}_{\{d(x,y) \leq \rho\}} dj$ and observing that Q_t does not propagate from the inside of any ball of radius ρ to the outside, which follows from the ultrametric property (Lemma 12.3). Combining (2.15) with the on-diagonal upper bound (2.11), we obtain (wUE) (Lemma 12.6).

Notation. The letters $C, C', c, c', \cdots$ denote positive constants whose values are unimportant and can change at any occurrence. However, the value of all such constants depends only on the parameters in the hypotheses in question. The letters α, β and $\overline{R}$ denote the global parameters that have the same meaning all over the paper except for Section 15.

The essential supremum and infimum are always taken with respect to the measure μ. We use the expression "μ-almost all $x, y \in M$" as a shorthand for "$\mu \times \mu$-almost all $(x,y) \in M \times M$". We also use L^p as a shorthand for $L^p(M,\mu)$.

3. Examples

In this section, we give an example of an ultrametric space (M,d) and a jump kernel J on $M \times M$ that satisfies (TJ) and (PI) but does not satisfy ($J_\leq$) or ($J_\geq$) (Proposition 3.5). The idea is that we first take in $M = \mathbb{Q}_p$ the jump kernel $\|x-y\|_p^{-(1+\beta)}$, that satisfies ($J_\leq$) and ($J_\geq$), and then reduce it on some set $N \subset M \times M$ to zero so that ($J_\geq$) is no longer valid, whereas the Poincaré inequality (PI) still holds. The choice of the set N is quite subtle – we use for that some arithmetic properties of $\mathbb{Q}_p$ (see (3.8)).

Let (M,d) be so far any metric space with a measure μ that is finite and positive on all balls. Let $J(x,y)$ be a symmetric non-negative function on $M \times M$, and let Φ be an increasing positive function on $(0,+\infty)$. We say that J satisfies a Φ-Poincaré inequality if, for any ball B in M of radius r and for any $f \in L^2(B)$,

$$\int_{B\times B} (f(x)-f(y))^2 J(x,y) d\mu(x) d\mu(y) \geq \frac{1}{\Phi(r)} \int_{B\times B} (f(x)-f(y))^2 d\mu(x) d\mu(y). \tag{3.1}$$

Lemma 3.1. *The inequality* (3.1) *is equivalent to*

$$\int_{B\times B}(f(x)-f(y))^2 J(x,y)d\mu(x)d\mu(y) \ge \frac{2\mu(B)}{\Phi(r)}\int_B (f-f_B)^2 d\mu. \tag{3.2}$$

Clearly, if $\mu(B)\simeq r^\alpha$ and $\Phi(r)\simeq r^{\alpha+\beta}$ then (3.2) coincides with the Poincaré inequality (PI).

Proof. Let us verify that the right hand sides of (3.1) and (3.2) coincide. We have

$$\begin{aligned}\int_B\int_B (f(x)-f(y))^2 d\mu(x)d\mu(y) &= \int_B\int_B \left(f(x)^2-2f(x)f(y)+f(y)^2\right)d\mu(x)d\mu(y)\\ &= 2\mu(B)\int_B f^2 d\mu - 2\left(\int_B f d\mu\right)^2\\ &= 2\mu(B)\left(\int_B f^2 d\mu - f_B^2\mu(B)\right)\end{aligned}$$

and

$$\int_B (f-f_B)^2 d\mu = \int_B f^2 d\mu - 2f_B\int_B f d\mu + f_B^2\mu(B) = \int_B f^2 d\mu - f_B^2\mu(B),$$

whence

$$\int_{B\times B}(f(x)-f(y))^2 d\mu(x)d\mu(y) = 2\mu(B)\int_B (f-f_B)^2 d\mu, \tag{3.3}$$

which was to be proved. □

Remark 3.2. It is clear that if, for all $x,y\in M$,

$$J(x,y)\ge \frac{1}{\Phi(d(x,y))},$$

then (3.1) holds because for all $x,y\in B$ we have $d(x,y)\le r$ and, hence, $J(x,y)\ge \frac{1}{\Phi(r)}$. Hence, also (3.2) holds. Consequently, ($V_\ge$) and ($J_\ge$) imply (PI).

Now let us fix a prime p, consider a finite field $\mathbb{F}_p := \{0,1,2,\cdots,p-1\}$ and the following set

$$M_p := \{x=\{x_k\}_{k\in\mathbb{Z}} : x_k\in\mathbb{F}_p \text{ and } x_k=0 \text{ for all } k<-K \text{ for some } K\in\mathbb{Z}\},$$

that consists of double sequences of elements of $\mathbb{F}_p$ that are vanishing near $-\infty$. Consider M_p as a linear space over $\mathbb{F}_p$ with linear operations

$$x+y=\{x_k+y_k\}_{k\in\mathbb{Z}} \text{ and } ax=\{ax_k\}_{k\in\mathbb{Z}},$$

for all $x,y\in M_p$ and $a\in\mathbb{F}_p$. Define in M_p the usual p-adic norm by

$$\|x\|_p = p^{-n}, \quad \text{where } n:=\min\{k\in\mathbb{Z}: x_k\neq 0\}.$$

For all $x,y\in M_p$, set $d(x,y)=\|x-y\|_p$ and observe that (M_p,d) is an ultrametric space. Furthermore, (M_p,d) is obviously separable and every ball $B(x,r)$ in M_p is compact. As a metric space, M_p can be identified with $\mathbb{Q}_p$, but the operations in M_p are different from those in $\mathbb{Q}_p$.

The Haar measure μ on M_p can be constructed as follows. For any $n\in\mathbb{Z}$ and for any ball B of radius p^{-n}, set

$$\mu(B) := p^{-n}. \tag{3.4}$$

Since each ball of radius p^{-n} is a disjoint union of p balls of radii $p^{-(n+1)}$, it is easy to see that μ is σ-additive and σ-finite on the semi-ring of all balls in M_p. By Carathéodory's extension theorem, μ extends to a Borel measure on M_p. It follows easily from (3.4) that the measure μ is 1-regular, that is,

$$\mu(B(x,r))\simeq r \tag{3.5}$$

for all $x\in M_p$ and $r>0$.

For any set $A \subset M_p$, any $u \in M_p$ and $a \in \mathbb{F}_p$, define the sets $A+u=\{y+u : y \in A\}$ and $aA=\{ay : y \in A\}$.

Lemma 3.3. *For any $a \in \mathbb{F}_p \setminus \{0\}$ and $u \in M_p$, the map $y \mapsto ay+u$ preserves the metric d and the measure μ. Consequently, for any nonnegative measurable function f on M_p,*

$$\int_{M_p} f(y)d\mu(y) = \int_{M_p} f(ay+u)d\mu(y). \tag{3.6}$$

Proof. For any $y \in M_p$ we have $\|y\|_p = \|ay\|_p$ because $y_k \neq 0 \Leftrightarrow ay_k \neq 0$. It follows that, for all $x, y \in M_p$,

$$d(ax+u, ay+u) = \|ay-ax\|_p = \|y-x\|_p = d(x,y),$$

so that the metric d is preserved by $y \mapsto ay+u$. Let us show that the measure μ is also preserved, that is, for any Borel set $A \subset M_p$,

$$\mu(A) = \mu(aA+u). \tag{3.7}$$

It suffices to prove this for $A = B(x,r)$. Since by the first part $aA+u$ is a ball of the same radius r, (3.7) follows from the construction of the Haar measure. The identity (3.6) is a consequence of (3.7). □

Define a function $S : M_p \mapsto \mathbb{F}_p$ as follows: for any $x \in M_p$ with $\|x\|_p = p^{-n}$, set

$$S(x) = x_n.$$

In other words, $S(x)$ is equal to the non-zero digit x_n of x with the smallest n. Define the following subset N of $M_p \times M_p$:

$$N = \{(x,y) \in M_p \times M_p : S(x) = S(x+y) = 1 \text{ or } S(y) = S(x+y) = 1\}. \tag{3.8}$$

Fix some $n \in \mathbb{Z}$ and let $w \in M_p$ be such that $w_{n-1} = 1$ and $w_k = 0$ for all $k < n-1$. We claim that

$$B(w, p^{-n}) \times B(\mathbf{0}, p^{-n}) \subset N,$$

where $\mathbf{0}$ is the zero element of M_p. Indeed, if $x \in B(w, p^{-n})$ and $y \in B(\mathbf{0}, p^{-n})$ then $x_k - w_k = y_k = 0$ for all $k < n$, which implies that the first non-zero component of x is $x_{n-1} = 1$ and the same is true for $x+y$, whence $S(x) = S(x+y) = 1$ and, hence, $(x,y) \in N$.

It follows that

$$(\mu \times \mu)(N) \geq \mu\left(B(w, p^{-n})\right) \mu\left(B(\mathbf{0}, p^{-n})\right) = p^{-2n}.$$

Since $n \in \mathbb{Z}$ is arbitrary, we see that $(\mu \times \mu)(N) = \infty$.

Proposition 3.4. *Let $p > 2$. Then the jump kernel*

$$J(x,y) = \frac{\mathbf{1}_{N^c}(x,y)}{\Phi(d(x,y))} = \begin{cases} 0, & (x,y) \in N, \\ \frac{1}{\Phi(d(x,y))}, & (x,y) \in N^c, \end{cases} \tag{3.9}$$

satisfies the 5Φ*-Poincaré inequality.*

Proof. Fix a ball $B := B(w,r) \subset M_p$. If $(x,y) \in (B \times B) \setminus N$ then $\mathbf{1}_{N^c}(x,y) = 1$ and

$$\begin{aligned} (f(x)-f(y))^2 J(x,y) &= \frac{1}{\Phi(d(x,y))} (f(x)-f(y))^2 \\ &\geq \frac{1}{\Phi(r)} (f(x)-f(y))^2, \end{aligned}$$

which implies

$$\begin{aligned} \int_{B\times B} (f(x)-f(y))^2 J(x,dy)d\mu(x) &= \int_{(B\times B)\setminus N} (f(x)-f(y))^2 J(x,y)d\mu(x)d\mu(y) \\ &\geq \frac{1}{\Phi(r)} \int_{(B\times B)\setminus N} (f(x)-f(y))^2 d\mu(x)d\mu(y). \end{aligned}$$

We will prove that

$$\int\limits_{(B\times B)\cap N} (f(x)-f(y))^2\,d\mu(x)d\mu(y) \le 4 \int\limits_{(B\times B)\setminus N} (f(x)-f(y))^2\,d\mu(x)d\mu(y), \tag{3.10}$$

which will then imply that

$$\int_{B\times B} (f(x)-f(y))^2\,d\mu(x)d\mu(y) \le 5 \int_{(B\times B)\setminus N} (f(x)-f(y))^2\,d\mu(x)d\mu(y)$$

and, hence,

$$\int_{B\times B} (f(x)-f(y))^2\,J(x,y)d\mu(x)d\mu(y) \ge \frac{1}{5\Phi(r)} \int_{B\times B} (f(x)-f(y))^2\,d\mu(x)d\mu(y),$$

that is, the 5Φ-Poincaré inequality.

For any pair $x,y\in M_p$ consider

$$z = z\,(x,y) := \frac{p+1}{2}(x+y) \in M_p,$$

where $\frac{p+1}{2}\in\mathbb{F}_p$ since $p>2$. Observe that

$$z-x = \frac{p-1}{2}x + \frac{p+1}{2}y = \frac{p+1}{2}\,(y-x)$$

and, hence,

$$\|z-x\|_p = \|y-x\|_p\,.$$

Consequently, if x,y belong to B then also $z\in B$ because x can be regarded as a center of B.

If $(x,y)\in N$ then we have by (3.8)

$$S(z) = S(\frac{p+1}{2}(x+y)) = \frac{p+1}{2} \ne 1.$$

Therefore, no pair $(\cdot,z)$ can lie in N, and we conclude that

$$\text{if } (x,y)\in(B\times B)\cap N \quad \text{then } (x,z),(y,z)\in(B\times B)\setminus N. \tag{3.11}$$

Next, we have, for all x,y and $z=\frac{p+1}{2}(x+y)$,

$$(f(x)-f(y))^2 \le 2\,(f(x)-f(z))^2 + 2\,(f(y)-f(z))^2\,,$$

and

$$\begin{aligned}
&\int_{(B\times B)\cap N} (f(x)-f(y))^2\,d\mu(x)d\mu(y)\\
&\le 2\int_{M_p\times M_p} \mathbf{1}_{\{(B\times B)\cap N\}}(x,y)\,(f(x)-f(z))^2\,d\mu(x)d\mu(y)\\
&\quad + 2\int_{M_p\times M_p} \mathbf{1}_{\{(B\times B)\cap N\}}(x,y)\,(f(y)-f(z))^2\,d\mu(x)d\mu(y).
\end{aligned} \tag{3.12}$$

Furthermore, by (3.11) and Lemma 3.3 with $a=\frac{p+1}{2}$ and $u=ax$ we have

$$\begin{aligned}
&\int_{M_p\times M_p} \mathbf{1}_{\{(B\times B)\cap N\}}(x,y)\,(f(x)-f(z))^2\,d\mu(x)d\mu(y)\\
\le &\int_{M_p}\left(\int_{M_p} \mathbf{1}_{(B\times B)\setminus N}\,(x,a(x+y))\,(f(x)-f(a(x+y)))^2\,d\mu(y)\right)d\mu(x)\\
= &\int_{M_p}\left(\int_{M_p} \mathbf{1}_{(B\times B)\setminus N}(x,y)\,(f(x)-f(y))^2\,d\mu(y)\right)d\mu(x)
\end{aligned}$$

$$= \int_{(B\times B)\setminus N} (f(x)-f(y))^2\, d\mu(x)d\mu(y).$$

Estimating similarly the integral in (3.12), we obtain (3.10), which finishes the proof. □

Set now $\Phi(r) = r^{1+\beta}$. The jump kernel J from (3.9) satisfies (PI) by Lemma 3.1 and Proposition 3.4. By [19, Prop. 6.4], we have, for any α-regular space,

$$\int_{B(x,r)^c} \frac{d\mu(y)}{d(x,y)^{\alpha+\beta}} \le Cr^{-\beta}, \tag{3.13}$$

that is, ($J_\le$) implies (TJ). Clearly, J satisfies ($J_\le$) and, hence, (TJ) but J obviously does not satisfy ($J_\ge$).

Now we construct a new jump kernel $\widetilde{J} \ge J$ that satisfies (TJ) and (PI) but not ($J_\le$) or ($J_\ge$).

For any integer $n \ge 1$, define the set $E_n \subset M_p$ to consist of all sequences of the form

$$x = \underbrace{...}_{k\ge 0}\ \underbrace{1...1}_{k=-1,...-n}\ \underbrace{0...0...}_{k<-n}$$

that is, $x_k = 1$ for $k = -1, ..., -n$, $x_k = 0$ for all $k < -n$ and x_k is arbitrary for all $k \ge 0$. Similarly define a set $F_n \subset M_p$ to consist of all the sequences

$$x = \underbrace{...}_{k\ge 0}\ \underbrace{2...2}_{k=-1,...-n}\ \underbrace{0...0...}_{k<-n}.$$

Clearly, E_n and F_n are balls of radii 1, so that $\mu(E_n) = \mu(F_n) = 1$, and all the balls E_n, F_n, $n \ge 1$, are pairwise disjoint. It follows that also all the sets

$$(E_n \times F_n),\ (F_n \times E_n),\ n \ge 1,$$

are pairwise disjoint in $M_p \times M_p$. Define

$$E = \bigcup_{n=1}^{\infty} (E_n \times F_n) \cup (F_n \times E_n),$$

so that the set E is symmetric and $(\mu \times \mu)(E) = \infty$.

It follows from the definition (3.8) of the set N that, for any $n \ge 1$, the sets $E_n \times F_n$ and $F_n \times E_n$ are disjoint with N. Indeed, if $x \in E_n$ and $y \in F_n$ then $S(x+y) = 3 \ne 1$ in $\mathbb{F}_p$ so that $(x,y) \notin N$. Consequently, the sets N and E are disjoint.

Let $J(x,y)$ be the jump kernel from Proposition 3.4 with $\Phi(r) = r^{1+\beta}$, that is,

$$J(x,y) = \frac{1}{d(x,y)^{1+\beta}} \mathbf{1}_{N^c}(x,y).$$

Fix $\varepsilon > 0$ and define further the kernels

$$J_0(x,y) := \frac{d(x,y)^\varepsilon}{d(x,y)^{1+\beta}} \mathbf{1}_E(x,y) \quad \text{and} \quad \widetilde{J}(x,y) := J(x,y) + J_0(x,y).$$

Proposition 3.5. *For any $\varepsilon \in (0,1)$, the jump kernel $\widetilde{J}$ satisfies (TJ) and (PI) but neither ($J_\ge$) nor ($J_\le$).*

Proof. As we have already mentioned, $\widetilde{J}$ satisfies (PI) since $\widetilde{J} \ge J$ and J satisfies (PI).

Since both J and J_0 vanish on N, we have $\widetilde{J} = 0$ on N so that $\widetilde{J}$ does not satisfy ($J_\ge$). To disprove ($J_\le$) observe that $d(x,y) = p^n$ for all $(x,y) \in E_n \times F_n$, which implies for such pairs (x,y)

$$\widetilde{J}(x,y)d(x,y)^{1+\beta} \ge J_0(x,y)d(x,y)^{1+\beta} = d(x,y)^\varepsilon = p^{\varepsilon n},$$

that can be arbitrarily large.

It remains to prove that $\widetilde{J}$ satisfies (TJ). Since J satisfies (TJ), it suffices to prove that J_0 satisfies (TJ). By symmetry, it suffices to prove that, for any $x \in E_n$ and $r > 0$,

$$\int_{B(x,r)^c} J_0(x,y)d\mu(y) \leq Cr^{-\beta}. \tag{3.14}$$

Consider two cases.

(i) Let $r \geq 1$. By the definition of E, we see that if $x \in E_n$ and $(x,y) \in E$ then $y \in F_n$. Hence, we have, for $x \in E_n$,

$$\begin{aligned}\int_{B(x,r)^c} J_0(x,y)d\mu(y) &= \int_{B(x,r)^c} \mathbf{1}_E\,(x,y)\,\frac{d\mu(y)}{d(x,y)^{1+\beta-\varepsilon}} \\ &\leq \int_{B(x,r)^c\cap F_n} \frac{d\mu(y)}{d(x,y)^{1+\beta-\varepsilon}} \leq \frac{\mu(F_n)}{r^{1+\beta-\varepsilon}} \leq \frac{1}{r^\beta}.\end{aligned}$$

(ii) Let $r < 1$. By (i), (3.5) and (3.13), we obtain

$$\begin{aligned}\int_{B(x,r)^c} J_0(x,y)d\mu(y) &\leq \int_{B(x,1)^c} J_0(x,y)d\mu(y) + \int_{B(x,1)\setminus B(x,r)} \frac{d\mu\,(y)}{d\,(x,y)^{1+\beta}} \\ &\leq 1 + Cr^{-\beta} \leq (1+C)\,r^{-\beta},\end{aligned}$$

which finishes the proof. □

4. Construction of non-local Dirichlet forms

The purpose of this section is to prove Theorem 2.2.

For any open set $\Omega \subset M$, we regard $L^2\,(\Omega)$ as a subset of $L^2\,(M)$ by extending any function $f \in L^2\,(\Omega)$ by constant zero outside Ω. Fix a kernel $J(x,E)$ on $M \times \mathcal{B}(M)$ satisfying (j.1) and (j.2), and consider the bilinear form $(\mathcal{E}, \mathcal{F}_{\max})$ on $L^2(M,\mu)$ given by (2.2). Let $\mathcal{F}$ be defined by (2.4). Recall that $\mathcal{D}$ denotes the space of all locally constant functions on M with compact supports. Denote by $\mathcal{D}\,(\Omega)$ the subspace of $\mathcal{D}$ that consists of functions with supports in Ω.

Lemma 4.1. *Under the above hypotheses, the following are true.*

(I) *For any compact ball B, the indicator function $\mathbf{1}_B$ belongs to $\mathcal{F}_{\max}$. Moreover, $\mathcal{D} \subset \mathcal{F}_{\max}$.*

(II) *For any open set $\Omega \subset M$, $\mathcal{D}(\Omega)$ is dense in $C_0(\Omega)$ with respect to* sup*-norm and in $L^2(\Omega)$ with respect to L^2-norm. In particular, $\mathcal{D}$ is dense in $C_0(M)$ and in $L^2(M)$.*

Proof. **(I).** Denote $\phi = \mathbf{1}_B$ for a ball of radius r and prove first that if B is compact and if $r < \overline{R}$ then $\mathcal{E}(\phi,\phi) < \infty$. Since $\phi(x) - \phi(y) = 0$ provided x, y are both in B or in B^c, we obtain by (2.2) and (j.1)

$$\begin{aligned}\mathcal{E}(\phi,\phi) &= \int_M\int_M (\phi(x)-\phi(y))^2 dj(x,y) \\ &= 2\int_B\int_{B^c} (\phi(x)-\phi(y))^2 dj(x,y) \\ &= 2j(B,B^c) = 2\int_B J(x,B(x,r)^c)d\mu(x) < \infty,\end{aligned} \tag{4.1}$$

where we also use the property that $B = B(x,r)$.

Let f be any function from $\mathcal{D}$ (in particular, f can be $\mathbf{1}_B$ for a compact ball B). Since f is locally constant, for any $x \in M$, there exists $r_x \in (0,\overline{R})$ such that $f = \text{const}$ in $B(x,r_x)$. Since the family $\{B(x,r_x)\}_{x\in B}$ is an open covering of $\operatorname{supp} f$, there exists a finite subcovering $\{B(x_i,r_{x_i})\}_{i=1}^N$. By the properties of ultrametric balls, we may further assume that all the balls $B(x_i,r_{x_i})$ are mutually disjoint. It follows that f is a finite linear combination of functions $\mathbf{1}_{B(x_i,r_{x_i})}$ (cf. (2.3)), which implies $\mathcal{E}\,(f,f) < \infty$.

(II) Fix an open set $\Omega \subset M$, a function $f \in C_0(\Omega)$ and set $K = \operatorname{supp} f$. Since f is uniformly continuous, for any $\varepsilon > 0$ there exists $r > 0$ such that any ball $B\,(x,r)$ with $x \in K$

lies in $V \subset \Omega$, where V is a precompact open set such that $K \subset V \subset \overline{V} \subset \Omega$, and the oscillation of f in $B(x,r)$ is bounded by ε. Choose a finite covering $\{B(x_i,r)\}_{i=1}^N$ of K. As above, we can assume that all the balls $B(x_i,r)$ are mutually disjoint. Clearly, the function

$$f_\varepsilon := \sum_{i=1}^N f(x_i)\mathbf{1}_{B(x_i,r)}$$

belongs to $\mathcal{D}(\Omega)$ and

$$\sup|f_\varepsilon - f| < \varepsilon,$$

which proves that $\mathcal{D}(\Omega)$ is dense in $C_0(\Omega)$ in sup-norm. Since also

$$\|f_\varepsilon - f\|_{L^2}^2 \leq \varepsilon^2 \sum_{i=1}^N \mu(B(x_i,r)) \leq \varepsilon^2\mu(V),$$

$\mathcal{D}(\Omega)$ is dense in $C_0(\Omega)$ also in L^2-norm, whence all the claims follow. □

Proof of Theorem 2.2. **(I)**. Under conditions ($j.1$) and ($j.2$), it follows easily from (2.2) that $(\mathcal{E},\mathcal{F}_{\max})$ is a bilinear, symmetric, non-negative and Markovian form. Moreover, by the arguments in [14, Example 1.2.4, p. 14], $(\mathcal{E},\mathcal{F}_{\max})$ is also closed. It remains to show that the domain $\mathcal{F}_{\max}$ is dense in $L^2(M)$. Indeed, by Lemma 4.1, $\mathcal{D}$ is a subset of $\mathcal{F}_{\max}$ and $\mathcal{D}$ is dense in $L^2(M)$, whence also $\mathcal{F}_{\max}$ is dense in $L^2(M)$, and $(\mathcal{E},\mathcal{F}_{\max})$ is a Dirichlet form.

(II). By Lemma 4.1(II), $\mathcal{F}$ is dense in $L^2(M)$ so that $(\mathcal{E},\mathcal{F})$ is a Dirichlet form. To prove the regularity of $(\mathcal{E},\mathcal{F})$, we need to verify that $\mathcal{F}\cap C_0(M)$ is dense in $C_0(M)$ in sup-norm and in $\mathcal{F}$ in $\mathcal{E}_1$-norm. Since $\mathcal{F}\cap C_0(M)$ contains $\mathcal{D}$, the regularity of $(\mathcal{E},\mathcal{F})$ also follows from Lemma 4.1(II). □

Corollary 4.2. *Assume that* (TJ) *is satisfied. Then* $(\mathcal{E},\mathcal{F})$ *defined in* (2.2) *and* (2.4) *is a regular Dirichlet form. Besides, for any compact ball* B *of radius* $r \in (0,\overline{R})$*, the indicator* $\phi := \mathbf{1}_B$ *of* B *belongs to* $\mathcal{D} \subset \mathcal{F}$ *and satisfies:*

$$\mathcal{E}(\phi,\phi) \leq C\frac{\mu(B)}{r^\beta}. \tag{4.2}$$

Proof. Clearly, (TJ) implies ($j.1$), and the first claim follows from Theorem 2.2. By (4.1) and (TJ), we obtain

$$\mathcal{E}(\phi,\phi) = 2\int_B J(x,B(x,r)^c)d\mu(x) \leq C\frac{\mu(B)}{r^\beta},$$

which proves the second claim. □

In the subsequent sections we will need also the following statement.

Proposition 4.3. *Under the hypotheses of* Theorem 2.2*, for any open set* $A \subset M$ *and for any Borel function* $v \in \mathcal{F}$*, that is non-negative on* A*, we have*

$$\int_A v(y)J(x,dy) \leq \operatorname*{ess\,sup}_A v \int_A J(x,dy), \tag{4.3}$$

for μ-a.a. $x \in M$.

Proof. By [14, Lemma 4.5.4(i), p. 184] and [14, Theorem 4.2.1(ii), p. 161], the measure j charges no part of $M \times M \setminus \operatorname{diag}$ whose projection on the factor M has capacity 0.

It follows that if

$$u = 0 \;\text{ q.e. in } A$$

then

$$\int_{M\times A\setminus \operatorname{diag}} u\,(y)\,J\,(x,dy)\,d\mu\,(x) = 0$$

and, hence,

$$\int_A u(y)\, J(x, dy) = 0$$

for μ-a.a. $x \in M$.

Hence, the function v in (4.3) can be replaced by its quasi-continuous version $\widetilde{v}$. Set

$$a = \operatorname*{ess\,sup}_A v = \operatorname*{ess\,sup}_A \widetilde{v}.$$

By [14, Lemma 2.1.4, p. 70] we have

$$\widetilde{v} \le a \quad \text{q.e. in } A,$$

which implies

$$\int_A v(y) J(x, dy) = \int_A \widetilde{v}(y) J(x, dy) \le a \int_A J(x, dy).$$

□

5. Nash inequality

From this section, we start preparation for the proof of Theorem 2.8. From now on, we always assume (M, d) is an ultrametric space that is proper and separable, and μ is a Radon measure on M with full support. Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form defined by Theorem 2.2. Other hypotheses will be stated explicitly.

Definition 5.1. We say the *Nash inequality* ($Nash$) holds for the Dirichlet form $(\mathcal{E}, \mathcal{F})$ if there exist positive constants ν and C such that

$$\|f\|_{L^2}^{2(1+\nu)} \le C \left(\mathcal{E}(f, f) + \overline{R}^{-\beta} \|f\|_{L^2}^2 \right) \|f\|_{L^1}^{2\nu} \tag{$Nash$}$$

for all $f \in \mathcal{F} \cap L^1$.

The following lemma was proved in [28, Theorem 2.1] for a local Dirichlet form on a Riemannian manifold. We extend this result to non-local Dirichlet forms on ultrametric spaces.

Lemma 5.2. *We have* (V) + (PI) ⇒ ($Nash$) *where* $\nu = \beta/\alpha$.

Proof. The proof is divided into three steps. For any $f \in L^1(M)$ and $s > 0$, define a function f_s on M by

$$f_s(x) := \frac{1}{\mu(B(x, s))} \int_{B(x,s)} f(z) d\mu(z).$$

Step I. Let us prove that, for any $f \in L^1$ and for all $s \in (0, \overline{R})$,

$$\|f_s\|_{L^2}^2 \le C s^{-\alpha} \|f\|_{L^1}^2, \tag{5.1}$$

where the constant C depends only on the constants in hypotheses. Indeed, for all $z \in M$, $s > 0$ and $x \in B(z, s)$, we have $B(x, s) = B(z, s)$ and, hence,

$$\begin{aligned}
\|f_s\|_{L^1} \le \||f|_s\|_{L^1} &= \int_M \frac{1}{\mu(B(x, s))} \int_{B(x,s)} |f(z)| d\mu(z) d\mu(x) \\
&= \int_M |f(z)| d\mu(z) \int_M \frac{\mathbf{1}_{B(z,s)(x)}}{\mu(B(z, s))} d\mu(x) \\
&= \|f\|_{L^1}.
\end{aligned}$$

On the other hand, we have by ($V_\ge$),

$$\|f_s\|_{L^\infty} \le C s^{-\alpha} \sup_{x \in M} \int_{B(x,s)} |f(z)| d\mu(z) \le C s^{-\alpha} \|f\|_{L^1}.$$

It follows that

$$\|f_s\|_{L^2}^2 \le \|f_s\|_{L^\infty} \|f_s\|_{L^1} \le C s^{-\alpha} \|f\|_{L^1}^2,$$

which proves (5.1).

Step II. Let us prove that, for all $f \in \mathcal{F} \cap L^1$ and $s \in (0, \kappa \overline{R})$,

$$\|f - f_s\|_{L^2}^2 \leq C s^\beta \mathcal{E}(f, f), \tag{5.2}$$

where κ is the constant from (PI), and C depends only on the constants in hypotheses. Indeed, since all distinct balls of radii s in M are disjoint and M is separable, there exists a (at most) countable family $\{B_i := B(x_i, s)\}$ of disjoint balls of radii s such that $M = \sqcup_i B_i$. Note that, for any $x \in B_i$, we have $B(x, s) = B_i$ and, hence,

$$f_s(x) = \fint_{B(x,s)} f \, d\mu = \fint_{B_i} f \, d\mu = f_{B_i}.$$

Hence, we obtain by (PI) that

$$\begin{aligned}\|f - f_s\|_{L^2}^2 &= \sum_i \int_{B_i} |f - f_s|^2 d\mu = \sum_i \int_{B_i} |f - f_{B_i}|^2 d\mu \\ &\leq C s^\beta \sum_i \int_{(\kappa^{-1} B_i) \times (\kappa^{-1} B_i)} (f(x) - f(y))^2 dj(x, y).\end{aligned}$$

Each ball $\kappa^{-1} B_j$ is a disjoint union of at most N balls B_i where N depends on the constants in (V). It follows that, for each index i, there is at most N indices j so that $B_i \subset \kappa^{-1} B_j$. Hence, we obtain

$$\begin{aligned}\|f - f_s\|_{L^2}^2 &\leq C N s^\beta \sum_i \int_{B_i \times M} (f(x) - f(y))^2 dj(x, y) \\ &= C N s^\beta \int_{M \times M} (f(x) - f(y))^2 dj(x, y),\end{aligned}$$

which proves (5.2).

Step III. Now we can prove ($Nash$). Indeed, by (5.1) and (5.2), we have, for any $f \in \mathcal{F} \cap L^1$ and $s \in (0, \kappa \overline{R})$,

$$\|f\|_{L^2}^2 \leq 2\|f - f_s\|_{L^2}^2 + 2\|f_s\|_{L^2}^2 \leq C s^\beta \mathcal{E}(f, f) + C s^{-\alpha} \|f\|_{L^1}^2.$$

On the other hand, if $s \in [\kappa \overline{R}, \infty)$ (in the case $\overline{R} < \infty$) then

$$\|f\|_{L^2}^2 \leq (s / (\kappa \overline{R}))^\beta \|f\|_{L^2}^2.$$

Combining the above two inequalities and assuming that $C > 1$, we obtain that, for any $s > 0$,

$$\|f\|_{L^2}^2 \leq C s^\beta \left(\mathcal{E}(f, f) + \overline{R}^{-\beta} \|f\|_{L^2}^2 \right) + C s^{-\alpha} \|f\|_{L^1}^2.$$

Choosing s so that the two terms on the right hand side are equal, that is,

$$s^{\alpha+\beta} = \frac{\|f\|_{L^1}^2}{\mathcal{E}(f, f) + \overline{R}^{-\beta} \|f\|_{L^2}^2},$$

we obtain

$$\|f\|_{L^2}^2 \leq 2 C s^{-\alpha} \|f\|_{L^1}^2 = 2C \left(\mathcal{E}(f, f) + \overline{R}^{-\beta} \|f\|_{L^2}^2 \right)^{\frac{\alpha}{\alpha+\beta}} \|f\|_{L^1}^{2\frac{\beta}{\alpha+\beta}},$$

which yields ($Nash$) with $\nu = \beta / \alpha$. □

Lemma 5.3. *Assume that ($V_\leq$) and ($Nash$) hold with $\nu = \beta/\alpha$. Then there exists $\sigma \in (0, 1)$ such that, for any ball $B \subset M$ of radius $R \in (0, \sigma \overline{R})$, for any measurable set $E \subset B$, and for any function $f \in \mathcal{F}$ such that $f = 0$ a.e. in E^c, we have*

$$\|f\|_{L^2}^2 \leq C \mu(E)^\nu \mathcal{E}(f, f). \tag{5.3}$$

Proof. Indeed, by Cauchy-Schwarz inequality,

$$\|f\|_{L^1}^2 \leq \mu(E)\, \|f\|_{L^2}^2$$

whence by ($Nash$)

$$\|f\|_{L^2}^{2(1+\nu)} \leq C\left(\mathcal{E}(f,f) + \overline{R}^{-\beta} \|f\|_{L^2}^2\right) \|f\|_{L^2}^{2\nu} \mu(E)^{\nu}$$

whence

$$\|f\|_{L^2}^2 \leq C\mathcal{E}(f,f)\mu(E)^{\nu} + C\overline{R}^{-\beta}\mu(E)^{\nu}\, \|f\|_{L^2}^2.$$

Choosing σ so small that

$$C\overline{R}^{-\beta}\mu(E)^{\nu} \leq C\overline{R}^{-\beta}\mu(B)^{\alpha/\beta} \leq C'\left(\frac{R}{\overline{R}}\right)^{\beta} \leq C'\sigma^{\beta} < \frac{1}{2},$$

we obtain (5.3). □

The inequality (5.3) is called the *Faber-Krahn inequality*. It follows from Lemmas 5.2 and 5.3 that the hypotheses (V) and (PI) imply the Faber-Krahn inequality (5.3) with $\nu = \beta/\alpha$ and for some $\sigma \in (0,1)$. This parameter σ will be used in the rest of this paper alongside with α and β. Without loss of generality, we can assume that σ is small enough, in particular, $\sigma \leq \kappa$ where κ is the parameter from (PI).

For a non-empty open set $\Omega \subset M$, let $\mathcal{F}(\Omega)$ be the closure of $\mathcal{F} \cap C_0(\Omega)$ in $\mathcal{F}$ with respect to the norm $\mathcal{E}_1(u,u) = \mathcal{E}(u,u) + \|u\|_{L^2}^2$. It is well known (see [14]) that if $(\mathcal{E},\mathcal{F})$ is regular, then $(\mathcal{E},\mathcal{F}(\Omega))$ is a regular Dirichlet form on $L^2(\Omega)$. Denote the corresponding generator, heat semigroup and heat kernel (if it exists) respectively by $\mathcal{L}^{\Omega}$, $\{P_t^{\Omega}\}$ and $p_t^{\Omega}(x,y)$.

Denote by $\lambda_1(\Omega)$ the bottom of the spectrum of the operator $\mathcal{L}^{\Omega}$ in $L^2(\Omega)$. It is known that

$$\lambda_1(\Omega) = \inf_{f \in \mathcal{F}(\Omega)\setminus\{0\}} \frac{\mathcal{E}(f,f)}{\|f\|_{L^2}^2}.$$

It follows from (5.3) that if Ω is contained in a ball B of radius $R < \sigma\overline{R}$ then

$$\lambda_1(\Omega) \geq c\mu(\Omega)^{-\nu}. \tag{5.4}$$

Definition 5.4. We say that the condition (DUE) is satisfied if the heat kernel $p_t(x,y)$ exists and satisfies the following *diagonal upper estimate*

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}}, \tag{DUE}$$

for any $t \in (0, \overline{R}^{\beta})$ and for μ-almost all $x, y \in M$.

A very useful consequence of the Nash inequality is stated in the next lemma.

Lemma 5.5. *If $(\mathcal{E},\mathcal{F})$ satisfies ($Nash$) with $\nu = \alpha/\beta$ then, for all $t \in (0, \overline{R}^{\beta})$,*

$$\|P_t\|_{L^2 \to L^\infty} \leq \frac{C}{t^{\alpha/(2\beta)}}. \tag{5.5}$$

Consequently, (DUE) is satisfied.

For the proof see [4, Theorem 2.1] and [20, Lemma 3.7].

The converse is also true: (DUE) implies the ultracontractive estimate (5.5), while the latter implies ($Nash$) (see [10]).

6. Lemma of growth

The main result of this section is Lemma of growth (Lemma 6.4) and its consequences. A similar lemma in general metric spaces was proved [19] but in the present setting we have significant simplifications due to the ultrametric properties. In particular, we do not need to use a generalized capacity condition as in [19].

Consider the space

$$\mathcal{F}' := \mathcal{F} + \{\text{const}\}$$

and extend $\mathcal{E}$ from $\mathcal{F}$ to $\mathcal{F}'$ as follows: for all $u, v \in \mathcal{F}$ and $a, b \in \mathbb{R}$, set

$$\mathcal{E}\left(u+a, v+b\right) := \mathcal{E}\left(u, v\right).$$

Definition 6.1. Let Ω be an open subset of M. We say that a function $u \in \mathcal{F}'$ is *subharmonic* (resp. *superharmonic*) in Ω if

$$\mathcal{E}(u, \varphi) \leq 0 \quad (\text{resp. } \mathcal{E}(u, \varphi) \geq 0) \tag{6.1}$$

for any $0 \leq \varphi \in \mathcal{F}(\Omega)$. A function $u \in \mathcal{F}'$ is called *harmonic* in Ω if it is both subharmonic and superharmonic in Ω.

Let v be a Borel function on M. Define its *tail* $T_B(v)$ outside a ball B by

$$T_B(v) := \operatorname*{ess\,sup}_{x \in B} \int_{B^c} |v(y)| J(x, dy). \tag{6.2}$$

Lemma 6.2. *Let B be a compact ball of radius $R \in (0, \overline{R})$. Set*

$$\phi := \mathbf{1}_B.$$

Then, for any $v \in \mathcal{F}' \cap L^\infty$, that is non-negative and subharmonic in B, we have

$$\mathcal{E}(v\phi, v\phi) \leq 2T_B(v) \int_B v d\mu. \tag{6.3}$$

Proof. By Lemma 4.1, $\phi \in \mathcal{F}(B)$. Hence, both $v\phi$ and $v\phi^2$ belong to $\mathcal{F}(B)$ (cf. [19, Proposition 6.5 (i)-(ii)]). By a direct computation, we obtain the identity

$$\mathcal{E}(v\phi, v\phi) = \mathcal{E}(v, v\phi^2) + \int_{M \times M} v(x)v(y) \left(\phi(x) - \phi(y)\right)^2 dj \tag{6.4}$$

(see also [19, (3.19)]). Since $v\phi^2 \in \mathcal{F}(B)$, we conclude by the definition of subharmonic functions, that

$$\mathcal{E}(v, v\phi^2) \leq 0.$$

Splitting the domain of the integration in (6.4) and using symmetrization and $\phi = \mathbf{1}_B$, we obtain

$$\begin{aligned} \mathcal{E}(v\phi, v\phi) &\leq \left(\int_{B\times B} + \int_{B^c \times B} + \int_{B \times B^c} + \int_{B^c \times B^c}\right) v(x)v(y) \left(\phi(x) - \phi(y)\right)^2 dj \\ &= 2\int_{B \times B^c} v(x)v(y) \left(\phi(x) - \phi(y)\right)^2 dj \quad \text{(by symmetrization)} \\ &\leq 2\int_B v(x) d\mu(x) \cdot \operatorname*{ess\,sup}_{x \in B} \int_{B^c} |v(y)| J(x, dy), \end{aligned} \tag{6.5}$$

which is equivalent to (6.3). □

Remark 6.3. In a setting of jump Dirichlet forms in general metric measure spaces, a similar lemma was proved in [19, Lemma 3.10]. However, the proof in [19] is much more involved because in general the indicator function $\mathbf{1}_B$ is not in $\mathcal{F}$ and, hence, one must use a cutoff function ϕ of B in a larger ball. In order to do so, one has to assume an additional complicated hypothesis: the generalized capacity condition, that we do not need in the ultrametric setting.

Lemma 6.4 (Lemma of growth). *Assume that* (V), (TJ) *and* (PI) *are satisfied. If a function $u \in \mathcal{F}' \cap L^\infty$ is superharmonic and non-negative in a ball B of radius $R \in (0, \sigma \overline{R})$, and if, for some $a > 0$,*

$$\frac{\mu(B \cap \{u < a\})}{\mu(B)} \leq \varepsilon_0 \left(1 + \frac{R^\beta T_B(u_-)}{a}\right)^{-\alpha/\beta}, \tag{6.6}$$

then

$$\operatorname*{ess\,inf}_B u \geq \frac{a}{2}. \tag{6.7}$$

Here $\varepsilon_0 \in (0,1)$ is a constant that depends only on the constants in the hypotheses.

Recall that the tail function $T_B(v)$ was defined by (6.2). Observe also that if $u \geq 0$ on M then $T_B(u_-) = 0$ and the condition (6.6) simplifies. The statement of Lemma 6.4 means the following: if the set $\{u < a\}$ occupies in the ball B a small enough portion (where the smallness is determined by the right hand side of (6.6)) then the set $\{u < a/2\} \cap B$ has measure zero (see Fig. 1).

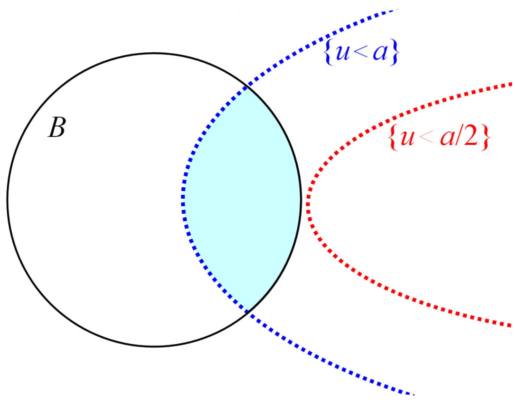

Figure 1. Level sets $\{u < a\}$ and $\{u < a/2\}$

The notion of Lemma of growth was introduced by Landis [26], [27] in the context of elliptic second order PDEs in $\mathbb{R}^n$, where it was used in order to obtain the Hölder continuity and the Harnack inequality for solutions. An earlier version of this type of argument goes back to De Giorgi [11]. The proof of the Lemma of growth for the non-divergence form second order elliptic and parabolic PDEs was a key part of the work of Krylov and Safonov [25].

Here we use the Lemma of growth for non-local operators that appeared in this form in [19]. However, in the presence of an ultrametric, the statement and the proof noticeably simplify.

The most essential part of the proof of Lemma 6.4 is contained in the following lemma.

Lemma 6.5. *Assume that* (V), (TJ) *and* (PI) *are satisfied. Let a function $u \in \mathcal{F}' \cap L^\infty$ be superharmonic and non-negative in a ball B of radius $R \in (0, \sigma \overline{R})$. Fix some $0 < a < b$ and set*

$$m_a = \frac{\mu(B \cap \{u < a\})}{\mu(B)} \quad \text{and} \quad m_b = \frac{\mu(B \cap \{u < b\})}{\mu(B)}.$$

Then

$$m_a \leq CA \left(\frac{b}{b-a}\right)^2 m_b^{1+\beta/\alpha}, \tag{6.8}$$

where

$$A := 1 + \frac{R^\beta T_B(u_-)}{b}, \tag{6.9}$$

and the constant $C > 0$ depends only on the constants in (V), (TJ) *and* (PI).

Proof. Denote

$$\widetilde{m}_a := \mu(B \cap \{u < a\}) \quad \text{and} \quad \widetilde{m}_b := \mu(B \cap \{u < b\})$$

and consider the functions

$$\phi := \mathbf{1}_B \quad \text{and} \quad v := (b-u)_+.$$

Since $v \geq b-a$ on the set $\{u < a\}$, we obtain

$$\widetilde{m}_a = \int_{B\cap\{u<a\}} \phi^2 d\mu \leq \frac{1}{(b-a)^2} \int_B (\phi v)^2 d\mu. \tag{6.10}$$

Consider the set $E = B \cap \{u < b\}$ (see Fig. 2). Since $\phi = 0$ outside B and $v = 0$ outside $\{u < b\}$, we see that $\phi v = 0$ in E^c.

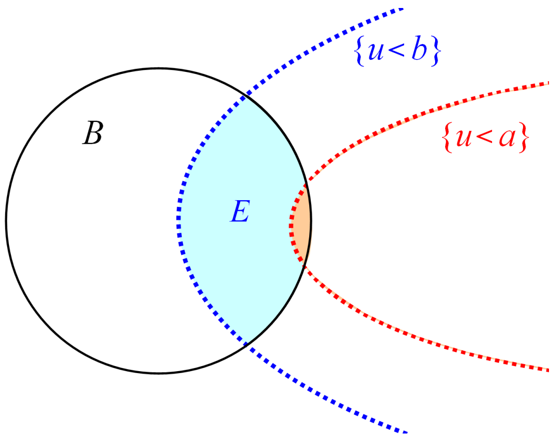

FIGURE 2. Set $E = B \cap \{u < b\}$

Since $\phi \in \mathcal{F}(B)$ and, hence, $\phi v \in \mathcal{F}(B)$, we conclude by the Faber-Krahn inequality (see Lemmas 5.2, 5.3) that

$$\int_B (\phi v)^2 d\mu = \int_E (\phi v)^2 d\mu \leq \mathcal{E}(\phi v, \phi v)\mu(E)^\nu = \mathcal{E}(\phi v, \phi v)\widetilde{m}_b^\nu. \tag{6.11}$$

Combining this inequality with (6.10), we obtain

$$\widetilde{m}_a \leq \frac{\mathcal{E}(\phi v, \phi v)}{(b-a)^2}\widetilde{m}_b^\nu. \tag{6.12}$$

Let us now estimate $\mathcal{E}(\phi v, \phi v)$ from above. Since u is superharmonic in B, the function $b-u$ is subharmonic in B. Then the function $v = (b-u)_+$ is also subharmonic in B (cf. [19, Lemma 3.2(ii)]). By Lemma 6.2, we have

$$\mathcal{E}(v\phi, v\phi) \leq 2T_B(v) \int_B v d\mu. \tag{6.13}$$

Using $v \leq b\mathbf{1}_{\{u<b\}}$, the definitions of E, v, $\widetilde{m}_b$, A, and the hypothesis (TJ), we obtain

$$\begin{aligned} \mathcal{E}(v\phi, v\phi) &\leq 2T_B(v)b\mu(E) \leq 2\left(T_B(b) + T_B(u_-)\right)b\widetilde{m}_b \\ &\leq 2\left(CbR^{-\beta} + T_B(u_-)\right)b\widetilde{m}_b \leq CAb^2R^{-\beta}\widetilde{m}_b. \end{aligned} \tag{6.14}$$

Combining (6.12) and (6.14) yields

$$\widetilde{m}_a \leq CA\left(\frac{b}{b-a}\right)^2 R^{-\beta}\widetilde{m}_b^{\nu+1}.$$

Finally, dividing this inequality by $\mu(B) \simeq R^\alpha$, we obtain (6.8), which finishes the proof. □

As we see from the proof, Lemma 6.5 does not use (PI) directly, only its consequence – the Faber-Krahn inequality in (6.11), whereas the hypothesis (TJ) is used explicitly in (6.14). The ultrametric property was most essentially used via Lemma 6.2 in (6.13).

Proof of Lemma 6.4. Let $u \in \mathcal{F}' \cap L^\infty$ be superharmonic and non-negative in a ball B of radius $R < \sigma \overline{R}$ and let $a > 0$. Consider the following sequence

$$a_k := \frac{1}{2}(1 + 2^{-k})a, \quad k = 0, 1, 2, ...,$$

so that $a_0 = a$ and $a_k \searrow \frac{1}{2}a$ as $k \to \infty$. Set also

$$m_k := \frac{\mu(B \cap \{u < a_k\})}{\mu(B)}.$$

Applying the inequality (6.8) of Lemma 6.5 with $a = a_k$ and $b = a_{k-1}$, we obtain, for any $k \geq 1$,

$$m_k \leq C\left(1 + \frac{R^\beta T_B(u_-)}{a_{k-1}}\right)\left(\frac{a_{k-1}}{a_{k-1} - a_k}\right)^2 m_{k-1}^{1+\beta/\alpha}.$$

Since $a_{k-1} \geq \frac{1}{2}a$ and

$$\frac{a_{k-1}}{a_{k-1} - a_k} = \frac{1 + 2^{-(k-1)}}{2^{-(k-1)} - 2^{-k}} \leq 2^{k+1},$$

it follows that

$$m_k \leq CA \cdot 4^k \cdot m_{k-1}^q, \tag{6.15}$$

where

$$A = 1 + \frac{R^\beta T_B(u_-)}{a} \quad \text{and} \quad q = 1 + \beta/\alpha.$$

Iterating (6.15), we obtain

$$\begin{aligned} m_k &\leq (CA)^{1+q+\cdots+q^{k-1}} \cdot 4^{k+q(k-1)+\cdots+q^{k-1}} \cdot m_0^{q^k} \\ &\leq \left((CA)^{\frac{1}{q-1}} \cdot 4^{\frac{q}{(q-1)^2}} \cdot m_0\right)^{q^k}, \end{aligned} \tag{6.16}$$

where in the second line we have used that

$$k + q(k-1) + \cdots + q^{k-1} = \frac{q^{k+1} - (k+1)q + k}{(q-1)^2} \leq \frac{q}{(q-1)^2}q^k,$$

and $C > 1$. It follows from (6.16) and $q > 1$ that if

$$(CA)^{\frac{1}{q-1}} \cdot 4^{\frac{q}{(q-1)^2}} \cdot m_0 \leq \frac{1}{2}, \tag{6.17}$$

then

$$\lim_{k\to\infty} m_k = 0. \tag{6.18}$$

Clearly, (6.17) is equivalent to

$$m_0 \leq 2^{-\frac{2q}{(q-1)^2}-1} \cdot (CA)^{-\frac{1}{q-1}}. \tag{6.19}$$

Since $\frac{1}{q-1} = \frac{\alpha}{\beta}$, we see that (6.19) is equivalent to the hypothesis (6.6) with

$$\varepsilon_0 := 2^{-\frac{2q}{(q-1)^2}-1} C^{-\frac{1}{q-1}}. \tag{6.20}$$

Assuming that ε_0 is defined by (6.20), we see that (6.17) is satisfied and, hence, we have (6.18). It follows that

$$\mu(B \cap \{u \leq \frac{a}{2}\}) = 0,$$

which implies $\operatorname{ess\,inf}_B u \geq \frac{a}{2}$, that is, (6.7). □

The following lemma is an easy consequence of Lemma 6.4.

Lemma 6.6. *Assume that* (V), (TJ) *and* (PI) *are satisfied. Then, for any ball* B *of radius* $R \in (0, \sigma \overline{R})$, *and for any non-negative function* $u \in \mathcal{F}' \cap L^\infty$ *that is superharmonic in* B, *the following is true:*

$$\operatorname*{ess\,inf}_B u \geq \frac{\varepsilon_0}{2} \left(\fint_B \frac{1}{u} d\mu \right)^{-1},$$

where ε_0 *is the same as in Lemma 6.4.*

Proof. We will apply Lemma 6.4 with a suitable value of a. Indeed, for any $a > 0$, we have

$$\mu(B \cap \{u < a\}) = \mu(B \cap \{\frac{1}{u} > \frac{1}{a}\}) \leq a \int_B \frac{1}{u} d\mu = a\mu(B) \fint_B \frac{1}{u} d\mu.$$

Since u is non-negative, we have that $R^\beta T_B(u_-) = 0$. Setting

$$a := \varepsilon_0 \left(\fint_B \frac{1}{u} d\mu \right)^{-1},$$

we obtain that the condition (6.6) of Lemma 6.4 is fulfilled. Hence, by Lemma 6.4, we conclude that

$$\operatorname*{ess\,inf}_B u \geq \frac{a}{2} = \frac{\varepsilon_0}{2} \left(\fint_B \frac{1}{u} d\mu \right)^{-1},$$

which was to be proved. □

7. Some auxiliary inequalities

In this section we prove some preparatory lemmas to be used in Sections 8 and 10. We frequently use the notation

$$u_\lambda := u + \lambda$$

where u is a function on M and $\lambda > 0$ is a constant.

Lemma 7.1. *Let a function* $u \in \mathcal{F}' \cap L^\infty$ *be non-negative in a ball* $B \subset M$. *Set* $\phi := \mathbf{1}_B$. *Then, for any* $\lambda > 0$, *we have* $\frac{\phi^2}{u_\lambda} \in \mathcal{F}(B)$ *and*

$$\mathcal{E}(u, \frac{\phi^2}{u_\lambda}) + \frac{1}{2} \int_{B \times B} \left| \ln \frac{u_\lambda(y)}{u_\lambda(x)} \right|^2 dj\,(x,y) \leq 3\mathcal{E}(\phi, \phi) + 2 \int_{B \times B^c} \frac{(u_\lambda\,(y))_-}{u_\lambda(x)} dj\,(x,y), \tag{7.1}$$

where in the last integral $x \in B$ *and* $y \in B^c$. *If in addition* u *is superharmonic in* B *then*

$$\int_{B \times B} \left| \ln \frac{u_\lambda(y)}{u_\lambda(x)} \right|^2 dj \leq 6\mathcal{E}\,(\phi, \phi) + 4 \int_{B \times B^c} \frac{(u_\lambda(y))_-}{u_\lambda(x)} dj. \tag{7.2}$$

Proof. By [19, Lemma 3.7], for any $\phi \in \mathcal{F} \cap C_0\,(B)$, we have $\frac{\phi^2}{u_\lambda} \in \mathcal{F} \cap L^\infty$ and

$$\mathcal{E}(u, \frac{\phi^2}{u_\lambda}) + \frac{1}{2} \int_{B \times B} \left(\phi^2(x) \wedge \phi^2(y)\right) \left| \ln \frac{u_\lambda(y)}{u_\lambda(x)} \right|^2 dj\,(x,y)$$
$$\leq 3\mathcal{E}(\phi, \phi) - 2 \int_{B \times B^c} u_\lambda(y) \frac{\phi^2(x)}{u_\lambda(x)} dj\,(x,y).$$

Substituting $\phi = \mathbf{1}_B \in \mathcal{F} \cap C_0\,(B)$ into this inequality, we obtain (7.1). Observe that $\frac{\phi^2}{u_\lambda} = \phi \frac{\phi^2}{u_\lambda} \in \mathcal{F}\,(B)$ (cf. [14]).

If u is superharmonic in B then $\mathcal{E}(u, \frac{\phi^2}{u_\lambda}) \geq 0$ because $\frac{\phi^2}{u_\lambda} \in \mathcal{F}\,(B)$ is non-negative. Hence, (7.2) follows from (7.1). □

Lemma 7.2. *Assume that* (V), (TJ) *and* (PI) *are satisfied. Let a function* $u \in \mathcal{F}' \cap L^\infty$ *be non-negative and superharmonic in a ball* B *of radius* $R \in (0, \overline{R})$. *Fix three positive numbers* a, b, λ *and consider in* B *the function:*

$$v := \left(\ln \frac{a}{u_\lambda} \right)_+ \wedge b.$$

Then

$$\fint_{\kappa B}\fint_{\kappa B}(v(x)-v(y))^2 d\mu(x)d\mu(y) \le C\left(1+\frac{R^\beta T_B((u_\lambda)_-)}{\lambda}\right), \tag{7.3}$$

where κ is the constant from (PI) and C depends only on the constants in the hypotheses.

Proof. Let us extend v to B^c by setting $v=0$ and show that $v\in\mathcal{F}(B)$. Indeed, the function $F(t)=\ln\frac{a}{|t|+\lambda}$ is a bounded Lipschitz function on $\mathbb{R}$. Therefore, $F(u)\in\mathcal{F}\cap L^\infty$ and, hence, $\mathbf{1}_B F(u)\in\mathcal{F}(B)$. Consequently, by the Markov property, $v=(\mathbf{1}_B F(u))_+\wedge b$ is also in $\mathcal{F}(B)$.

Applying the identity (3.3), the hypothesis (*PI*), and the obvious inequality

$$|v(x)-v(y)|\le\left|\ln\frac{u_\lambda(y)}{u_\lambda(x)}\right|$$

that holds for all $x,y\in B$, we obtain

$$\begin{aligned}\fint_{\kappa B}\fint_{\kappa B}(v(x)-v(y))^2 d\mu(x)d\mu(y) &= \frac{2}{\mu(\kappa B)}\int_{\kappa B}(v-v_{\kappa B})^2 d\mu\\ &\le \frac{CR^\beta}{\mu(\kappa B)}\int_{B\times B}(v(x)-v(y))^2 dj(x,y)\\ &\le \frac{CR^\beta}{\mu(\kappa B)}\int_{B\times B}\left|\ln\frac{u_\lambda(y)}{u_\lambda(x)}\right|^2 dj(x,y). \end{aligned}\tag{7.4}$$

Applying the inequality (7.2) of Lemma 7.1, estimating $\mathcal{E}(\phi,\phi)$ by Corollary 4.2, and using $u_\lambda(x)\ge\lambda$ in B, we obtain

$$\begin{aligned}\int_{B\times B}\left|\ln\frac{u_\lambda(y)}{u_\lambda(x)}\right|^2 dj(x,y) &\le C\frac{\mu(B)}{R^\beta}+4\int_{B\times B^c}\frac{(u_\lambda(y))_-}{u_\lambda(x)}dj(x,y)\\ &\le C\frac{\mu(B)}{R^\beta}+4\frac{\mu(B)}{\lambda}\operatorname*{ess\,sup}_{x\in B}\int_{B^c}(u_\lambda(y))_- J(x,dy)\\ &\le C\frac{\mu(B)}{R^\beta}\left(1+\frac{R^\beta}{\lambda}T_B((u_\lambda)_-)\right). \end{aligned}\tag{7.5}$$

Combining (7.4) and (7.5), we obtain (7.3). □

Note that Lemma 7.2 is the only place in the entire proof where we use directly the Poincaré inequality (*PI*) (except for Lemma 5.2, where we derive the Nash inequality from (*PI*)). Through Lemma 7.2, (*PI*) is used in the derivation of the weak Harnack inequality in the next section.

8. Weak Harnack inequality

Lemma 8.1. *Assume that (V), (TJ) and (PI) are satisfied. Then, for any ball B of radius $R\in(0,\sigma\overline{R})$, for any function $u\in\mathcal{F}'\cap L^\infty$ that is superharmonic and non-negative in B, and for any $a>0$, such that*

$$\frac{\mu(\kappa B\cap\{u\ge a\})}{\mu(\kappa B)}\ge\frac{1}{2} \tag{8.1}$$

and

$$R^\beta T_B(u_-)\le\varepsilon a, \tag{8.2}$$

we have

$$\operatorname*{ess\,inf}_{\kappa B} u\ge\varepsilon a \tag{8.3}$$

(see Fig. 3). Here $\varepsilon\in(0,1)$ is a constant that depends only on the constants in the hypotheses.

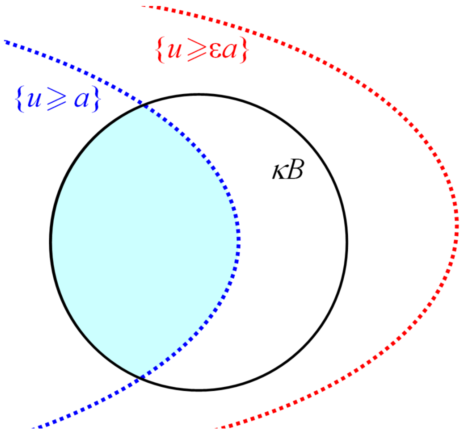

FIGURE 3. Level sets $\{u \geq a\}$ and $\{u \geq \varepsilon a\}$

If $u \geq 0$ on M then the condition (8.2) is trivially satisfied. A (strong) Harnack inequality for non-negative harmonic functions (should it be true) would say that

$$\operatorname*{ess\,inf}_{\kappa B} u \geq \varepsilon \operatorname*{ess\,sup}_{\kappa B} u.$$

In particular, for any $a < \operatorname{ess\,sup}_{\kappa B} u$, we would have (8.3). Thus, the hypothesis (8.1) could have been relaxed in this case to $\mu\left(\kappa B \cap \{u \geq a\}\right) > 0$. Hence, Lemma 8.1 can be regarded as a weak version of the Harnack inequality.

However, in the literature the term "weak Harnack inequality" is frequently used for a stronger statement containing a lower bound of $\operatorname{ess\,inf}_{\kappa B} u$ via some L^p-norm of u (see [15, Theorem 8.18], [13, Section 1.3]). Our "weak Harnack inequality" is really "very weak".

Proof of Lemma 8.1. Let λ, b be two positive parameters to be determined later. Consider in B the function

$$v := \left(\ln \frac{a+\lambda}{u_\lambda}\right)_+ \wedge b,$$

where $u_\lambda = u + \lambda$. Note that $0 \leq v \leq b$ and

$$\begin{aligned} v = 0 \quad &\Leftrightarrow \quad \frac{a+\lambda}{u_\lambda} \leq 1 \quad \Leftrightarrow \quad u \geq a \\ v = b \quad &\Leftrightarrow \quad \frac{a+\lambda}{u_\lambda} \geq e^b \quad \Leftrightarrow \quad u_\lambda \leq (a+\lambda)e^{-b} =: q. \end{aligned}$$

We will apply Lemma 6.4 to u_λ instead of u. For that, set

$$\omega := \frac{\mu(\kappa B \cap \{u \geq a\})}{\mu(\kappa B)} = \frac{\mu(\kappa B \cap \{v = 0\})}{\mu(\kappa B)} \tag{8.4}$$

and

$$m := \frac{\mu(\kappa B \cap \{u_\lambda \leq q\})}{\mu(\kappa B)} = \frac{\mu(\kappa B \cap \{v = b\})}{\mu(\kappa B)}. \tag{8.5}$$

By Lemma 6.4, if

$$m \leq \varepsilon_0 \left(1 + \frac{(\kappa R)^\beta \, T_{\kappa B}((u_\lambda)_-)}{q}\right)^{-\alpha/\beta}, \tag{8.6}$$

then

$$\operatorname*{ess\,inf}_{\kappa B} u_\lambda \geq \frac{q}{2}. \tag{8.7}$$

Since $u \geq 0$ in B, we have

$$A := R^\beta T_B(u_-) \geq (\kappa R)^\beta \, T_{\kappa B}\left((u_\lambda)_-\right).$$

Hence, in order to have (8.6), it suffices to ensure that

$$m \le \varepsilon_0 \left(1 + \frac{A}{q}\right)^{-\alpha/\beta}. \tag{8.8}$$

Using (8.4), (8.5), and Lemma 7.2, we obtain

$$\begin{aligned} b^2 m\omega &= \frac{1}{\mu(\kappa B)^2} \int_{\kappa B \cap \{v=0\}} \int_{\kappa B \cap \{v=b\}} b^2 d\mu(x) d\mu(y) \\ &= \frac{1}{\mu(\kappa B)^2} \int_{\kappa B \cap \{v=0\}} \int_{\kappa B \cap \{v=b\}} (v(x) - v(y))^2 d\mu(x) d\mu(y) \\ &\le \fint_{\kappa B} \fint_{\kappa B} (v(x) - v(y))^2 d\mu(x) d\mu(y) \\ &\le C \left(1 + \frac{R^\beta T_B((u_\lambda)_-)}{\lambda}\right) \le C \left(1 + \frac{A}{\lambda}\right). \end{aligned}$$

It follows that

$$m \le \frac{C}{b^2 \omega} \left(1 + \frac{A}{\lambda}\right) \le \frac{2C}{b^2} \left(1 + \frac{A}{\lambda}\right),$$

where we have used that $\omega \ge 1/2$, which is true by (8.1). Hence, the condition (8.8) will be satisfied provided

$$\frac{2C}{b^2} \left(1 + \frac{A}{\lambda}\right) \le \varepsilon_0 \left(1 + \frac{A}{q}\right)^{-\alpha/\beta}$$

that is equivalent to

$$b^2 \ge \frac{2C}{\varepsilon_0} \left(1 + \frac{A}{\lambda}\right) \left(1 + \frac{A}{q}\right)^{\alpha/\beta}. \tag{8.9}$$

Fix $\varepsilon > 0$ to be determined later, and specify the parameters λ, b as follows:

$$\lambda := \varepsilon a, \quad b := \ln \frac{1+\varepsilon}{4\varepsilon}.$$

Then we have

$$q = (a + \lambda) e^{-b} = 4\varepsilon a,$$

and the inequality (8.9) is equivalent to

$$\left(\ln \frac{1+\varepsilon}{4\varepsilon}\right)^2 \ge \frac{2C}{\varepsilon_0} \left(1 + \frac{A}{\varepsilon a}\right) \left(1 + \frac{A}{4\varepsilon a}\right)^{\alpha/\beta}. \tag{8.10}$$

Since $A \le \varepsilon a$ by (8.2), the inequality (8.10) will follow from

$$\left(\ln \frac{1+\varepsilon}{4\varepsilon}\right)^2 \ge \frac{4C}{\varepsilon_0} \left(\frac{5}{4}\right)^{\alpha/\beta}$$

that can be achieved by choosing ε small enough. With this choice of ε we conclude that (8.7) holds, which implies

$$\operatorname*{ess\,inf}_{\kappa B} u \ge \frac{q}{2} - \lambda = 2\varepsilon a - \varepsilon a = \varepsilon a,$$

thus finishing the proof. □

9. Oscillation properties for harmonic functions

Here we use the weak Harnack inequality of Lemma 8.1 in order to obtain Hölder estimates for harmonic functions in Lemmas 9.3 and 9.4. In the case of local Dirichlet forms (and for solutions of second order elliptic PDEs) this is quite simple as was demonstrated in [27, Theorem 7.2]. The non-local case is much more involved because of the tail condition (8.2). We use an enhanced version of the argument that originated from [12] and that was also used in [19]. Earlier versions of this argument were used in [30] and [24]. The fact that a stronger version of a weak Harnack inequality implies the Hölder continuity in the framework of non-local operators in $\mathbb{R}^n$ was first observed in [13].

Given a ball $B \subset M$ and a function $u \in \mathcal{F}' \cap L^\infty(M)$, set

$$m_* = \operatorname*{ess\,inf}_B u, \qquad m^* = \operatorname*{ess\,sup}_B u \tag{9.1}$$

and define the following notations:

$$\operatorname*{osc}_B u := m^* - m_*$$

and

$$T_B^*(u) := T_B\left((u - m_*)_- + (m^* - u)_-\right) = \operatorname*{ess\,sup}_{x \in B} \int_{B^c} \left((u - m_*)_- + (m^* - u)_-\right) J(x, dy).$$

It is easy to see that $T_B^*(u)$ is monotone decreasing with respect to B.

Lemma 9.1 (Oscillation inequality)**.** *Assume that* (V)*,* (TJ) *and* (PI) *are satisfied. Let* $u \in \mathcal{F}' \cap L^\infty$ *be harmonic in a ball* B *of radius* $R \in (0, \sigma \overline{R})$*. Then we have either*

$$\operatorname*{osc}_{\kappa B} u \le (1 - \varepsilon) \operatorname*{osc}_B u, \tag{9.2}$$

or

$$\operatorname*{osc}_B u \le \varepsilon^{-1} R^\beta T_B^*(u), \tag{9.3}$$

where $\varepsilon \in (0, 1)$ *is a constant depending only on the constants from the hypotheses.*

Proof. Let us use the notations (9.1). By adding to u a constant, we can assume without loss of generality that $m_* + m^* = 0$, that is,

$$a := m^* = -m_*.$$

Clearly, one of the sets $\kappa B \cap \{u \ge 0\}$ and $\kappa B \cap \{u \le 0\}$ takes at least $\frac{1}{2}$ of the measure of κB. Without loss of generality, we can assume that this is the first one (otherwise change u to $-u$), which is equivalent to

$$\frac{\mu(\kappa B \cap \{u + a \ge a\})}{\mu(\kappa B)} \ge \frac{1}{2}. \tag{9.4}$$

Since the function $u + a$ belongs to $\mathcal{F}' \cap L^\infty$ and is non-negative and harmonic in B, we conclude by Lemma 8.1 that if

$$R^\beta T_B((u + a)_-) \le \varepsilon a, \tag{9.5}$$

then

$$\operatorname*{ess\,inf}_{\kappa B} (u + a) \ge \varepsilon a.$$

In this case we obtain

$$\operatorname*{osc}_{\kappa B} u = \operatorname*{ess\,sup}_{\kappa B} u - \operatorname*{ess\,inf}_{\kappa B} u \le a - (\varepsilon a - a) = \left(1 - \frac{\varepsilon}{2}\right) 2a = \left(1 - \frac{\varepsilon}{2}\right) \operatorname*{osc}_B u.$$

On the other hand, if (9.5) fails then

$$R^\beta T_B^*(u) = R^\beta T_B((u + a)_- + (a - u)_-) \ge R^\beta T_B((u + a)_-) \ge \varepsilon a = \frac{\varepsilon}{2} \operatorname*{osc}_B u.$$

Renaming $\varepsilon/2$ to ε, we obtain that one of the inequalities (9.2), (9.3) is satisfied. □

Lemma 9.2. *Let* $\{B_j\}_{j=0}^k$ *be a sequence of* $k+1$ *balls such that* $B_{j+1} \subset B_j$ *for all* $j = 0, 1, ..., k-1$. *For any function* $u \in \mathcal{F}' \cap L^\infty$, *the following inequality holds:*

$$T^*_{B_k}(u) \le \sum_{j=0}^{k-1} T_{B_j}(1)(\operatorname*{osc}_{B_j} u - \operatorname*{osc}_{B_k} u) + T_{B_0}(u) + T_{B_0}(1)\, \|u\|_{L^\infty(B_0)}. \tag{9.6}$$

Proof. Denote

$$m_j = \operatorname*{ess\,inf}_{B_j} u, \quad M_j = \operatorname*{ess\,sup}_{B_j} u, \quad Q_j = \operatorname*{osc}_{B_j} u = M_j - m_j.$$

Set

$$v = (u - m_k)_- + (M_k - u)_- = (m_k - u)_+ + (u - M_k)_+$$

so that $T^*_{B_k}(u) = T_{B_k}(v)$. Let us first prove that, for any $j = 0, ..., k-1$,

$$T_{B_{j+1}}(v) \le (Q_j - Q_k) T_{B_{j+1}}(1) + T_{B_j}(v) \tag{9.7}$$

(Fig. 4). Since B^c_{j+1} is the union of $B_j \setminus B_{j+1}$ and B^c_j, we have

$$T_{B_{j+1}}(v) \le \operatorname*{ess\,sup}_{x \in B_{j+1}} \int_{B_j \setminus B_{j+1}} v(y) J(x, dy) + \operatorname*{ess\,sup}_{x \in B_j} \int_{B^c_j} v(y) J(x, dy). \tag{9.8}$$

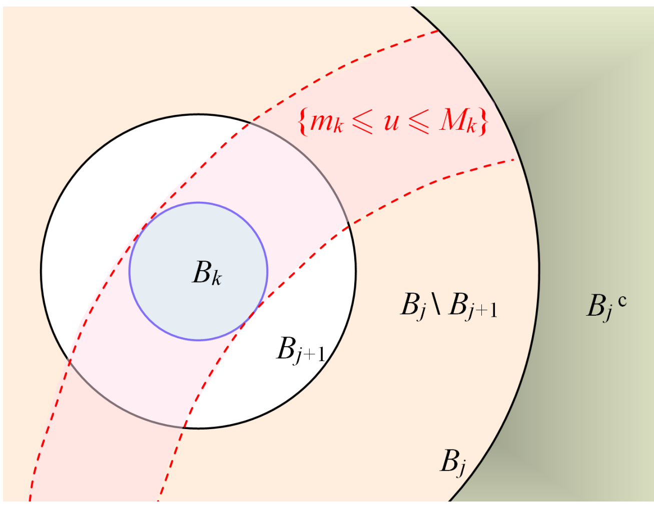

FIGURE 4.

Let us verify that

$$v \le Q_j - Q_k \quad \mu\text{-a.e. in } B_j. \tag{9.9}$$

Indeed, in the set $\{m_k \le u \le M_k\}$ we have $v = 0$ and (9.9) is trivial. In the set $\{u < m_k\} \cap B_j$ we have

$$v = m_k - u \le m_k - m_j \le (m_k - m_j) + (M_j - M_k) = Q_j - Q_k.$$

The same argument works in the set $\{u > M_k\} \cap B_j$ as in this case

$$v = u - M_k \le M_j - M_k \le Q_j - Q_k.$$

Using (9.9) and (4.3) we obtain, for μ-a.a. $x \in B_{j+1}$,

$$\int_{B_j \setminus B_{j+1}} v(y) J(x, dy) \le (Q_j - Q_k) \int_{B^c_{j+1}} J(x, dy) \le (Q_j - Q_k) T_{B_{j+1}}(1). \tag{9.10}$$

Substituting this into (9.8), we obtain (9.7).

Iterating (9.7), we obtain

$$T^*_{B_k}(u) = T_{B_k}(v) \le \sum_{j=0}^{k-1} (Q_j - Q_k)\, T_{B_{j+1}}(1) + T_{B_0}(v)\,.$$

Observing that

$$v \le |u| + \max\{|m_k|\,, |M_k|\} \le |u| + \|u\|_{L^\infty(B_0)}\,,$$

and, hence,

$$T_{B_0}(v) \le T_{B_0}(u) + T_{B_0}(1)\,\|u\|_{L^\infty(B_0)}\,,$$

we obtain (9.6). □

In the rest of this section we use the notation $B_r := B(x_0, r)$ assuming that x_0 is a fixed point on M.

Lemma 9.3 (Iterated Oscillation Inequality)**.** *Assume that* (V), (TJ) *and* (PI) *are satisfied. For any function* $u \in \mathcal{F}' \cap L^\infty$ *that is harmonic in a ball* B_R *of radius* $R \in (0, \sigma\overline{R})$*, the following inequality holds for any non-negative integer* k*:*

$$\operatorname*{osc}_{B_{q^{-k}R}} u \le C_0 q^{-\gamma k} A, \tag{9.11}$$

where $q > 1$, $C_0 > 1$ *and* $0 < \gamma < 1$ *are constants depending on the hypotheses and*

$$A := R^\beta T_{B_R}(u) + \|u\|_{L^\infty(B_R)}. \tag{9.12}$$

Proof. In this proof all constants are important and will be denoted by designated letters. The letter ε denotes the constant from Lemma 9.1, C is reserved for the constant from (TJ), and C_0 is the constant from (9.11).

Fix a large number $q > 1$ to be specified below and set, for any non-negative integer k,

$$R_k = q^{-k} R \quad \text{and} \quad Q_k = \operatorname*{osc}_{B_{R_k}} u.$$

The inequality (9.11) is equivalent to

$$Q_k \le C_0 q^{-\gamma k} A, \tag{9.13}$$

that will be proved by induction in k, where $\gamma > 0$ is a small number and C_0 is a large number, to be chosen below.

For $k = 0$ and $k = 1$ we have

$$Q_1 \le Q_0 = \operatorname*{osc}_{B_R} u \le 2\,\|u\|_{L^\infty(B_R)} \le 2A = 2q^\gamma \left(q^{-\gamma} A\right),$$

so that (9.13) holds provided $C_0 \ge 2q^\gamma$. Let us make the inductive step from $\le k$ to $k+1$, assuming $k \ge 1$.

Taking q so big that $q^{-1} \le \kappa$, we obtain by Lemma 9.1, that

$$\text{either} \quad Q_{k+1} \le (1-\varepsilon) Q_k \quad \text{or} \quad Q_k \le \varepsilon^{-1} R_k^\beta T^*_{B_{R_k}}(u).$$

Assuming first that $Q_{k+1} \le (1-\varepsilon)Q_k$, we obtain by the inductive hypothesis

$$Q_{k+1} \le (1-\varepsilon) C_0 q^{-\gamma k} A = (1-\varepsilon) q^\gamma C_0 q^{-\gamma(k+1)} A \le C_0 q^{-\gamma(k+1)} A,$$

provided

$$(1-\varepsilon) q^\gamma \le 1. \tag{9.14}$$

After we specify below a large enough q, we can always choose $\gamma > 0$ so small that (9.14) is satisfied. Thus, we complete the inductive step in this case.

Consider now the second case, that is, let

$$Q_k \le \varepsilon^{-1} R_k^\beta T^*_{B_k}(u). \tag{9.15}$$

By Lemma 9.2, we have

$$T^*_{B_{R_k}}(u) \le \sum_{j=0}^{k-1} (Q_j - Q_k)\, T_{B_{R_{j+1}}}(1) + T_{B_R}(u) + T_{B_R}(1)\, \|u\|_{L^\infty(B_R)} .$$

Clearly,

$$T_{B_R}(u) + T_{B_R}(1)\, \|u\|_{L^\infty(B_R)} \le T_{B_R}(u) + CR^{-\beta}\, \|u\|_{L^\infty(B_R)} = CAR^{-\beta},$$

and, by (TJ), we have

$$T_{B_{R_j}}(1) \le CR_j^{-\beta} = Cq^{\beta j} R^{-\beta}.$$

It follows that

$$T^*_{B_{R_k}}(u) \le C \sum_{j=0}^{k-1} (Q_j - Q_k)\, q^{\beta(j+1)} R^{-\beta} + CAR^{-\beta},$$

whence

$$\begin{aligned} R_k^\beta T^*_{B_{R_k}}(u) &\le C \sum_{j=0}^{k-1} (Q_j - Q_k)\, q^{\beta(j-k+1)} + CAq^{-\beta k} \\ &\le C \left(\sum_{j=0}^{k-1} Q_j q^{\beta(j-k+1)} - Q_k + Aq^{-\beta k} \right), \end{aligned}$$

where we have used that $\sum_{j=0}^{k-1} q^{\beta(j-k+1)} \ge 1$. By the inductive hypothesis we have

$$Q_j \le C_0 q^{-\gamma j} A, \quad j = 0, 1, \dots, k. \tag{9.16}$$

Substituting into the previous estimate and assuming $\gamma < \beta$, we obtain

$$\begin{aligned} R_k^\beta T^*_{B_{R_k}}(u) &\le C \left(C_0 q^{-\beta(k-1)} \sum_{j=0}^{k-1} q^{(\beta-\gamma)j} A - Q_k + Aq^{-\beta k} \right) \\ &\le C \left(C_0 q^{-\beta(k-1)} \frac{q^{(\beta-\gamma)k}}{q^{\beta-\gamma} - 1} A - Q_k + Aq^{-\beta k} \right) \\ &= C \left(\frac{C_0 q^\beta q^{-\gamma k} A}{q^{\beta-\gamma} - 1} - Q_k + Aq^{-\beta k} \right). \end{aligned}$$

Combining with (9.15), we obtain

$$Q_k \le \varepsilon^{-1} C \left(\frac{C_0 q^\beta q^{-\gamma k} A}{q^{\beta-\gamma} - 1} - Q_k + Aq^{-\beta k} \right),$$

whence

$$\begin{aligned} Q_k &\le \frac{C}{C+\varepsilon} \left(\frac{C_0 q^\beta q^{-\gamma k}}{q^{\beta-\gamma} - 1} + q^{-\beta k} \right) A \\ &\le \frac{C}{C+\varepsilon} \left(\frac{q^\gamma}{1 - q^{-(\beta-\gamma)}} + \frac{1}{C_0} \right) C_0 q^{-\gamma k} A. \end{aligned} \tag{9.17}$$

In order to complete the inductive step, it suffices to verify that

$$Q_k \le C_0 q^{-(k+1)\gamma} A$$

because $Q_{k+1} \le Q_k$. Clearly, this will follow from (9.17) provided

$$\frac{C}{C+\varepsilon} \left(\frac{q^\gamma}{1 - q^{-(\beta-\gamma)}} + \frac{1}{C_0} \right) q^\gamma \le 1. \tag{9.18}$$

Assuming that $\gamma < \beta/2$, (9.18) will follow from

$$\frac{q^{2\gamma}}{1-q^{-\beta/2}} + \frac{q^{\gamma}}{C_0} \leq 1 + \frac{\varepsilon}{C}. \tag{9.19}$$

Clearly, by choosing first q large enough, then $\gamma > 0$ small enough, and then C_0 large enough, we achieve both (9.14) and (9.19), thus finishing the proof. □

Lemma 9.4 (Oscillation Lemma)**.** *Assume that* (V), (TJ) *and* (PI) *are satisfied. Let* $u \in \mathcal{F}' \cap L^{\infty}$ *be harmonic in a ball* $B_R = B(x_0, R)$ *with* $R < \sigma\overline{R}$*. Then, for any* $\rho \in (0, R]$,

$$\operatorname*{osc}_{B_\rho} u \leq C\left(\frac{\rho}{R}\right)^{\gamma}\left(R^{\beta} T_{B_R}(|u|) + \|u\|_{L^{\infty}(B_R)}\right), \tag{9.20}$$

where $\gamma > 0$ *is the constant from Lemma 9.3 and* C *depends only on the constants from the hypotheses.*

Proof. We use the notation from Lemma 9.3. Since $\rho \in (0, R]$, there exists an integer $k \geq 0$ such that

$$q^{-(k+1)} < \frac{\rho}{R} \leq q^{-k}.$$

Hence, by Lemma 9.3,

$$\operatorname*{osc}_{B_\rho} u \leq \operatorname*{osc}_{B_{R/q^k}} u \leq C_0 q^{-k\gamma} A = C_0 q^{\gamma} (q^{-(k+1)})^{\gamma} A \leq C_0 q^{\gamma} \left(\frac{\rho}{R}\right)^{\gamma} A,$$

which is exactly (9.20) with $C = C_0 q^{\gamma}$. □

10. Conditions (E) and (S)

Recall that, for any open set $\Omega \subset M$, $(\mathcal{E}, \mathcal{F}(\Omega))$ is a regular Dirichlet form in $L^2(\Omega)$, $\mathcal{L}^{\Omega}$ is its generator, and $P_t^{\Omega} = e^{-t\mathcal{L}^{\Omega}}$is the corresponding heat semigroup acting in $L^2(\Omega)$ (see Section 7). By the functional calculus, for any $f \in L^2(\Omega)$, the function $t \mapsto P_t^{\Omega} f$ is a continuous mapping from $[0, \infty)$ to $L^2(\Omega)$. Hence, it can be integrated in t on any bounded time interval $[0, T]$ as an L^2-valued function. Assuming further that $f \geq 0$, we can let $T \to \infty$ and define the *Green operator* G^{Ω} by

$$G^{\Omega} f = \int_0^{\infty} P_t^{\Omega} f \, dt := \lim_{T \to \infty} \int_0^T P_t^{\Omega} f \, dt$$

where the limit is understood a.e.. Note that the function $G^{\Omega} f$ takes values in $[0, +\infty]$. By the monotonicity of $G^{\Omega} f$ in f, we can extend the operator G^{Ω} to all non-negative measurable functions f, in particular, to $f \equiv 1$.

Definition 10.1. We say that the condition ($E_{\leq}$) holds if there exist $\sigma \in (0, 1)$ and $C > 0$ such that, for any ball B of radius $r \in (0, \sigma\overline{R})$,

$$\operatorname*{ess\,sup}_B G^B 1 \leq C r^{\beta}. \tag{$E_{\leq}$}$$

We say that the condition ($E_{\geq}$) holds[1] if, for any ball B of radius $r \in (0, \overline{R})$,

$$\operatorname*{ess\,inf}_B G^B 1 \geq C^{-1} r^{\beta}. \tag{$E_{\geq}$}$$

We say that the condition (E) holds if both ($E_{\leq}$) and ($E_{\geq}$) are satisfied.

In what follows, the parameter σ in the above definition will coincide with the parameter σ in the Faber-Krahn inequality (cf. Lemma 5.3).

[1]On an arbitrary metric space, the condition ($E_{\geq}$) should look as follows:

$$\operatorname*{ess\,inf}_{\varepsilon B} G^B 1 \geq C^{-1} r^{\beta},$$

where $\varepsilon > 0$ is small enough, to ensure that the points where we estimate $G^B 1$ from below are far enough from B^c. However, in ultra-metric space, the distance from any point of B to B^c is at least r, so we can take $\varepsilon = 1$.

Lemma 10.2. *We have*

$$(V_{\leq}) + (Nash) \Rightarrow (E_{\leq})$$

where ν and σ are the same as in Lemma 5.3.

Proof. Let B be a ball of radius $r \in \left(0, \sigma\overline{R}\right)$. Let us first prove that, for all $t \geq \frac{1}{3}r^{\beta}$,

$$\left\|P_t^B\right\|_{L^2\to L^\infty} \leq \frac{C}{r^{\alpha/2}}.$$

Indeed, if $t < \overline{R}^{\beta}$ then, by the inequality (5.5) of Lemma 5.5,

$$\|P_t^B\|_{L^2\to L^\infty} \leq \|P_t\|_{L^2\to L^\infty} \leq \frac{C}{t^{\alpha/(2\beta)}} \leq \frac{C'}{r^{\alpha/2}}.$$

The same inequality holds also for $t \geq \overline{R}^{\beta}$ as $\left\|P_t^B\right\|_{L^2\to L^\infty}$ is monotone decreasing in t.

Next, we have by (5.4) and ($V_{\leq}$)

$$\lambda_1(B) \geq c\mu\left(B\right)^{-\beta/\alpha} \geq cr^{-\beta}.$$

Hence, by the spectral theory of self-adjoint operators, we obtain, for all $t > 0$,

$$\|P_t^B\|_{L^2\to L^2} \leq e^{-\lambda_1(B)t} \leq e^{-cr^{-\beta}t}.$$

It follows that, for all $t \geq r^{\beta}$,

$$\begin{aligned}\|P_t^B\|_{L^1\to L^\infty} &\leq \|P_{t/3}^B\|_{L^1\to L^2}\|P_{t/3}^B\|_{L^2\to L^2}\|P_{t/3}^B\|_{L^2\to L^\infty}\\ &= \|P_{t/3}^B\|^2_{L^2\to L^\infty}\|P_{t/3}^B\|_{L^2\to L^2}\\ &\leq \frac{C}{r^\alpha}e^{-cr^{-\beta}t}.\end{aligned}$$

Consequently, we have, for all $t \geq r^{\beta}$,

$$\left\|P_t^B\mathbf{1}_B\right\|_{L^\infty} \leq \frac{C}{r^\alpha}e^{-cr^{-\beta}t}\left\|\mathbf{1}_B\right\|_{L^1} = \frac{C}{r^\alpha}e^{-cr^{-\beta}t}\mu\left(B\right) \leq Ce^{-cr^{-\beta}t}.$$

For any nonnegative $f \in L^1(B)$, we obtain

$$\begin{aligned}(G^B\mathbf{1}_B, f) = \int_0^\infty (P_t^B\mathbf{1}_B, f)dt &\leq \int_0^{r^\beta}(\mathbf{1}_B, f)dt + \int_{r^\beta}^\infty (P_t^B\mathbf{1}_B, f)dt\\ &\leq \int_0^{r^\beta}\|f\|_{L^1}\,dt + \int_{r^\beta}^\infty Ce^{-cr^{-\beta}t}\|f\|_{L^1}dt\\ &\leq C'r^\beta\|f\|_{L^1}\end{aligned}$$

where $C' = 1 + C\int_1^\infty e^{-cs}ds$. Since $0 \leq f \in L^1(B)$ is arbitrary, we obtain ($E_{\leq}$). □

Lemma 10.3 ([19, Lemma 5.1])**.** *If $G^\Omega 1 \in L^\infty(\Omega)$ then, for any $f \in L^2(\Omega)$, the function $G^\Omega f$ belongs to $\mathcal{F}(\Omega)$ and satisfies the identity*

$$\mathcal{E}(G^\Omega f, \varphi) = (f, \varphi) \quad \forall\varphi \in \mathcal{F}(\Omega).$$

If in addition $f \geq 0$ then $G^\Omega f$ is superharmonic in Ω.

Lemma 10.4. *We have*

$$(V) + (TJ) + (PI) \Rightarrow (E_{\geq}).$$

Proof. By Lemmas 5.2 and 10.2, we have ($E_{\leq}$). Let B be a ball of radius $r \in (0, \overline{R})$. We need to prove that

$$\operatorname*{ess\,inf}_B G^B 1 \geq cr^\beta. \tag{10.1}$$

Assume first that $r \in \left(0, \sigma \overline{R}\right)$. By ($E_{\leq}$) we have $G^B 1 \in L^\infty$. Hence, by Lemma 10.3, the function $u := G^B 1$ is superharmonic in B. Since $u \geq 0$ in M, we obtain by Lemma 6.6 that

$$\operatorname*{ess\,inf}_B u \geq c \left(\fint_B \frac{1}{u} d\mu \right)^{-1}. \tag{10.2}$$

Recall that the function $\phi := \mathbf{1}_B$ belongs to $\mathcal{F}(B)$ (cf. Lemma 4.1). For any $\lambda > 0$, set $u_\lambda = u + \lambda$. Since $\frac{\phi^2}{u_\lambda} \in \mathcal{F}(B)$ (cf. [19, Propostion 6.5(iii)]), we obtain by Lemma 10.3

$$\int_B \frac{1}{u_\lambda} d\mu = (\mathbf{1}_B, \frac{\phi^2}{u_\lambda}) = \mathcal{E}(G^B 1, \frac{\phi^2}{u_\lambda}) = \mathcal{E}(u, \frac{\phi^2}{u_\lambda}).$$

Since $u_\lambda \geq 0$ on M, we obtain by Lemma 7.1 and (4.2)

$$\mathcal{E}(u, \frac{\phi^2}{u_\lambda}) \leq 3\mathcal{E}(\phi, \phi) \leq C \frac{\mu(B)}{r^\beta}.$$

Combining the two previous lines, dividing by $\mu(B)$ and letting $\lambda \to 0$, we obtain

$$\fint_B \frac{1}{u} d\mu \leq C r^{-\beta},$$

which together with (10.2) proves (10.1).

Now let $r \in [\sigma \overline{R}, \overline{R})$ (in the case $\overline{R} < \infty$). For any ball $B' \subset B$ of radius $r' = \frac{1}{2}\sigma \overline{R}$, we have by the first part of the proof that

$$G^B 1 \geq G^{B'} 1 \geq c \left(r'\right)^\beta \geq c' r^\beta \quad \text{a.e. in } B'.$$

Since B can be covered by a countable family of balls of radii r', we obtain (10.1), which was to be proved. □

Definition 10.5. We say that a *survival condition* (S) is satisfied if there exist constants $\varepsilon, \delta > 0$ such that, for any ball $B \subset M$ of radius $r \in (0, \overline{R})$, the following inequality holds:

$$\operatorname*{ess\,inf}_B P_t^B 1 \geq \varepsilon, \tag{S}$$

provided $t^{1/\beta} \leq \delta r$.

Lemma 10.6. *We have* (E) $\Rightarrow$ (S).

Proof. Let us first prove that, for any ball B with $\left\|G^B 1\right\|_{L^\infty} < \infty$ and for any $t > 0$,

$$P_t^B 1(x) \geq \frac{G^B 1(x) - t}{\|G^B 1\|_{L^\infty}} \text{ for } \mu\text{-a.a. } x \in B. \tag{10.3}$$

Indeed, we have

$$\begin{aligned} G^B 1 &= \left(\int_0^t + \int_t^\infty \right) P_s^B 1 \, ds \\ &= \int_0^t P_s^B 1 \, ds + \int_0^\infty P_t^B P_\tau^B 1 \, d\tau \\ &\leq t + P_t^B \left(G^B 1\right) \leq t + \left\|G^B 1\right\|_{L^\infty} P_t^B 1, \end{aligned}$$

whence (10.3) follows.

Let now B be a ball of radius $r \in \left(0, \overline{R}\right)$. If $r < \sigma \overline{R}$ then by (E) we have $G^B 1 \simeq r^\beta$ a.e. in B. Substituting into (10.3), we obtain $P_t^B 1 \geq \varepsilon$ a.e. in B for all $t \leq (\delta r)^\beta$ with small enough $\varepsilon, \delta > 0$. If $r \in [\sigma \overline{R}, \overline{R})$ then we obtain (S) by covering of B by smaller balls as in the proof of Lemma 10.4. □

A version of Lemma 10.6 for general metric spaces was proved in [19, Lemma 5.6], but the present proof is simpler.

11. Oscillation inequality for $\mathcal{L}u = f$

We apply the oscillation inequalities for harmonic functions in Section 9 to prove the Hölder continuity of the solutions to the equation $\mathcal{L}u = f$ for $f \in L^2$.

Definition 11.1. For a non-empty open set $\Omega \subset M$ and $f \in L^2(\Omega)$, we say that a function $u \in \mathcal{F}$ solves weakly the equation

$$\mathcal{L}u = f \ \text{ in } \Omega,$$

if, for any $\phi \in \mathcal{F}(\Omega)$,

$$\mathcal{E}(u, \phi) = (f, \phi).$$

We continue using the notation $B_r = B(x_0, r)$ where $x_0 \in M$ is fixed.

Lemma 11.2. *Assume that* (V), (TJ) *and* (PI) *are satisfied. Let* Ω *be any open subset of* M. *Let* $f \in L^2 \cap L^\infty(\Omega)$ *and assume that a function* $u \in \mathcal{F} \cap L^\infty$ *solves the equation* $\mathcal{L}u = f$ *weakly in* Ω. *Then, for any ball* $B_r = B(x_0, r) \subset \Omega$ *of radius* $r \in (0, \sigma\overline{R})$ *and for any* $0 < \rho \leq r$, *we have*

$$\operatorname*{osc}_{B_\rho} u \leq C\left(\frac{\rho}{r}\right)^\gamma \|u\|_{L^\infty(M)} + Cr^\beta \|f\|_{L^\infty(B_r)}, \tag{11.1}$$

where positive constants σ,γ, C *depend only on the constants in the hypotheses.*

Proof. By ($E_\leq$) and $r < \sigma\overline{R}$ we have $\left\|G^B 1\right\|_{L^\infty} < \infty$. By Lemma 10.3, the function $w = G^B f$ belongs to $\mathcal{F}(B)$ and solves the equation $\mathcal{L}w = f$ weakly in B. Clearly, the function $v := u - G^B f$ belongs $\mathcal{F} \cap L^\infty$ and is harmonic in B (cf. [19, Prop. 5.8]). Hence, by Lemma 9.4,

$$\operatorname*{osc}_{B_\rho} v \leq C\left(\frac{\rho}{r}\right)^\gamma \left(r^\beta T_{B_r}(v) + \|v\|_{L^\infty(B_r)}\right). \tag{11.2}$$

Using (TJ) and (4.3), we obtain

$$r^\beta T_{B_r}(v) = r^\beta \operatorname*{ess\,sup}_{x \in B_r} \int_{B_r^c} |v| \, J(x, dy) \leq \ C\|v\|_{L^\infty(M)}. \tag{11.3}$$

Clearly, we have

$$\|v\|_{L^\infty(M)} \leq \|u\|_{L^\infty(M)} + \left\|G^{B_r} f\right\|_{L^\infty(M)} = \|u\|_{L^\infty(M)} + \left\|G^{B_r} f\right\|_{L^\infty(B_r)}.$$

Combining the above three lines, we obtain

$$\operatorname*{osc}_{B_\rho} v \leq C\left(\frac{\rho}{r}\right)^\gamma \left(\|u\|_{L^\infty(M)} + \left\|G^{B_r} f\right\|_{L^\infty(B_r)}\right). \tag{11.4}$$

By ($E_\leq$), we have

$$\left\|G^{B_r} f\right\|_{L^\infty(B_r)} \leq \left\|G^B 1\right\|_{L^\infty} \|f\|_{L^\infty(B_r)} \leq Cr^\beta \|f\|_{L^\infty(B_r)}.$$

Combining this with (11.4), we obtain

$$\begin{aligned}
\operatorname*{osc}_{B_\rho} u &\leq \operatorname*{osc}_{B_\rho} v + \operatorname*{osc}_{B_\rho} G^{B_r} f \\
&\leq C\left(\frac{\rho}{r}\right)^\gamma \left(\|u\|_{L^\infty(M)} + \left\|G^{B_r} f\right\|_{L^\infty(B_r)}\right) + 2\left\|G^{B_r} f\right\|_{L^\infty(B_r)} \\
&\leq C\left(\frac{\rho}{r}\right)^\gamma \|u\|_{L^\infty(M)} + Cr^\beta \|f\|_{L^\infty(B_r)},
\end{aligned}$$

which proves (11.1). □

12. Heat kernel

12.1. **Existence and the Hölder continuity of the heat kernel.** Let the hypotheses (V), (TJ) and (PI) be satisfied. By the results of Section 5, the heat kernel $p_t(x,y)$ exists and satisfies the diagonal upper bound (DUE). The same argument applies to any open set $\Omega\subset M$ so that the heat kernel $p_t^\Omega(x,y)$ exists and satisfies the same upper bound.

For any $f\in L^1\cap L^2(\Omega)$, the function $u_t=P_t^\Omega f$ satisfies for any $t>0$ the equation $\mathcal{L}u_t=-\partial_t u$ in Ω understood weakly as above. Lemma 5.5 and the duality argument yield an upper bound of $\|u_t\|_{L^\infty}$ via $\|f\|_{L^1(\Omega)}$, which implies by the standard argument of the spectral theory also an upper bound for $\|\partial_t u\|_{L^\infty}$ (see [19, Lemma 5.10]). Combining with Lemma 11.2, one obtains an oscillation inequality for u_t, which implies the same for the heat kernel. In particular, one obtains the Hölder continuity of the heat kernel. This argument has been carried out in details in [19, Lemmas 5.10, 5.11, 5.12, 5.13] and results in the following statement.

Lemma 12.1. *Assume that (V), (TJ) and (PI) are satisfied.*

For any non-empty open set $\Omega\subset M$, there exists the heat kernel $p_t^\Omega(x,y)$ that is jointly continuous in $(x,y,t)\in\Omega\times\Omega\times(0,\infty)$ and locally Hölder continuous in (x,y). Besides, it satisfies the following upper bound for all $x,y\in\Omega$ and $t\in(0,\overline{R}^\beta)$

$$p_t^\Omega(x,y)\le\frac{C}{t^{\alpha/\beta}}. \tag{12.1}$$

In the case $\Omega=M$, the heat kernel $p_t(x,y)$ satisfies the following estimate, for all $x,y,y'\in M$ and $t\in(0,\overline{R}^\beta)$

$$\left|p_t(x,y)-p_t(x,y')\right|\le\frac{C}{t^{\alpha/\beta}}\left(\frac{d(y,y')}{t^{1/\beta}}\right)^\theta, \tag{12.2}$$

where $\theta=\frac{\beta\gamma}{\beta+\gamma}$, γ is the constant from Lemma 11.2, and C depends only on the constants in the hypotheses.

12.2. **Near-diagonal lower estimate.**

Lemma 12.2. *We have $(V)+(TJ)+(PI)\Rightarrow(nLE)$.*

Proof. By Lemma 12.1, there is a continuous heat kernel $p_t(x,y)$. Fix $t<(\delta\overline{R})^\beta$ and set $r=\delta^{-1}t^{1/\beta}$ where δ is the constant from (S). Using the semigroup identity and (S), we obtain, for any $x\in M$,

$$\begin{aligned}p_{2t}(x,x)&=\int_M p_t(x,y)^2\,d\mu(y)\ge\int_B p_t(x,y)^2\,d\mu(y)\\&\ge\frac{1}{\mu(B)}\left(\int_B p_t(x,y)\,d\mu(x)\right)^2\ge\frac{\left(P_t^B1(x)\right)^2}{\mu(B)}\\&\ge\frac{\varepsilon^2}{\mu(B)}\simeq r^{-\alpha}\simeq t^{-\alpha/\beta}.\end{aligned}$$

It follows that

$$p_t(x,x)\ge ct^{-\alpha/\beta}, \tag{12.3}$$

for all $x\in M$ and for all $t<\delta'\overline{R}^\beta$, for some $c,\delta'>0$.

By (12.2), we have, for all $t<\overline{R}^\beta$ and $x,y\in M$,

$$|p_t(x,x)-p_t(x,y)|\le Ct^{-\alpha/\beta}\left(\frac{d(x,y)}{t^{1/\beta}}\right)^\theta.$$

In particular, if $d(x,y)\le\delta t^{1/\beta}$ with small enough δ, then

$$|p_t(x,x)-p_t(x,y)|\le\frac{c}{2}t^{-\alpha/\beta},$$

whence

$$p_t(x,y) \geq c' t^{-\alpha/\beta} \tag{12.4}$$

where $c' = c/2$. Hence, we have proved this inequality provided $t < \delta' \overline{R}^{\beta}$ and $d(x,y) \leq \delta t^{1/\beta}$.

In order to complete the proof of (nLE), we need to extend (12.4) to all $t < \overline{R}^{\beta}$ that is, to achieve $\delta' = 1$. We have, for any $t < \delta' \overline{R}^{\beta}$,

$$p_{2t}(x,x) \geq \int_{B\left(x,\delta t^{1/\beta}\right)} p_t^2(x,y)\,d\mu(y) \geq \left(c' t^{-\alpha/\beta}\right)^2 c\left(\delta t^{1/\beta}\right)^{\alpha} = c''(2t)^{-\alpha/\beta}.$$

Hence, renaming $2t$ to t, we see that (12.3) holds for all $t < 2\delta' \overline{R}^{\beta}$. Setting $\delta'' = \min(1, 2\delta')$ and repeating the above argument with (12.2), we obtain that (12.4) holds for all $t < \delta'' \overline{R}^{\beta}$. Iterating this argument, we obtain (12.4) for all $t < \overline{R}^{\beta}$. □

A similar argument was carried out in [19, Sect. 5.6] in the setting of general metric spaces.

12.3. **Weak upper estimate.** In this section we prove the implication

$$(V) + (TJ) + (PI) \Rightarrow (wUE) \tag{12.5}$$

of Theorem 2.8. Let us fix an arbitrary $\rho \in (0,\infty)$ and define the following truncated bilinear form:

$$\mathcal{E}^{(\rho)}(f,g) := \iint_{M\times M\setminus \mathrm{diag}} (f(x)-f(y))(g(x)-g(y))dj^{(\rho)}(x,y),$$

where

$$dj^{(\rho)}(x,y) := \mathbf{1}_{\{d(x,y)\leq\rho\}} dj(x,y).$$

As always, we assume that $dj(x,y) = J(x,dy)\,d\mu(x)$ where J satisfies ($j.1$) and ($j.2$). Using the same argument as in [23, Propositions 4.1 and 4.2], we conclude that the form $(\mathcal{E}^{(\rho)},\mathcal{F})$ is closable and its closure $(\mathcal{E}^{(\rho)},\mathcal{F}^{(\rho)})$ is a regular Dirichlet form on $L^2(M)$. Moreover, $\mathcal{F}\cap C_0(M)$ is the core of the Dirichlet form $(\mathcal{E}^{(\rho)},\mathcal{F}^{(\rho)})$.

By [23, Proposition 4.6], we have the following estimate: for any nonnegative $f \in L^\infty(M)$ and for all $t > 0$,

$$\|P_t f - Q_t f\|_{L^\infty} \leq 2t\|f\|_{L^\infty} \sup_{x\in M} J(x, B(x,\rho)^c). \tag{12.6}$$

Lemma 12.3. *For any ball $B = B(x,\rho)$ of radius ρ and any $t > 0$, we have*

$$Q_t \mathbf{1}_B = \mathbf{1}_B \tag{12.7}$$

and

$$Q_t \mathbf{1}_{B^c} = \mathbf{1}_{B^c}.$$

Proof. Consider the killed heat semigroup Q_t^B. It is generated by the Dirichlet form $(\mathcal{E}^{(\rho)},\mathcal{F}^{(\rho)}(B))$. In particular, for $f \in \mathcal{F}\cap C_0(B)$ we have

$$\begin{aligned}\mathcal{E}^{(\rho)}(f,f) &= \int_{M\times M} (f(x)-f(y))^2\,dj^{(\rho)}(x,y)\\ &= \int_{B\times B} (f(x)-f(y))^2\,\mathbf{1}_{\{d(x,y)\leq\rho\}} dj(x,y),\end{aligned}$$

where we can replace $M\times M$ by $B\times B$ because of the following observation: if both $x,y \in B^c$ then $f(x) = f(y) = 0$ while if one of x,y is in B and the other is in B^c then $d(x,y) > \rho$ and, hence, $\mathbf{1}_{\{d(x,y)\leq\rho\}} = 0$. Hence, the Dirichlet form $(\mathcal{E}^{(\rho)},\mathcal{F}^{(\rho)}(B))$ has no killing part. Since B is compact, we conclude that $(\mathcal{E}^{(\rho)},\mathcal{F}^{(\rho)}(B))$ is conservative, that is, for all $t > 0$,

$$Q_t^B \mathbf{1}_B = \mathbf{1}_B.$$

It follows that

$$Q_t \mathbf{1}_B \geq \mathbf{1}_B \tag{12.8}$$

whence

$$(Q_t\mathbf{1}_B,\mathbf{1}) \geq (\mathbf{1}_B,\mathbf{1}) = \mu(B). \tag{12.9}$$

Since also

$$(Q_t\mathbf{1}_B,\mathbf{1}) = (\mathbf{1}_B,Q_t\mathbf{1}) \leq (\mathbf{1}_B,\mathbf{1}) = \mu(B),$$

we see that, in fact, equality is attained in (12.9) and, hence, in (12.8), which proves (12.7).

Since M is a disjoint union of all distinct balls of radius ρ, it follows from (12.7) that $Q_t\mathbf{1} = \mathbf{1}$. Finally, we have

$$Q_t\mathbf{1}_{B^c} = Q_t\mathbf{1} - Q_t\mathbf{1}_B = \mathbf{1} - \mathbf{1}_B = \mathbf{1}_{B^c}.$$

□

Corollary 12.4. *If* (TJ) *is satisfied then, for any ball B of radius $r \in (0,\overline{R})$ and for any $t>0$,*

$$P_t\mathbf{1}_{B^c} \leq C\frac{t}{r^\beta} \text{ in } B. \tag{12.10}$$

Proof. Choose $\rho = r$. Setting $f = \mathbf{1}_{B^c}$ in (12.6), using that by Lemma 12.3 $Q_t\mathbf{1}_{B^c} = 0$ in B, and applying (TJ), we obtain

$$P_t\mathbf{1}_{B^c} \leq 2t\sup_{x\in M} J(x,B(x,r)^c) \leq C\frac{t}{r^\beta} \quad \text{in } B,$$

which was to be proved. □

Remark 12.5. In fact, we have (12.10)⇔(TJ), since (TJ) follows from (12.10) by dividing by t and letting $t\to 0$.

Now we can prove the implication (12.5).

Lemma 12.6. (V) + (TJ) + (PI) ⇒ (wUE).

Proof. Under the conditions (V), (TJ) and (PI), by Lemma 12.1, the heat kernel $p_t(x,y)$ exists, is continuous and satisfies the estimate

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}}, \tag{12.11}$$

for all $x,y\in M$ and $t\in(0,\overline{R}^\beta)$. It suffices to prove that the heat kernel satisfies also

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}}\frac{t}{\left(d(x,y)\wedge\overline{R}\right)^\beta}, \tag{12.12}$$

which together with (12.11) will imply (wUE).

By Corollary 12.4, we have (12.10). Since $P_t\mathbf{1}_{B(x,r)^c}$ is monotone decreasing in r, it follows from (12.10) that, for all $r>0$ and $t>0$,

$$P_t\mathbf{1}_{B(x,r)^c} \leq \frac{Ct}{\left(r\wedge\overline{R}\right)^\beta} \quad \text{in } B(x,r). \tag{12.13}$$

Now fix $t\in(0,\overline{R}^\beta)$, $x,y\in M$ and set $r := \frac{1}{2}d(x,y)$. It suffices to prove (12.12) in the case $r > t^{1/\beta}$. By semigroup property of $p_t(x,y)$, (12.11) and (12.13), we have

$$\begin{aligned}
p_t(x,y) &= \int_M p_{t/2}(x,z)p_{t/2}(z,y)d\mu(z)\\
&\leq \left(\int_{B(x,r)^c} + \int_{B(y,r)^c}\right) p_{t/2}(x,z)p_{t/2}(z,y)d\mu(z)\\
&\leq \sup_{y,z\in M} p_{t/2}(z,y)\cdot P_{t/2}\mathbf{1}_{B(x,r)^c}(x) + \sup_{x,z\in M} p_{t/2}(x,z)\cdot P_{t/2}\mathbf{1}_{B(y,r)^c}(y)\\
&\leq \frac{C}{t^{\alpha/\beta}}\cdot\frac{t}{\left(r\wedge\overline{R}\right)^\beta},
\end{aligned}$$

which finishes the proof. $\square$

13. Derivation of (PI) from heat kernel estimates

Definition 13.1. We say that the *localized lower estimate* (LLE) is satisfied if there exist $c > 0$ and $\varepsilon \in (0,1)$ such that, for any ball B of radius $r \in (0, \overline{R})$, the heat kernel $p_t^B(x,y)$ exists and satisfies for any $t^{1/\beta} \le \varepsilon r$ the following inequality:

$$p_t^B(x,y) \ge ct^{-\alpha/\beta} \text{ for } \mu\text{-a.a. } x, y \in B \text{ such that } d(x,y) \le \varepsilon t^{1/\beta}. \tag{LLE}$$

Lemma 13.2. *We have* $(wUE) + (nLE) \Rightarrow (LLE)$.

Proof. Since (wUE) implies (DUE) and, hence, $(Nash)$, applying $(Nash)$ in any open set $\Omega \subset M$, we obtain by Lemma 5.5 that p_t^Ω exists for any Ω.

Let $\varepsilon \in (0,1)$ be a small enough number to be determined later. By [22, Thm. 5.1] with $\rho = \infty$, we have, for any $t > 0$ and for μ-a.a. $x, y \in B$,

$$p_t(x,y) \le p_t^B(x,y) + \sup_{t/2<s\le t} \operatorname*{ess\,sup}_{z\in K^c} p_s(x,z) + \sup_{t/2<s\le t} \operatorname*{ess\,sup}_{z\in K^c} p_s(y,z), \tag{13.1}$$

where K is any compact subset of B. Since B is compact, we can take here $K = B$. Note that, for any $t^{1/\beta} \le \varepsilon r$ and for all $x \in B$, $z \in B^c$, we have

$$d(x,z) \wedge \overline{R} \ge r \wedge \overline{R} = r \ge \varepsilon^{-1} t^{1/\beta}.$$

By (wUE), we have, for any $s \in [t/2, t]$ and μ-a.a. $x \in B$, $z \in B^c$ that

$$\begin{aligned} p_s(x,z) &\le \frac{C}{s^{\alpha/\beta}}\left(1 + \frac{d(x,z)\wedge \overline{R}}{s^{1/\beta}}\right)^{-\beta} \\ &\le \frac{C}{(t/2)^{\alpha/\beta}}\left(1 + \frac{\varepsilon^{-1}t^{1/\beta}}{t^{1/\beta}}\right)^{-\beta} \\ &\le C'\varepsilon^\beta t^{-\alpha/\beta}. \end{aligned}$$

Estimating in the same way $p_s(y,z)$ in (13.1), we obtain, for any $t^{1/\beta} \le \varepsilon r$ and μ-a.a. $x, y \in B$ that

$$p_t^B(x,y) \ge p_t(x,y) - C\varepsilon^\beta t^{-\alpha/\beta}. \tag{13.2}$$

Let δ be the constant from (nLE). Assuming that $\varepsilon \le \delta$ and applying (nLE), we obtain that, for all $t^{1/\beta} \le \varepsilon r$ and for μ-a.a. $x, y \in B$ such that $d(x,y) \le \varepsilon t^{1/\beta}$,

$$p_t(x,y) \ge ct^{-\alpha/\beta}.$$

Substituting into (13.2), we obtain

$$p_t^B(x,y) \ge ct^{-\alpha/\beta} - C\varepsilon^\beta t^{-\alpha/\beta} = (c - C\varepsilon^\beta)t^{-\alpha/\beta}.$$

Finally, choosing ε small enough, we obtain (LLE). $\square$

Proposition 13.3. *Let B be a ball such that*

$$\operatorname*{ess\,sup}_{x\in B} J(x, B^c) < \infty. \tag{13.3}$$

Set

$$\begin{cases} \overline{\mathcal{E}}(u,v) = \iint_{B\times B\setminus \mathrm{diag}} (u(x)-u(y))(v(x)-v(y))dj(x,y), \\ \overline{\mathcal{F}} = \mathcal{F}(B). \end{cases}$$

Then, for all $u \in \overline{\mathcal{F}}$,

$$\overline{\mathcal{E}}_1(u,u) := \overline{\mathcal{E}}(u,v) + (u,v)_{L^2} \simeq \mathcal{E}_1(u,u). \tag{13.4}$$

Consequently, $(\overline{\mathcal{E}}, \overline{\mathcal{F}})$ is a regular Dirichlet form in $L^2(B)$.

Proof. Clearly, if J satisfies ($j.1$) and ($j.2$), then any kernel of the form

$$\varphi(x,y)\,J(x,dy)$$

with a bounded non-negative symmetric Borel function φ also satisfies ($j.1$) and ($j.2$). In particular, this is the case for $\varphi(x,y)=\mathbf{1}_{B\times B}$. Consequently, $\overline{\mathcal{E}}$ is well-defined on $\mathcal{F}_{\max}$ (cf. (2.2)).

It is easy to see that the form $(\overline{\mathcal{E}},\overline{\mathcal{F}})$ is bilinear, symmetric, non-negative and Markovian. By definition, $\overline{\mathcal{F}}$ is dense in $L^2(B)$. Hence, the fact that $(\overline{\mathcal{E}},\overline{\mathcal{F}})$ is closed and regular will follow from (13.4).

For all $u,v\in\overline{\mathcal{F}}$, we compute $\mathcal{E}(u,v)-\overline{\mathcal{E}}(u,v)$ by splitting the domain of integration $M\times M$ into the union of $B\times B$, $B\times B^c$, $B^c\times B$, $B^c\times B^c$ and using symmetry of the jump measure, as follows:

$$\begin{aligned}\mathcal{E}(u,v)-\overline{\mathcal{E}}(u,v)&=\left(\int_{B\times B^c}+\int_{B^c\times B}+\int_{B^c\times B^c}\right)(u(x)-u(y))(v(x)-v(y))J(x,dy)d\mu(x)\\&=2\int_{x\in B}\int_{y\in B^c}u(x)v(x)J(x,dy)d\mu(x)\\&=2\int_B u(x)v(x)J(x,B^c)d\mu(x)\\&\le C\,\|u\|_{L^2}\,\|v\|_{L^2}\,,\end{aligned}\tag{13.5}$$

where $C=2\operatorname{ess\,sup}_{x\in B}J(x,B^c)<\infty$. It follows that, for any $u\in\overline{\mathcal{F}}$,

$$\overline{\mathcal{E}}(u,u)\le\mathcal{E}(u,u)\le\overline{\mathcal{E}}(u,u)+C\,\|u\|_{L^2}^2\tag{13.6}$$

and, hence,

$$\overline{\mathcal{E}}_1(u,u)\le\mathcal{E}_1(u,u)\le(C+1)\,\overline{\mathcal{E}}_1(u,u)\,,$$

whence (13.4) follows. □

Lemma 13.4. (Comparison principle) *Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form, $\lambda>0$ and $u\in\mathcal{F}$. Suppose that*

$$\mathcal{E}_\lambda(u,v):=\mathcal{E}(u,v)+(u,v)_{L^2}\ge 0\quad\textit{for all } 0\le v\in\mathcal{F}.$$

Then $u\ge 0$.

Proof. Indeed, take $v=G_\lambda f$ with $0\le f\in L^2$. Then we have $0\le v\in\mathcal{F}$ and then

$$(u,f)=\mathcal{E}_\lambda(u,G_\lambda f)=\mathcal{E}_\lambda(u,v)\ge 0.$$

Since f is arbitrary, we can conclude that $u\ge 0$. □

Proposition 13.5. *Let $B\subset M$ be a ball. Let $(\overline{\mathcal{E}},\overline{\mathcal{F}})$ be the regular Dirichlet form from Proposition 13.3 and let $\{\overline{P}_t\}_{t\ge 0}$ be its heat semigroup. Then, for all non-negative functions $f\in L^2(B)$ and for all $t>0$, we have*

$$\overline{P}_t f\ge P_t^B f\quad\textit{in } B.$$

Proof. We denote the resolvents of $(\mathcal{E},\mathcal{F}(B))$ and $(\overline{\mathcal{E}},\overline{\mathcal{F}})$ by G_λ^B and $\overline{G}_\lambda$, respectively. Fix $0\le f\in L^2(B)$ and $\lambda>0$. Since $\overline{\mathcal{F}}=\mathcal{F}(B)$, both functions $u=G_\lambda^B f$ and $\overline{u}=\overline{G}_\lambda f$ belong to $\overline{\mathcal{F}}$. For any $v\in\overline{\mathcal{F}}$, we have

$$\mathcal{E}_\lambda(u,v)=(f,v)_{L^2(B)}=\overline{\mathcal{E}}_\lambda(\overline{u},v),$$

which together with (13.5) yields

$$\begin{aligned}\overline{\mathcal{E}}_\lambda(\overline{u}-u,v)&=\overline{\mathcal{E}}_\lambda(\overline{u},v)-\overline{\mathcal{E}}_\lambda(u,v)\\&=\mathcal{E}_\lambda(u,v)-\overline{\mathcal{E}}_\lambda(u,v)\\&=\mathcal{E}(u,v)-\overline{\mathcal{E}}(u,v)\\&=2\int_B u(x)v(x)J(x,B^c)d\mu(x)\ge 0.\end{aligned}$$

Hence, by Lemma 13.4, we conclude that $\overline{u}-u\geq 0$ in B, that is,

$$\overline{G}_\lambda f \geq G_\lambda^B f \text{ in } B.$$

Since this is true for all $\lambda>0$, applying the formula [14, (1.3.5)], we obtain

$$\overline{P}_t f=\lim_{\lambda\to\infty} e^{-t\lambda}\sum_{n=0}^{\infty}\frac{(t\lambda)^n}{n!}(\lambda\overline{G}_\lambda)^n f\geq \lim_{\lambda\to\infty} e^{-t\lambda}\sum_{n=0}^{\infty}\frac{(t\lambda)^n}{n!}(\lambda G_\lambda^B)^n f=P_t^B f.$$

□

Lemma 13.6. *We have (V) + (wUE) + (nLE) ⇒ (PI).*

Proof. Let $B=B(x_0,r)\subset M$ be a ball of radius $r\in(0,\overline{R})$. Consider the heat semigroup $\{\overline{P}_t\}$ in B defined as above. Let us show that it has a heat kernel $\overline{p}_t(x,y)$. Indeed, by (wUE), we have, for all $t<\overline{R}^\beta$ and μ-a.a. $x,y\in M$,

$$p_t(x,y)\leq Ct^{-\alpha/\beta}.$$

By [4, Thm. 2.1], we have the Nash inequality

$$\|f\|_{L^2}^{2(1+\nu)}\leq C\left(\mathcal{E}(f,f)+\overline{R}^{-\beta}\|f\|_{L^2}^2\right)\|f\|_{L^1}^{2\nu}$$

for all $f\in\mathcal{F}\cap L^1$, where $\nu=\beta/\alpha$. By (13.6), we obtain, for all $f\in\overline{\mathcal{F}}\cap L^1$,

$$\|f\|_{L^2}^{2(1+\nu)}\leq C\left(\overline{\mathcal{E}}(f,f)+\left(1+\overline{R}^{-\beta}\right)\|f\|_{L^2}^2\right)\|f\|_{L^1}^{2\nu}.$$

Applying again [4, Thm. 2.1] and [20, Cor. 3.8], we conclude that the heat semigroup $\{\overline{P}_t\}$ has a heat kernel that we denote by $\overline{p}_t(x,y)$. It follows from Proposition 13.5 that, for all $t>0$ and μ-a.a. $x,y\in B$,

$$\overline{p}_t(x,y)\geq p_t^B(x,y).$$

By Lemma 13.2, we have the lower bound (*LLE*) for p_t^B, which implies the same lower bound for $\overline{p}_t(x,y)$, that is,

$$\overline{p}_t(x,y)\geq ct^{-\alpha/\beta}, \tag{13.7}$$

for all $t^{1/\beta}\leq\varepsilon r$ and μ-a.a. $x,y\in B$ such that $d(x,y)\leq\varepsilon t^{1/\beta}$. Choose here

$$t=(\varepsilon r)^\beta.$$

It follows from (13.7) that, for this t,

$$\overline{p}_t(x,y)\geq cr^{-\alpha}, \tag{13.8}$$

for μ-a.a. $x,y\in B\left(x_0,\varepsilon^2 r\right)$.

For any $f\in\overline{\mathcal{F}}$, we have by [14, eq. (1.3.17)]

$$\overline{\mathcal{E}}(f,f)\geq\frac{1}{t}\left(f-\overline{P}_t f,f\right)\geq\frac{1}{2t}\int_B\int_B\overline{p}_t(x,y)\left(f(x)-f(y)\right)^2d\mu(x)\,d\mu(y),$$

which together with (13.8) implies that

$$\overline{\mathcal{E}}(f,f)\geq\frac{c}{r^{\alpha+\beta}}\int_{B(x_0,\varepsilon^2 r)}\int_{B(x_0,\varepsilon^2 r)}(f(x)-f(y))^2d\mu(x)\,d\mu(y).$$

Using the definition of $\overline{\mathcal{E}}$, we rewrite this inequality in the form

$$\int_{B(x_0,\varepsilon^2 r)}\int_{B(x_0,\varepsilon^2 r)}(f(x)-f(y))^2d\mu(x)d\mu(y)\leq Cr^{\alpha+\beta}\int_B\int_B(f(x)-f(y))^2dj(x,y), \tag{13.9}$$

where f is any function from $\overline{\mathcal{F}}=\mathcal{F}(B)$. Let now f be any function from $\mathcal{F}\cap L^\infty(M)$. Applying (13.9) to the function $f\mathbf{1}_B\in\mathcal{F}(B)$ (cf. Lemma 4.1), we conclude that (13.9) holds also for all $f\in\mathcal{F}\cap L^\infty(M)$. By a standard approximation argument, it follows that (13.9) holds also for all $f\in\mathcal{F}$, which proves (*PI*) with $\kappa=\varepsilon^2$. □

14. Completion of proof of the main results

Proof of Theorem 2.8. Assume that (V), (TJ) and (PI) are satisfied. By Lemma 12.1, $(\mathcal{E},\mathcal{F})$ has a Hölder continuous heat kernel $p_t(x,y)$. The heat kernel satisfies the upper bound (wUE) by Lemma 12.6, and the lower bound (nLE) by Lemma 12.2.

Conversely, if (V), (wUE) and (nLE) are satisfied, then we obtain (PI) by Lemma 13.6, which completes the proof. □

Proof of Corollary 2.13. By Lemmas 5.2, 10.2, 10.4, 10.6, we obtain condition (S). By [18, Lemma 4.6], conditions (V) and (S) imply that $(\mathcal{E},\mathcal{F})$ is conservative.

(a) Note that (wUE) implies (DUE). By Theorem 2.8, it suffices to prove the following implications:

$$(V_{\geq}) + (DUE) + (J_{\leq}) + (S) \Rightarrow (UE), \tag{14.1}$$

and

$$(UE) \Rightarrow (J_{\leq}),$$

which follow from [23, Theorem 2.1] and the first implication of [18, Lemma 4.9], respectively. Although the implication (14.1) was derived under the assumption that $\overline{R} = \infty$, its proof also works for $\overline{R} < \infty$ (see also [18, Sect. 4.4, proof of Thm. 2.9]).

(b) By Theorem 2.8, it suffices to prove the following implications:

$$(nLE) + (V_{\geq}) + (J_{\geq}) + (S) \Rightarrow (LE),$$

and

$$(LE) \Rightarrow (J_{\geq}),$$

which follow from [18, Theorem 2.8] and the first implication of [18, Lemma 4.9], respectively. □

15. Optimality of heat kernel bounds under (TJ) and (PI)

In this section we give examples to show that, under the conditions (V), (PI) and (TJ), the heat kernel estimates (wUE) and (nLE) are sharp in certain sense.

Fix a positive integer n, a sequence of positive reals $\{\alpha_i\}_{i=1}^n$ and a positive real β. Let (M_i, d_i, μ_i), $1 \leq i \leq n$, be n ultrametric spaces satisfying the conditions

$$\mu(B_i(x,r)) \simeq r^{\alpha_i}, \quad \text{for all } x \in M_i,\ r > 0.$$

For example, M_i can be taken to be $\mathbb{Q}_p$ with the distance $d_i(x,y) = \|x-y\|_p^{1/\alpha_i}$ and the Haar measure μ_i.

For each $1 \leq i \leq n$, consider on M_i the jump kernel

$$J_i(x,y) = \frac{1}{d_i(x,y)^{\alpha_i+\beta}}.$$

Since this jump kernel satisfies (TJ) (cf. (3.13)), it determines by Corollary 4.2 a regular Dirichlet form $(\mathcal{E}_i,\mathcal{F}_i)$ with the jump kernel J_i. By Theorem 2.8, the heat kernel $p_t^{(i)}(x,y)$ of $(\mathcal{E}_i,\mathcal{F}_i)$ satisfies (UE) and (LE):

$$p_t^{(i)}(x,y) \simeq \frac{1}{t^{\alpha_i/\beta}}\left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha_i+\beta)}.$$

Now let us consider the product space $M := M_1 \times M_2 \times ... \times M_n$ equipped with the metric d and the product measure μ as follows:

$$d(x,y) := \max_{1\leq i\leq n}\{d_i(x_i,y_i)\}, \qquad \mu := \mu_1 \times \mu_2 \times \cdots \times \mu_n,$$

where $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$. Clearly, (M,d) is an ultrametric space and μ satisfies

$$\mu(x,r) \simeq r^{\alpha}, \quad \text{for all } x \in M,\ r > 0,$$

where

$$\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n. \tag{15.1}$$

Consider the jump measure $J(x, dy)$ on M that is defined on non-negative continuous functions f on M by

$$\int_M f(y)\, J(x, dy) = \sum_{i=1}^{n} \int_{M_i} f(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)\, J_i(x_i, y_i)\, d\mu_i(y_i). \tag{15.2}$$

Let us verify that the jump measure J satisfies (TJ). Indeed, since each J_i satisfies (TJ), we obtain using (15.2) and

$$B(x, r) = B_1(x_1, r) \times B_2(x_2, r) \times \cdots \times B_n(x_n, r), \tag{15.3}$$

that

$$\begin{aligned} J(x, B(x,r)^c) &= \int_M \mathbf{1}_{B(x,r)^c}(y)\, J(x, dy) \\ &= \sum_{i=1}^{n} \int_{M_i} \mathbf{1}_{B(x,r)^c}(x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)\, J_i(x_i, y_i)\, d\mu_i(y_i) \\ &= \sum_{i=1}^{n} \int_{M_i} \mathbf{1}_{B_i(x_i,r)^c}(y_i) J_i(x_i, y_i)\, d\mu_i(y_i) \\ &= \sum_{i=1}^{n} \int_{B_i(x_i,r)^c} J_i(x_i, y_i)\, d\mu_i(y_i) \le \sum_{i=1}^{n} \frac{C_i}{r^\beta} = \frac{C}{r^\beta}. \end{aligned}$$

Hence, by Theorem 2.2, the jump measure J determines a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ with the jump measure J. Clearly, we have

$$\begin{aligned} \mathcal{E}(f, f) := \sum_{i=1}^{n} \int_M \int_{M_i} &(f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) - f(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n))^2 \\ &\times J_i(x_i, y_i)\, d\mu_i(y_i) d\mu(x), \end{aligned}$$

for any $f \in L^2(M, \mu)$. The generator of $(\mathcal{E}, \mathcal{F})$ is given by

$$\begin{aligned} \mathcal{L}f(x) &= \int_M (f(x) - f(y))\, J(x, dy) \\ &= \sum_{i=1}^{n} \int_{M_i} (f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) - f(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)) \\ &\qquad \times J_i(x_i, y_i)\, d\mu_i(y_i) \\ &= \sum_{i=1}^{n} \mathcal{L}_i f(x), \end{aligned}$$

where $\mathcal{L}_i$ is the generator of $(\mathcal{E}_i, \mathcal{F}_i)$ acting on the component x_i of x. It follows that

$$e^{-t\mathcal{L}} = \prod_{i=1}^{n} e^{-t\mathcal{L}_i}$$

and, consequently, the heat kernel $p_t(x, y)$ of $(\mathcal{E}, \mathcal{F})$ satisfies:

$$p_t(x, y) = \prod_{i=1}^{n} p_t^{(i)}(x_i, y_i) \simeq \frac{1}{t^{\alpha/\beta}} \prod_{i=1}^{n} \left(1 + \frac{d_i(x_i, y_i)}{t^{1/\beta}}\right)^{-(\alpha_i+\beta)}. \tag{15.4}$$

By a direct computation, we see the following.

(i) $p_t(x,y)$ satisfies (wUE), since by (15.4)

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}}\left(1+\frac{\max_i d_i(x_i,y_i)}{t^{1/\beta}}\right)^{-\beta} = \frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-\beta}.$$

(ii) $p_t(x,y)$ satisfies (nLE). Indeed, for any $x,y\in M$ and $t>0$ with $d(x,y)\le t^{1/\beta}$, we have also $d_i(x_i,y_i)\le t^{1/\beta}$ and, hence,

$$p_t(x,y) \ge \frac{c}{t^{\alpha/\beta}}\prod_{i=1}^{n}\left(1+\frac{d_i(x_i,y_i)}{t^{1/\beta}}\right)^{-(\alpha_i+\beta)} \ge \frac{c}{t^{\alpha/\beta}}\prod_{i=1}^{n}(1+1)^{-(\alpha_i+\beta)} = \frac{c'}{t^{\alpha/\beta}}.$$

By Theorem 2.8, we conclude that $(\mathcal{E},\mathcal{F})$ satisfies also (PI).

Hence, the ultrametric space M with the reference measure μ and the jump measure J satisfies (V), (PI), (TJ) with parameters α,β and $\overline{R}=\infty$. Note that α and β can take arbitrary positive values. Even if we fix α and β, there are still "hidden" parameters $n,\alpha_1,...,\alpha_n$ with the only constraint (15.1), and they can be varied to obtain desired properties of M.

Let us show that (wUE) is optimal in the sense that the exponent β cannot be replaced by any larger number. More precisely, let us show that, for any $\varepsilon>0$, there is a choice of "hidden" parameters such that, in some range of the variables (t,x,y),

$$p_t(x,y) \ge \frac{c}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\beta+\varepsilon)},$$

while $\frac{d(x,y)}{t^{1/\beta}}$ can take arbitrarily large values. Indeed, we can assume that ε is small enough and set $\alpha_1=\varepsilon$, while $\alpha_2,...,\alpha_n$ should only satisfy (15.1). Set

$$E := \{(t,x,y)\in\mathbb{R}_+\times M\times M : d_i(x_i,y_i)\le t^{1/\beta} < d_1(x_1,x_1),\ 2\le i\le n\}.$$

Then, for any $(t,x,y)\in E$, we have

$$d(x,y) = \max_{1\le i\le n}\{d_i(x_i,y_i)\} = d_1(x_1,y_1),$$

and hence, by (15.4),

$$\begin{aligned} p_t(x,y) &\ge \frac{c}{t^{\alpha/\beta}}\prod_{i=1}^{n}\left(1+\frac{d_i(x_i,y_i)}{t^{1/\beta}}\right)^{-(\alpha_i+\beta)} \\ &\ge \frac{c}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha_1+\beta)}\prod_{i=2}^{n}(1+1)^{-(\alpha_i+\beta)} \\ &= \frac{c'}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\beta+\varepsilon)}, \end{aligned}$$

which was to be proved.

Now let us show that (nLE) is optimal in the following sense: for any large number $N>0$, there is a choice of "hidden" parameters such that, in some range of the variables (t,x,y),

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-N}, \tag{15.5}$$

while $\frac{d(x,y)}{t^{1/\beta}}$ can be arbitrarily large. Indeed, set

$$F := \{(x,y)\in M\times M : \frac{1}{2}d_1(x_1,y_1)\le d_i(x_i,y_i)\le d_1(x_1,y_1),\ 2\le i\le n\}.$$

Then, for all $t>0$ and $(x,y)\in F$,

$$d(x,y) = d_1(x_1,y_1) \text{ and } d_i(x_i,y_i)\ge\frac{d(x,y)}{2},\quad 2\le i\le n.$$

Hence, we obtain by (15.4) that, for all $t>0$ and $(x,y)\in F$,

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \prod_{i=1}^{n} \left(1+\frac{d_i(x_i,y_i)}{t^{1/\beta}}\right)^{-(\alpha_i+\beta)} \le \frac{C}{t^{\alpha/\beta}} \prod_{i=1}^{n} \left(1+\frac{d(x,y)}{2t^{1/\beta}}\right)^{-(\alpha_i+\beta)}$$
$$= \frac{C}{t^{\alpha/\beta}} \left(1+\frac{d(x,y)}{2t^{1/\beta}}\right)^{-(\alpha+n\beta)} \le \frac{C'}{t^{\alpha/\beta}} \left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+n\beta)}.$$

Choosing n large enough such that $\alpha+n\beta>N$ we obtain (15.5).

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