Analysis on manifolds and volume growth

Alexander Grigor'yan

Contents

1. Setup

Let M be a Riemannian manifold that is geodesically complete and non-compact. Let $d(x,y)$ denote the geodesic distance on M and μ be the Riemannian measure. Consider geodesic balls

$$B(x,r)=\{y\in M: d(x,y)<r\},$$

that are necessarily precompact, and their volumes:

$$V(x,r)=\mu(B(x,r)).$$

In this survey we collect some old and new results relating the rate growth of $V(x,r)$ as $r\to\infty$ to the properties of elliptic and parabolic PDEs on M.

Recall that the *Laplace-Beltrami operator* Δ on M is given in the local coordinates $x_1,...,x_n$ as follows:

$$\Delta=\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x_i}\left(\sqrt{\det g}g^{ij}\frac{\partial}{\partial x_j}\right)$$

where $g=(g_{ij})$ is the Riemannian metric tensor and $(g^{ij})=(g_{ij})^{-1}$. Equivalently, we have $\Delta=\operatorname{div}\circ\nabla$ where ∇ is the Riemannian gradient and div – the corresponding divergence.

Supported by SFB1283 of the German Research Foundation.

The *heat kernel* $p_t(x,y)$ of M is the minimal positive fundamental solution of the heat equation

$$\frac{\partial}{\partial t}u = \Delta u$$

on $M \times \mathbb{R}_+$. It is known that the heat kernel exists on any manifold and is a smooth, positive function of $x, y \in M$ and $t > 0$ ([18]). For example, in $\mathbb{R}^n$ we have

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right). \tag{1.1}$$

The heat kernel satisfies the semigroup identity

$$p_{t+s}(x,y) = \int_M p_t(x,z)\, p_s(z,y)\, d\mu(z)$$

and, hence, can be used as a transition density for constructing a diffusion process on M (see [17]). This diffusion process is called *Brownian motion* on M.

If $M = \mathbb{R}^n$ then one obtains in this way the classical Brownian motion in $\mathbb{R}^n$ with the time scaled by the factor 2.

2. Parabolicity and recurrence

A function $u \in C^2(M)$ is called *superharmonic* if $\Delta u \leq 0$. A manifold M is called *parabolic* if any positive superharmonic function on M is constant, and *non-parabolic* otherwise.

For any compact set $K \subset M$ define its capacity by

$$\operatorname{cap}(K) = \inf_{\varphi \in C_0^\infty(M),\ \varphi|_K \equiv 1} \int_M |\nabla\varphi|^2 d\mu.$$

The following theorem gives equivalent characterizations of the parabolicity

Theorem 2.1. *([16, Thm. 5.1]) The following properties are equivalent:*

- *M is parabolic.*
- *Any bounded superharmonic function on M is constant.*
- *There exists no positive fundamental solution of $-\Delta$ on M.*
- *For all/some $x, y \in M$ we have*

$$\int_1^\infty p_t(x,y)\, dt = \infty. \tag{2.1}$$

- *For any compact set $K \subset M$, we have* $\operatorname{cap}(K) = 0$.
- *Brownian motion on M is recurrent.*

The *Green function* of Δ is defined by

$$g(x,y) = \int_0^\infty p_t(x,y)\, dt.$$

The condition (2.1) is equivalent to the fact that $g(x,y) \equiv \infty$. If M is non-parabolic then $g(x,y) < \infty$ for all $x \neq y$ and, moreover, $g(x,y)$ is the minimal positive fundamental solution of $-\Delta$.

A celebrated theorem of Polya (1921) says that Brownian motion in $\mathbb{R}^n$ is recurrent if and only if $n \leq 2$. Indeed, one can see from the explicit formula (1.1) for the heat kernel that the condition (2.1) holds if and only if $n \leq 2$.

Surprisingly enough, there exist rather good sufficient conditions for the recurrence of Brownian motion in terms of the volume function. Let us fix a reference point x_0 and set

$$V(r) = V(x_0, r).$$

THEOREM 2.2. (Cheng-Yau [4]) *If there exists a sequence $r_k \to \infty$ such that, for some $C > 0$ and all k*

$$V(r_k) \le C r_k^2, \tag{2.2}$$

then M is parabolic.

THEOREM 2.3. ([8], [29], [37]) *If*

$$\int^{\infty} \frac{r\,dr}{V(r)} = \infty \tag{2.3}$$

then M is parabolic.

One can show that (2.2) implies (2.3) so that Theorem 2.2 follows from Theorem 2.3.

The condition (2.3) is sharp: if $f(r)$ is a smooth convex function such that $f'(r) > 0$ and

$$\int^{\infty} \frac{r\,dr}{f(r)} < \infty,$$

then there is a non-parabolic manifold such that $V(r) = f(r)$ for large r. On the other hand, the condition (2.3) is not necessary for parabolicity: there exist parabolic manifolds with arbitrarily large volume function $V(r)$ as it follows from [16, Prop. 3.1].

3. Stochastic completeness

A manifold M is called *stochastically complete* if for all $x \in M$ and $t > 0$

$$\int_M p_t(x, y)\, d\mu(y) = 1.$$

Here are some equivalent characterizations of the stochastic completeness.

THEOREM 3.1. ([16, Thm. 6.2]) *The following conditions are equivalent.*

- *M is stochastically complete.*
- *For some/any $\lambda > 0$, any bounded solution u to $\Delta u - \lambda u = 0$ on M is identical zero.*
- *For some/any $T \in (0, \infty]$, the Cauchy problem*

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in } M \times (0, T) \\ u\big|_{t=0} = 0 \end{cases} \tag{3.1}$$

  has the only bounded solution $u \equiv 0$.
- *The lifetime of Brownian motion on M is equal to ∞ almost surely.*

The following theorem provides a volume test for stochastic completeness.

THEOREM 3.2. ([9]) *If*

$$\int^{\infty} \frac{r\,dr}{\log V(r)} = \infty \tag{3.2}$$

then M is stochastically complete.

In particular, M is stochastically complete provided

$$V(r) \le \exp\left(Cr^2\right) \tag{3.3}$$

or even if

$$V(r_k) \le \exp\left(Cr_k^2\right) \tag{3.4}$$

for a sequence $r_k \to \infty$. That (3.3) implies the stochastic completeness was also proved by different methods also in [6], [30], [36].

The condition (3.2) is sharp: if $f(r)$ is a smooth convex function such that $f'(r) > 0$ and

$$\int^{\infty} \frac{rdr}{f(r)} < \infty$$

then there exists a stochastically incomplete manifold with $V(r) = \exp(f(r))$. On there other hand, there are stochastically complete manifolds with arbitrarily large volume function as it follows from [16, Prop. 3.2].

4. Liouville properties

We say that M satisfies L^p-*Liouville property* if any harmonic function $u \in L^p(M, \mu)$ is identically equal to a constant.

THEOREM 4.1. ([38]) *Any complete Riemannian manifold satisfies L^P-Liouville property for any $1 < p < \infty$.*

For $p = 1$ and $p = \infty$ this is not true: there are manifolds with non-trivial L^1 harmonic functions (cf. [10]) as well as those with non-trivial L^∞ harmonic functions: for example, hyperbolic spaces $\mathbb{H}^n$ and connected sums $\mathbb{R}^n \# \mathbb{R}^n$, $n \ge 3$, – see [31].

Usually a volume information is not enough in order to decide whether L^1- or L^∞-Liouville property holds as the latter are sensitive to existence of bottlenecks on the manifold in question. However, analogous properties for *super*harmonic functions can be derived from the volume growth. For L^∞-superharmonic function that was Theorem 2.3, and a similar result for L^1-superharmonic functions is stated in the next theorem.

THEOREM 4.2. ([10]) *Assume that*

$$\int^{\infty} \frac{rdr}{\log V(r)} = \infty. \tag{4.1}$$

Then any positive superharmonic function $u \in L^1(M, \mu)$ is identical zero.

PROOF. Indeed, (4.1) implies by Theorem 3.2 the stochastic completeness of M. Hence, for the Green function $g(x, y)$ we obtain

$$\int_M g(x,y)\, d\mu(y) = \int_M \int_0^\infty p_t(x,y)\, dt\, d\mu(y) = \int_0^\infty \int_M p_t(x,y)\ d\mu(y)\ dt = \int_0^\infty 1\, dt = \infty. \tag{4.2}$$

Fix some $x \in M$. Since the Green function $g(x, \cdot)$ is locally integrable, (4.2) implies that $g(x, \cdot) \notin L^1(U^c)$ where U^c is the exterior of a neighborhood U of x. The the maximum principle implies that any positive superharmonic function u is bounded from below by const $g(x, \cdot)$ in U^c, which implies that $u \notin L^1(M)$. □

Combining this argument with Theorem 3.1 we obtain that the L^∞-Liouville property for the equation $\Delta u - \lambda u = 0$ (where $\lambda > 0$) implies the L^1-Liouville property for superharmonic functions. It may be interesting to investigate further relations between different types of Liouville properties.

However, the main open questions in this area are these.

Open Questions. *Find optimal conditions in geometric terms for*

(*a*) L^1*-Liouville property for harmonic functions;*
(*b*) L^∞*-Liouville property for harmonic functions.*

5. Bounded solutions of Schrödinger equations

Let $Q(x)$ be a nonnegative continuous function on M, $Q \not\equiv 0$. Consider the equation

$$\Delta u - Qu = 0 \tag{5.1}$$

and ask if (5.1) has a non-trivial (≡non-zero) bounded solution, that is, if L^∞-Liouville property holds for (5.1).

In the case $Q = \text{const}$ the L^∞-Liouville property for (5.1) is equivalent to the stochastic completeness of M. If Q is compactly supported then one can show that the L^∞-Liouville property for (5.1) is equivalent to the parabolicity of M.

In general, one can prove that (5.1) has a non-trivial bounded solution if and only if it has a positive solution.

Set $|x| = d(x, x_0)$ and denote

$$q(r) = \inf_{|x|=r} Q(x) \quad \text{and} \quad F(r) = \int_0^{r/2} \sqrt{q(t)}dt.$$

Theorem 5.1. ([**11**]) *If there is a sequence $r_k \to \infty$ such that for some $C > 0$ and all k*

$$V(r_k) \le C r_k^2 \exp\left(CF(r_k)^2\right) \tag{5.2}$$

then (5.1) has no bounded solution except for $u \equiv 0$.

Example 5.2. Let $Q \equiv 1$.Then we have $q \equiv 1$, $F(r) = r/2$, and (5.2) becomes $V(r_k) \le \exp(Cr_k^2)$, which coincides with the condition (3.4) for the stochastic completeness.

Example 5.3. Let Q have a compact support. Since $q(r) = 0$ for large enough r, we obtain that $F(r) = \text{const}$ for large r, and (5.2) becomes $V(r_l) \le Cr_k^2$, which coincides with the sufficient condition (2.2) for the parabolicity.

Example 5.4. Assume that, for all large $|x|$ and some $c > 0$,

$$Q(x) \ge \frac{c}{|x|^2 \log |x|}.$$

Then

$$F(r) \ge \int_2^{r/2} \frac{c}{t\sqrt{\log t}}dt \simeq \sqrt{\log r}$$

so that (5.2) is satisfied provided

$$V(r) \le Cr^N$$

for some $C, N > 0$ and all large r. Hence, in this case (5.1) has no bounded solution except for zero. For example, this is the case for $M = \mathbb{R}^n$.

On the other hand, if in $\mathbb{R}^n$

$$Q(x) \le \frac{C}{|x|^2 \log^{1+\varepsilon} |x|}$$

then (5.1) has a positive bounded solution in $\mathbb{R}^n$.

6. Semilinear PDEs

Consider on M the inequality

$$\Delta u + u^\sigma \leq 0 \tag{6.1}$$

and ask if it has a non-negative solution u on M except for $u \equiv 0$. Here $\sigma > 1$ is a given parameter. Note that any non-negative solution of (6.1) is superharmonic. Hence, if M is parabolic then u must be identical zero. In particular, this is the case if $V(r) \leq Cr^2$.

Otherwise (6.1) may have positive solutions. For example, in $\mathbb{R}^n$ with $n > 2$ the inequality (6.1) has a positive solution if and only if $\sigma > \frac{n}{n-2}$ (cf. [**33**]).

Theorem 6.1. ([**25**]) *Assume that, for all large r,*

$$V(r) \leq Cr^p \log^q r, \tag{6.2}$$

where

$$p = \frac{2\sigma}{\sigma - 1} \quad \text{and} \quad q = \frac{1}{\sigma - 1}. \tag{6.3}$$

Then any nonnegative solution of (6.1) is identical zero.

The values of the exponents p and q in (6.3) are sharp: if either $p > \frac{2\sigma}{\sigma-1}$ or $p = \frac{2\sigma}{\sigma-1}$ and $q > \frac{1}{\sigma-1}$ then there is a manifold satisfying (6.2) where the inequality (6.1) has a positive solution.

Theorem 6.1 can be equivalently reformulated as follows: if, for some $\alpha > 2$

$$V(r) \leq Cr^\alpha \log^{\frac{\alpha-2}{2}} r, \tag{6.4}$$

then, for any $\sigma \leq \frac{\alpha}{\alpha-2}$, any nonnegative solution of (6.1) is identical zero. In this form it contains the aforementioned result for $\mathbb{R}^n$ as in $\mathbb{R}^n$ (6.4) is satisfied with $\alpha = n$.

Conjecture. ([**26**]) *If*

$$\int^\infty \frac{r^{2\sigma-1}dr}{V(r)^{\sigma-1}} = \infty \tag{6.5}$$

then any nonnegative solution of (6.1) is identical zero.

In particular, the function (6.4) satisfies (6.5) with $\sigma = \frac{\alpha}{\alpha-2}$.

Similar results for a more general inequality $\Delta u + Qu^\sigma \leq 0$ with $Q(x) \geq 0$ were obtained in [**34**]. In the view of results of Section 5, it may be interesting to investigate the question of existence of positive solutions for a semilinear equation $\Delta u - Qu^\sigma = 0$.

Analogous problems for semilinear heat equation were addressed in [**35**].

7. Escape rate

Let $\{X_t\}_{t\geq 0}$ be Brownian motion on M. An increasing positive function $R(t)$ of $t \in \mathbb{R}_+$ is called *an upper rate function* for Brownian motion if we have $|X_t| < R(t)$ for all t large enough with probability 1. Similarly, an increasing positive function $r(t)$ is called a *lower rate function* if we have $|X_t| > r(t)$ for all t large enough with probability 1.

Hence, for large enough t, X_t is contained in the annulus $B(x_0, R(t)) \setminus B(x_0, r(t))$ almost surely, as on Fig. 1.

Note that an upper rate function may exist only on stochastically complete manifolds, and a lower rate function may exist only on non-parabolic manifolds.

For example, in $\mathbb{R}^n$ the following function

$$R(t) = \sqrt{(4 + \varepsilon)\, t \log \log t}$$

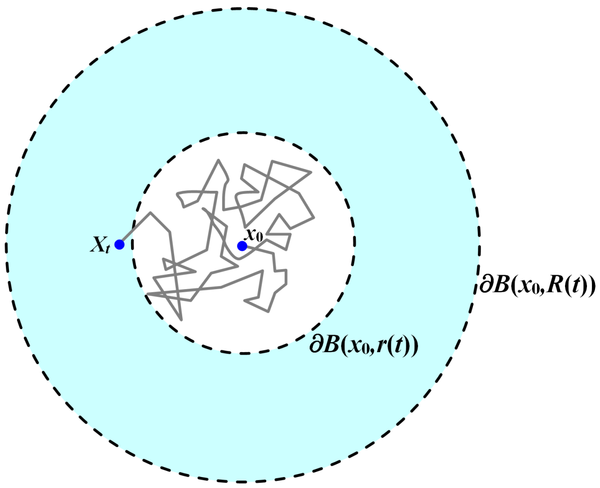

FIGURE 1. Upper and lower rate functions $R(t)$ and $r(t)$

is an upper rate function for any $\varepsilon > 0$ as it follows from Khinchin's law of iterated log. By the theorem of Dvoretzky-Erdös, if $r(t)/\sqrt{t}$ is decreasing then $r(t)$ is a lower rate function in $\mathbb{R}^n$, $n > 2$, if and only if

$$\int^{\infty}\left(\frac{r(t)}{\sqrt{t}}\right)^{n-2}\frac{dt}{t} < \infty \tag{7.1}$$

(cf. [7]). Here is an example of such a function:

$$r(t) = \frac{C\sqrt{t}}{\log^{\frac{1+\varepsilon}{n-2}} t}.$$

THEOREM 7.1. ([21], [15]) *Assume that, for all r large enough,*

$$V(r) \le Cr^N, \tag{7.2}$$

with some $N, C > 0$. Then the following function is an upper rate function:

$$R(t) = \sqrt{2Nt\log t}. \tag{7.3}$$

Under assumption (7.2), the upper rate function (7.3) is almost optimal (cf. [22]).

A similar result holds for simple random walks on graphs: it was proved by Hardy and Littlewood in 1914 for $\mathbb{Z}$ and in [3] for arbitrary graphs.

To state the next result, we need the notion of an *isoperimetric inequality.* We say that a manifold M satisfies the isoperimetric inequality if there exists $c > 0$ such that for any bounded domain $\Omega \subset M$ with smooth boundary,

$$\sigma(\partial\Omega) \ge c\mu(\Omega)^{\frac{n-1}{n}}, \tag{7.4}$$

where $n = \dim M$ and σ is the $(n-1)$-dimensional Riemannian measure on the hypersurface $\partial\Omega$. For example, (7.4) holds in $\mathbb{R}^n$ and, more generally, on any *Cartan-Hadamard* manifold that is a complete non-compact simply connected manifold with non-positive sectional curvature (cf. [27]). It is easy to see that (7.4) implies that

$$V(x,r) \ge c'r^n$$

for some $c' > 0$ and all $r > 0$.

THEOREM 7.2. ([20]) *Assume that M satisfies the isoperimetric inequality (7.4) and that*

$$\int^{\infty} \frac{rdr}{\log V(r)} = \infty. \tag{7.5}$$

Define a function $\varphi(t)$ as follows:

$$t = \int_{r_0}^{\varphi(t)} \frac{rdr}{\log V(r)}.$$

Then $R(t) = \varphi(Ct)$ is an upper rate function.

EXAMPLE 7.3. If $V(r) = Cr^N$ then

$$t \simeq \frac{\varphi^2(t)}{\log \varphi(t)}$$

whence $R(t) \simeq \varphi(t) \simeq \sqrt{t \log t}$ that matches (7.3).

EXAMPLE 7.4. If $V(r) = \exp(r^\alpha)$ where $0 < \alpha < 2$ then

$$t \simeq \varphi(t)^{2-\alpha}$$

whence $R(t) = Ct^{\frac{1}{2-\alpha}}$.

EXAMPLE 7.5. If $V(r) = \exp(r^2)$ then

$$t \simeq \log \varphi(t)$$

whence $R(t) = \exp(Ct)$.

The next result holds on any complete Riemannian manifold without assumption about an isoperimetric inequality.

THEOREM 7.6. ([28]) *On any complete manifold M, satisfying (7.5), define function $\varphi(t)$ as follows:*

$$t = \int_{r_0}^{\varphi(t)} \frac{rdr}{\log V(r) + \log\log r}.$$

Then $R(t) = C\varphi(Ct)$ is an upper rate function.

EXAMPLE 7.7. Let $V(r) \le C \log r$. Then

$$t \simeq \frac{\varphi^2(t)}{\log\log \varphi(t)}$$

and we obtain an upper rate function

$$R(t) = C\sqrt{t \log\log t}.$$

To state the next results about the lower rate function, we need the notion of a Faber-Krahn inequality. For any precompact domain $\Omega \subset M$, set

$$\lambda_{\min}(\Omega) = \inf_{f \in C_0^\infty(M) \setminus \{0\}} \frac{\int |\nabla f|^2 d\mu}{\int f^2 d\mu}.$$

In fact, $\lambda_{\min}(\Omega)$ is the minimal eigenvalue of the Laplace operator in Ω with the Dirichlet boundary value on $\partial\Omega$. By the theorem of Faber and Krahn, for $\Omega \subset \mathbb{R}^n$ we have

$$\lambda_{\min}(\Omega) \ge c_n \mu(\Omega)^{-2/n} \tag{7.6}$$

where $c_n > 0$ and the equality is attained if Ω is a ball.

We say that a manifold M satisfies a *relative Faber-Krahn* inequality if there exist $c,\nu>0$ such that, for any ball $B(x,r)$ and any open set $\Omega\subset B(x,r)$,

$$\lambda_{\min}(\Omega)\geq\frac{c}{r^2}\left(\frac{\mu(B(x,r))}{\mu(\Omega)}\right)^{\nu}. \tag{7.7}$$

As it follows from (7.6), in $\mathbb{R}^n$ the relative Faber-Krahn inequality (7.7) holds with $\nu=2/n$ and $c=c_n v_n^{-2/n}$ where v_n is the volume of the unit ball in $\mathbb{R}^n$.

More generally, the relative Faber-Krahn inequality holds on any complete manifold with non-negative Ricci curvature (see [**12**]).

THEOREM 7.8. ([**16**]) *If M satisfies the relative Faber-Krahn inequality then M is non-parabolic if and only if*

$$\int^{\infty}\frac{rdr}{V(r)}<\infty \tag{7.8}$$

PROOF. The necessity of (7.8) follows from Theorem 2.3, the sufficiency follows from the upper bound

$$p_t(x,x)\leq\frac{C}{V\left(x,\sqrt{t}\right)}$$

that holds under the relative Faber-Krahn inequality by [**13**]. □

THEOREM 7.9. ([**15**]) *Assume that a relative Faber-Krahn inequality holds on M. Assume also that (7.8) is satisfied so that M is non-parabolic. Denote*

$$\gamma(r):=\left(\int_r^{\infty}\frac{sds}{V(s)}\right)^{-1}. \tag{7.9}$$

Let $r(t)$ be an increasing positive function on $(0,\infty)$ such that

$$\int^{\infty}\frac{\gamma(r(t))}{V(\sqrt{t})}dt<\infty. \tag{7.10}$$

Then $r(t)$ is a lower rate function for Brownian motion on M.

EXAMPLE 7.10. Let $V(x,r)\simeq r^N$ for all large r and some $N>2$. We obtain from (7.9) $\gamma(t)\simeq t^{N-2}$, and (7.10) amounts to

$$\int^{\infty}\frac{r^{N-2}(t)dt}{t^{N/2}}<\infty,$$

which coincides with the Dvoretzky–Erdös condition (7.1).

8. Heat kernel lower bounds

Here we show some results on pointwise lower bounds of the heat kernel that use only the volume function. Recall that x_0 is a fixed point of M and $V(r)=V(x_0,r)$.

THEOREM 8.1. ([**5**]) *Assume that, for all $r\geq r_0>0$,*

$$V(r)\leq Cr^{\alpha}, \tag{8.1}$$

for some $C,\alpha>0$. Then, for all large enough t,

$$p_t(x_0,x_0)\geq\frac{1/4}{V(\sqrt{Kt\log t})}, \tag{8.2}$$

where $K = K(x_0, r_0, C, \alpha) > 0$. *Consequently, for some* $c > 0$,

$$p_t(x_0, x_0) \geq \frac{c}{(t \log t)^{\alpha/2}}. \tag{8.3}$$

If M has non-negative Ricci curvature then by the theorem of Li-Yau [**32**] the heat kernel satisfies on the diagonal the following two-sided estimate

$$p_t(x, x) \simeq \frac{1}{V(x, \sqrt{t})} \tag{8.4}$$

for all $x \in M$ and $t > 0$. Hence, the lower bound (8.3) differs from the best possible estimate (8.4) by the log-factor. However, under the hypothesis (8.1) alone, the lower bound (8.2) is optimal and cannot be essentially improved (cf. [**22**]).

Theorem 8.2. ([**5**]) *Assume that the function* $V(r)$ *is doubling, that is,*

$$V(2r) \leq CV(r),$$

and that, for all $t > 0$,

$$p_t(x_0, x_0) \leq \frac{C}{V(\sqrt{t})}.$$

Then, for all $t > 0$,

$$p_t(x_0, x_0) \geq \frac{c}{V(\sqrt{t})}.$$

Let Ω be an *end* of M, that is, an open connected proper subset of M such that $\overline{\Omega}$ is non-compact but $\partial\Omega$ is compact. Moreover, assume that $\partial\Omega$ is smooth hypersurface. Set also

$$B_\Omega(x, r) = B(x, r) \cap \Omega \qquad \text{and} \qquad V_\Omega(x, r) = \mu(B_\Omega(x, r)).$$

We consider the closure $\overline{\Omega}$ as a manifold with boundary $\partial\Omega$ and apply to this manifold the notion of parabolicity. We say that a function $u \in C^2(\overline{\Omega})$ is superharmonic if $\Delta u \leq 0$ in Ω and $\frac{\partial u}{\partial \nu}|_{\partial\Omega} \geq 0$ where ν is the exterior unit normal vector field on $\partial\Omega$. Then $\overline{\Omega}$ is called parabolic if every positive superharmonic function in $\overline{\Omega}$ is constant. Brownian motion in $\overline{\Omega}$ can be constructed by using the heat kernel in $\overline{\Omega}$ with the Neumann boundary condition, or, equivalently, from Brownian motion in M by imposing reflecting conditions on $\partial\Omega$. Similarly one handles other notions used in Section 2 so that Theorems 2.1, 2.2 and 2.3 remain true also for $\overline{\Omega}$ in place of M (see [**8**], [**16**, Section 5]).

In the same way one extends to $\overline{\Omega}$ the definition of stochastic completeness, which is equivalent to the fact that the lifetime of the reflected Brownian motion in Ω is equal to ∞. All the results of Section 8.83 remain true for $\overline{\Omega}$ in place of M as well.

The following theorem is new.

Theorem 8.3. *Let* Ω *be an end of* M *that satisfies the following two assumptions:*

- *there exists* $x_0 \in \Omega$, $C > 0$ *and* $N > 2$ *such that*

$$V_\Omega(x_0, r) \leq Cr^N \tag{8.5}$$

  for all large enough r.
- $\overline{\Omega}$ *is non-parabolic.*

Then, for any $x \in M$, *there exist* $t_x > 0$ *and* $c_x > 0$ *such that*

$$p_t(x, x) \geq \frac{c_x}{(t \log t)^{N/2}} \quad \text{for all } t \geq t_x. \tag{8.6}$$

Conjecture. *If* (8.5) *is satisfied with* $N \leq 2$ *(and, hence,* $\overline{\Omega}$ *is parabolic) then, for all* $x \in M$ *and* $t \geq t_x$*,*

$$p_t(x,x) \geq \frac{c_x}{t^\alpha}$$

with some $\alpha > 0$*. It is expected that if* $N < 2$ *then* $\alpha = \frac{4-N}{2}$*, whereas for* $N = 2$ *the value* α *can be taken arbitrarily close to* 2 *(cf.* [**24**, Example 6.11]*).*

Proof. Consider in Ω all functions $v \in C^2(\Omega) \cap C(\overline{\Omega})$ that satisfy the following conditions

- v is harmonic in Ω
- $0 \leq v \leq 1$ in Ω
- $v = 0$ on $\partial\Omega$.

Denote by v_Ω the maximal function that satisfies all these conditions – it exists as the supremum of all such v. This function is called a *subharmonic potential* of Ω (see [**16**, Def. 4.2]). In fact, $v_\Omega(x)$ is equal to the probability of the event that Brownian motion started at $x \in \Omega$ never hits $\partial\Omega$.

For example, if Ω is the exterior of the unit ball in $\mathbb{R}^n$ then $\overline{\Omega}$ is non-parabolic provided $n > 2$ and $v_\Omega(x) = 1 - |x|^{2-n}$ (cf. [**18**, Exercise 8.15]).

The set Ω is called *massive* if $v_\Omega > 0$ in Ω. The set Ω can be regarded as the exterior of a compact set $\partial\Omega$ on the manifold $\overline{\Omega}$. By [**16**, Corollary 4.6], Ω is massive provided $\overline{\Omega}$ is non-parabolic, which is the case now. Hence, we have $v_\Omega > 0$ in Ω.

Let $p_t^\Omega(x,y)$ denote the heat kernel in Ω with the Dirichlet boundary condition on $\partial\Omega$. It follows from [**23**, Rem. 2.1] that if M is stochastically complete then, for all $x \in \Omega$,

$$\int_\Omega p_t^\Omega(x,y)\,d\mu(y) \searrow v_\Omega(x) \quad \text{as } t \to \infty. \tag{8.7}$$

We apply this result with $\overline{\Omega}$ in place of M considering Ω is an open subset of $\overline{\Omega}$. By Theorem 3.2 and (8.5), $\overline{\Omega}$ is stochastically complete so that (8.7) is satisfied.

The rest of the argument is a modification of that in [**5**] and [**18**, Thm. 16.5]. We have, for all $t > 0$, $x, y \in \Omega$ and $r > 0$

$$\begin{aligned} p_{2t}^\Omega(x,x) &= \int_\Omega p_t^\Omega(x,y)^2\,d\mu(y) \geq \int_{B_\Omega(x,r)} p_t^\Omega(x,y)^2\,d\mu(y) \\ &\geq \frac{1}{V_\Omega(x,r)}\left(\int_{B_\Omega(x,r)} p_t^\Omega(x,y)\,d\mu(y)\right)^2 \\ &= \frac{1}{V_\Omega(x,r)}\left(\int_\Omega p_t^\Omega(x,y)\,d\mu(y) - \int_{\Omega\setminus B_\Omega(x,r)} p_t^\Omega(x,y)\,d\mu(y)\right)^2. \end{aligned} \tag{8.8}$$

Recall that

$$\int_\Omega p_t^\Omega(x,y)\,d\mu(y) \geq v_\Omega(x).$$

If r is chosen so large that

$$\int_{\Omega\setminus B_\Omega(x,r)} p_t^\Omega(x,y)\,d\mu(y) \leq \frac{1}{2} v_\Omega(x), \tag{8.9}$$

then it follows from (8.8) that

$$p_{2t}^\Omega(x,x) \geq \frac{v_\Omega^2(x)}{4V_\Omega(x,r)}. \tag{8.10}$$

Let us specify $r=r\left(t\right)$ that satisfies (8.9). Using $p_t^\Omega \le p_t$, we obtain

$$\begin{aligned}\int_{\Omega\setminus B_\Omega(x,r)} p_t^\Omega\left(x,y\right)d\mu(y) &\le \int_{\Omega\setminus B_\Omega(x,r)} p_t\left(x,y\right)e^{\frac{d^2(x,y)}{8t}}e^{-\frac{d^2(x,y)}{8t}}d\mu(y)\\ &\le \left(\int_M p_t^2\left(x,y\right)e^{\frac{d^2(x,y)}{4t}}d\mu(y)\right)^{1/2}\\ &\quad\times\left(\int_{\Omega\setminus B_\Omega(x,r)} e^{-\frac{d^2(x,y)}{4t}}d\mu(y)\right)^{1/2}. \end{aligned}\tag{8.11}$$

Next use the fact that, on any manifold M, the function

$$E\left(x,t\right):=\int_M p_t^2\left(x,y\right)e^{\frac{d^2(x,y)}{4t}}d\mu(y)$$

is finite and monotone decreasing in t ([**13**], [**18**, Thm. 12.1 and Cor. 15.9]). Hence, assuming $t\ge 1$, we obtain

$$\int_M p_t^2\left(x,y\right)e^{\frac{d^2(x,y)}{4t}}d\mu(y)\le E\left(x,1\right)=:E\left(x\right).$$

The integral in (8.11) can be estimated as follows:

$$\begin{aligned}\int_{\Omega\setminus B_\Omega(x,r)} e^{-\frac{d^2(x,y)}{4t}}d\mu(y) &= \sum_{k=0}^\infty \int_{B_\Omega\left(x,2^{k+1}r\right)\setminus B_\Omega\left(x,2^kr\right)} e^{-\frac{d^2(x,y)}{4t}}d\mu(y)\\ &\le \sum_{k=0}^\infty \exp\left(-\frac{\left(2^kr\right)^2}{4t}\right)V_\Omega\left(x,2^{k+1}r\right).\end{aligned}$$

The hypothesis (8.5) implies that, for $R>1$,

$$V_\Omega\left(x,R\right)\le C_xR^N \tag{8.12}$$

with a constant C_x depending on x. Hence, we obtain, for $r>1$ and $t>1$,

$$\int_{\Omega\setminus B_\Omega(x,r)} e^{-\frac{d^2(x,y)}{4t}}d\mu(y)\le\sum_{k=0}^\infty C_x\exp\left(-\frac{4^kr^2}{4t}\right)\left(2^{k+1}r\right)^N.$$

Assume now that

$$\frac{r^2}{4t}\ge 1. \tag{8.13}$$

Observing that

$$\begin{aligned}\exp\left(-\frac{4^kr^2}{4t}\right)\left(2^{k+1}r\right)^N &= 2^Nt^{N/2}\exp\left(-\frac{4^kr^2}{4t}\right)\left(\frac{4^kr^2}{t}\right)^{N/2}\\ &\le \operatorname{const} t^{N/2}\exp\left(-\frac{4^kr^2}{8t}\right)\end{aligned}$$

and summing up the geometric series, we obtain

$$\int_{\Omega\setminus B_\Omega(x,r)} e^{-\frac{d^2(x,y)}{4t}}d\mu(y)\le C_xt^{\frac N2}\exp\left(-\frac{r^2}{8t}\right).$$

It follows that

$$\int_{\Omega\setminus B_\Omega(x,r)} p_t^\Omega\left(x,y\right)d\mu(y)\le\left(E\left(x\right)C_xt^{\frac N2}\exp\left(-\frac{r^2}{8t}\right)\right)^{1/2}.$$

Hence, to ensure (8.9) it suffices to have

$$\left(E\left(x\right)C_x t^{\frac{N}{2}}\exp\left(-\frac{r^2}{8t}\right)\right)^{1/2}\leq\frac{1}{2}v_\Omega\left(x\right).$$

that is,

$$\frac{r^2}{8t}\geq\ln\frac{4E\left(x\right)C_x}{v_\Omega\left(x\right)^2}+\frac{N}{2}\ln t.$$

If $t\geq t_x$ where t_x is large enough, then this inequality is satisfied provided

$$\frac{r^2}{8t}=N\ln t,$$

that is, for

$$r=\sqrt{8Nt\ln t}. \tag{8.14}$$

For this r we clearly have also (8.13). Consequently, we obtain (8.9) and, hence, (8.10). Substituting (8.12) and (8.14) into (8.9), we obtain

$$p_{2t}^{\Omega}\left(x,x\right)\geq\frac{v_\Omega^2\left(x\right)}{4C_x\left(8Nt\ln t\right)^{N/2}},$$

which yields

$$p_t^{\Omega}\left(x,x\right)\geq\frac{c_x}{\left(t\log t\right)^{N/2}}\quad\text{for all } t\geq t_x. \tag{8.15}$$

Since $p_t\left(x,y\right)\geq p_t^{\Omega}\left(x,y\right)$ we obtain (8.6) for all $x\in\Omega$. Finally, by means of a local Harnack inequality, (8.6) extends to all $x\in M$. $\square$

9. Recurrence revisited

For any $\alpha\in(0,2)$, the operator $(-\Delta)^{\alpha/2}$ is the generator of a jump process on M that is called *the α-process*. It is a natural generalization of the symmetric stable Levy process of index α in $\mathbb{R}^d$. By a general semigroup theory, the Green function $g^{(\alpha)}(x,y)$ of $(-\Delta)^{\alpha/2}$ is given by

$$g^{(\alpha)}(x,y)=\int_0^\infty t^{\alpha/2-1}p_t(x,y)dt,$$

and the recurrence of the α-process is equivalent to $g^{(\alpha)}\equiv\infty$, that is, to

$$\int^\infty t^{\alpha/2-1}p_t(x,x)dt=\infty. \tag{9.1}$$

THEOREM 9.1. ([**16**, Thm. 16.2]) *If for all large enough r*

$$V(r)\leq Cr^\alpha, \tag{9.2}$$

then the α-process is recurrent.

PROOF. Indeed, by Theorem 8.1 we have

$$p_t(x_0,x_0)\geq\frac{c}{t^{\alpha/2}\log^{\alpha/2}t}.$$

Substituting into (9.1) we see that the integral diverges. $\square$

10. Heat kernel upper bounds

We say that a manifold M has *bounded geometry* if there exists $\varepsilon > 0$ such that all balls $B(x, \varepsilon)$ are uniformly quasi-isometric to a Euclidean ball B_ε of radius ε; that is, there is a constant C and, for any $x \in M$, a diffeomorphism $\varphi_x : B(x, \varepsilon) \to B_\varepsilon$ such that φ_x changes the Riemannian metric at most by the factor C

In particular, M has bounded geometry if its injectivity radius is positive and the Ricci curvature is bounded from below.

Theorem 10.1. ([2]) *Let M be a manifold of bounded geometry. Assume that, for all $x \in M$ and $r \geq r_0 > 0$,*

$$V(x, r) \geq cr^N, \tag{10.1}$$

where $c > 0$. Then, for all $x \in M$ and large enough t,

$$p_t(x, x) \leq Ct^{-\frac{N}{N+1}}. \tag{10.2}$$

For any $N \geq 1$, there exists an example of a manifold with $V(x, r) \simeq r^N$ and

$$p_t(x, x) \simeq t^{-\frac{N}{N+1}},$$

for all $x \in M$ and $t \geq 1$. Indeed, take any fractal graph where the volume function is of the order r^α and the on-diagonal decay of the heat kernel is of the order $t^{-\alpha/\beta}$. It is known that such a graph exists for any couple α, β satisfying

$$2 \leq \beta \leq \alpha + 1$$

(see [1]). Choose $\alpha = N$ and $\beta = N + 1$ and then inflate the graph into a manifold.

One of such graphs, the *Vicsek tree*, is shown on Fig. 2.

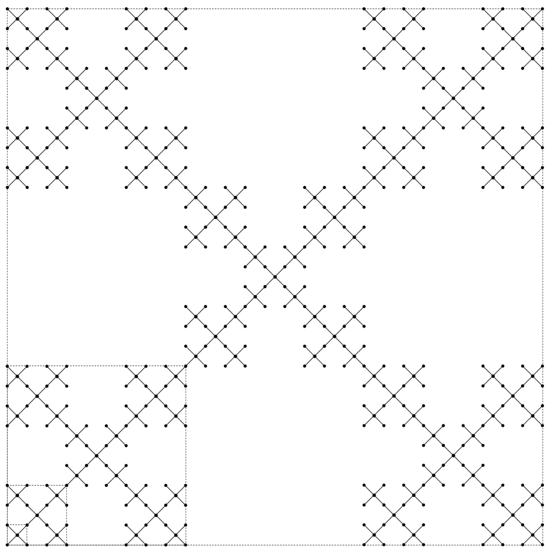

Figure 2. Vicsek tree

For this fractal we have

$$\alpha = \frac{\log 5}{\log 3} \quad \text{and} \quad \beta = \alpha + 1 = \frac{\log 15}{\log 3}.$$

It is possible to prove that on any manifold of bounded geometry there exists $c > 0$ such that

$$V(x, r) \geq cr,$$

for all $x \in M$ and large enough r, that is, (10.1) holds with $N = 1$ (this follows, for example, from [**14**, Theorem 2.1]). Hence, on any manifold of bounded geometry we have

$$p_t(x,x) \leq \frac{C}{\sqrt{t}},$$

for all $x \in M$ and large enough t. Note for comparison that, for a cylinder $M = K \times \mathbb{R}$ where K is a compact manifold, we have

$$V(x,r) \simeq r \quad \text{and} \quad p_t(x,x) \simeq t^{-1/2}$$

for all $x \in M$ and large enough r, t.

11. Biparabolic manifolds

A function $u \in C^4(M)$ is called *bi-superharmonic* if $\Delta u \leq 0$ and $\Delta^2 u \geq 0$.

For example, let M be nonparabolic and consider the Green operator

$$Gf = \int_0^\infty g(x,y) f(y) \, d\mu(y),$$

where $g(x,y)$ the Green function. If f is non-negative and superharmonic then the function $u = Gf$ is bi-superharmonic provided it is finite.

Here is another example of bi-superharmonic functions in a precompact domain $\Omega \subset M$. Let τ_Ω be the first exit time from Ω of Brownian motion X_t. If f is a non-negative continuous function on $\partial\Omega$ then the function

$$u(x) = \mathbb{E}_x\left(\tau_\Omega f\left(X_{\tau_\Omega}\right)\right)$$

solves the following boundary value problem

$$\begin{cases} \Delta^2 u = 0 \quad \text{in } \Omega \\ \Delta u|_{\partial\Omega} = -f, \\ u|_{\partial\Omega} = 0, \end{cases}$$

and, hence, is bi-superharmonic in Ω.

A manifold M is called *biparabolic*, if any positive bi-superharmonic function on M is harmonic, that is $\Delta u = 0$.

Note that the notion of parabolicity also admits a similar equivalent definition: M is parabolic if and only if any positive superharmonic function on M is harmonic.

One can show that $\mathbb{R}^n$ is biparabolic if and only if $n \leq 4$. For example, if $n > 4$ then $u(x) = |x|^{-(n-4)}$ is bi-superharmonic but not harmonic.

THEOREM 11.1. ([**19**]) *If for all large enough r*

$$V(r) \leq C \frac{r^4}{\log r} \tag{11.1}$$

then M is biparabolic.

The condition (11.1) is not far from optimal in the following sense: for any $\beta > 1$ there exists a manifold M with

$$V(r) \leq C\, r^4 \log^\beta r$$

that is not biparabolic.

Conjecture. *If*

$$V\left(r\right)\leq Cr^{4}\log r\quad or\quad even\quad\int^{\infty}\frac{r^{3}dr}{V\left(r\right)}=\infty,$$

then M is biparabolic.

Recall that M is parabolic if and only if $G\equiv\infty$ that is, $Gf\equiv\infty$ for any non-zero $f\geq0$. For the proof of Theorem 11.1 we use the following lemma.

Lemma 11.2. *The following conditions are equivalent:*
(i) M is biparabolic.
(ii) $G^{2}\equiv\infty$ (that is, $G^{2}f\equiv\infty$ for any non-zero functions $f\geq0$)

Proof of Theorem 11.1. Assuming (11.1), we prove that $G^{2}f\equiv\infty$ for any non-negative non-zero function f. It is easy to compute that

$$G^{2}f\left(x\right)=\int_{0}^{\infty}tP_{t}f\left(x\right)dt=\int_{0}^{\infty}\int_{M}tp_{t}(x,y)f(y)d\mu(y)dt.$$

Fix an arbitrary $x\in M$ and choose $R>0$ so big that the ball $B(x_{0},R)$ contains both $\operatorname{supp}f$ and x. By the local Harnack inequality, we have, for all $x,y\in B\left(x_{0},R\right)$ and $t>2R^{2}$

$$p_{t}\left(x,y\right)\geq cp_{t-R^{2}}\left(x_{0},x_{0}\right)\geq cp_{t}\left(x_{0},x_{0}\right),$$

where $c=c\left(x_{0},R\right)>0$. Hence, we obtain, for large enough t_{0},

$$G^{2}f(x)\geq\int_{t_{0}}^{\infty}\int_{B(x_{0},R)}tp_{t}(x,y)f(y)d\mu(y)dt\geq c||f||_{L^{1}}\int_{t_{0}}^{\infty}tp_{t}(x_{0},x_{0})dt.$$

By Theorem 8.1, we have, for large t,

$$p_{t}\left(x_{0},x_{0}\right)\geq\frac{1/4}{V\left(\sqrt{Kt\log t}\right)}\geq\frac{c}{v\left(\sqrt{t\log t}\right)},$$

where

$$v(r):=\frac{r^{4}}{\log r}.$$

For large t we have

$$v(\sqrt{t\log t})\simeq t^{2}\log t,$$

whence

$$\int_{t_{0}}^{\infty}tp_{t}(x_{0},x_{0})dt\geq c\int_{t_{0}}^{\infty}\frac{tdt}{v\left(\sqrt{t\log t}\right)}\simeq\int_{t_{0}}^{\infty}\frac{dt}{t\log t}=\infty.$$

We conclude that $G^{2}f\left(x\right)=\infty$, which was to be proved. □

Now let us construct for any $\beta>1$ an example of a manifold M that is not biparabolic and satisfies

$$V\left(r\right)\leq Cr^{4}\log^{\beta}r.$$

Fix $n\geq2$ and consider a smooth manifold $M=\mathbb{R}\times\mathbb{S}^{n-1}$, where any point $x\in M$ is represented in the polar form as (r,θ) where $r\in\mathbb{R}$ and $\theta\in\mathbb{S}^{n-1}$.

Define the Riemannian metric g on M by

$$\mathrm{g}=dr^{2}+\psi^{2}(r)d\theta^{2},\tag{11.2}$$

where $d\theta^{2}$ is the standard Riemannian metric on $\mathbb{S}^{n-1}$ and $\psi(r)$ is a smooth positive function on $\mathbb{R}$. Define the area function $S\left(r\right)$, $r\in\mathbb{R}$, by

$$S\left(r\right)=\omega_{n}\psi\left(r\right)^{n-1},$$

where ω_n is the volume of $\mathbb{S}^{n-1}$. We choose $S(r)$ as follows:

$$S(r) = \begin{cases} e^{-r^\alpha}, & r > 2, \\ |r|^3 \log^\beta |r|, & r < -2, \end{cases} \tag{11.3}$$

where α, β are arbitrary real numbers such that

$$\alpha > 2 \quad \text{and} \quad \beta > 1. \tag{11.4}$$

The manifold M looks as on Fig. 3.

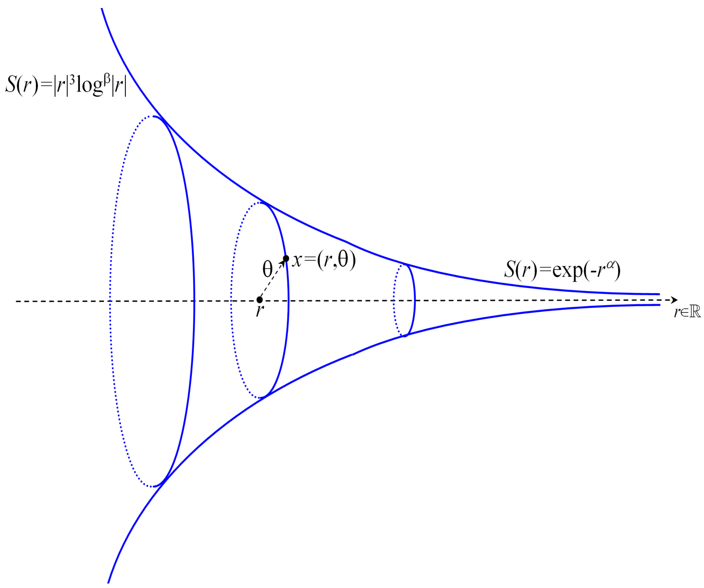

FIGURE 3. Manifold M with two ends

PROPOSITION 11.3. *Under the hypotheses* (11.3) *and* (11.4), *the manifold M is not biparabolic, and the volume growth function of M satisfies*

$$V(r) \le C r^4 \log^\beta r. \tag{11.5}$$

PROOF. Fix a reference point $x_0 = (0,0)$. The volume estimate (11.5) follows from

$$V(r) \simeq \int_{-r}^{r} S(t)\, dt.$$

In order to prove that M is not biparabolic, it suffices to construct a positive harmonic function h on M such that the function $u := Gh$ is finite at least at one point. Indeed, in this case we have $u \in C^\infty(M)$ and $\Delta u = -h$. Hence, $\Delta u < 0$ and $\Delta^2 u = \Delta h = 0$ so that u is bi-superharmonic, but not harmonic; hence, M is not biparabolic.

Choose h as follows:

$$h(r) = \int_{-\infty}^{r} \frac{dt}{S(t)}. \tag{11.6}$$

It is finite by (11.3) and harmonic on M because it depends only on r and

$$\Delta h = \frac{\partial^2 h}{\partial r^2} + \frac{S'(r)}{S(r)} \frac{\partial h}{\partial r} = \frac{1}{S(r)} \frac{\partial}{\partial r} \left(S(r) \frac{\partial h}{\partial r} \right) = 0.$$

Then one proves that, for any $x = (r, \theta)$,

$$g(x_0, x) \simeq \begin{cases} h(r), & \text{if } r < -2, \\ 1, & \text{if } r > 2. \end{cases}$$

We have

$$Gh(x_0) = \int_M g(x_0, x)h(x)d\mu(x) \simeq 1 + \int_{-\infty}^{-2} h^2(r)S(r)dr + \int_2^{\infty} h(r)S(r)dr.$$

For $r < -2$ we have

$$S\left(r\right) = |r|^3 \log^\beta |r| \quad \text{and} \quad h\left(r\right) \simeq \int_{-\infty}^{r} \frac{dt}{|t|^3 \log^\beta |t|} \simeq \frac{1}{|r|^2 \log^\beta |r|}.$$

Since $\beta > 1$, we obtain

$$\int_{-\infty}^{-2} h^2\left(r\right) S\left(r\right) dr \simeq \int_{-\infty}^{-2} \frac{1}{|r| \log^\beta |r|} dr < \infty.$$

For $r > 2$

$$S\left(r\right) = e^{-r^\alpha} \quad \text{and} \quad h\left(r\right) \simeq \int_0^r e^{t^\alpha} dt \simeq \frac{e^{r^\alpha}}{r^{\alpha-1}}.$$

Since $\alpha > 2$, we have

$$\int_2^{+\infty} h\left(r\right) S\left(r\right) dr \simeq \int_2^{\infty} \frac{dr}{r^{\alpha-1}} < \infty.$$

Hence, $Gh\left(x_0\right) < \infty$, which was to be proved. $\square$

References

[1] Barlow M.T., *Which values of the volume growth and escape time exponent are possible for graphs?*, *Revista Matemática Iberoamericana*, **40** (2004) 1-31.

[2] Barlow M.T., Coulhon T., Grigor'yan A., *Manifolds and graphs with slow heat kernel decay,* *Invent. Math.*, **144** (2001) 609-649.

[3] Barlow M.T., Perkins E.A., *Symmetric Markov chains in $\mathbb{Z}^d$: how fast can they move?,* *Probab. Th. Rel. Fields*, **82** (1989) 95–108.

[4] Cheng S.Y., Yau S.-T., *Differential equations on Riemannian manifolds and their geometric applications,* *Comm. Pure Appl. Math.*, **28** (1975) 333-354.

[5] Coulhon T., Grigor'yan A., *On-diagonal lower bounds for heat kernels and Markov chains,* *Duke Math. J.*, **89** (1997) no.1, 133-199.

[6] Davies E.B., *Heat kernel bounds, conservation of probability and the Feller property,* *J. d'Analyse Math.*, **58** (1992) 99-119.

[7] Dvoretzky A., Erdös P., *Some problems on random walk in space,* *Proc. Second Berkeley Symposium on Math. Stat. and Probability*, University of California Press, 1951. 353-368.

[8] Grigor'yan A., *On the existence of positive fundamental solution of the Laplace equation on Riemannian manifolds,* (in Russian) *Matem. Sbornik*, **128** (1985) no.3, 354-363. Engl. transl.: *Math. USSR Sb.*, **56** (1987) 349-358.

[9] Grigor'yan A., *On stochastically complete manifolds,* (in Russian) *DAN SSSR*, **290** (1986) no.3, 534-537. Engl. transl.: *Soviet Math. Dokl.*, **34** (1987) no.2, 310-313.

[10] Grigor'yan A., *Stochastically complete manifolds and summable harmonic functions,* (in Russian) *Izv. AN SSSR, ser. matem.*, **52** (1988) no.5, 1102-1108. Engl. transl.: *Math. USSR Izvestiya*, **33** (1989) no.2, 425-432.

[11] Grigor'yan A., *Bounded solutions of the Schrödinger equation on non-compact Riemannian manifolds,* (in Russian) *Trudy seminara I.G.Petrovskogo*, (1989) no.14, 66-77. Engl. transl.: *J. Soviet Math.*, **51** (1990) no.3, 2340-2349.

[12] Grigor'yan A., *The heat equation on non-compact Riemannian manifolds,* (in Russian) *Matem. Sbornik*, **182** (1991) no.1, 55-87. Engl. transl.: *Math. USSR Sb.*, **72** (1992) no.1, 47-77.

[13] Grigor'yan A., *Heat kernel upper bounds on a complete non-compact manifold,* *Revista Matemática Iberoamericana*, **10** (1994) no.2, 395-452.

[14] Grigor'yan A., *Heat kernel on a manifold with a local Harnack inequality,* *Comm. Anal. Geom.*, **2** (1994) no.1, 111-138.

[15] Grigor'yan A., *Escape rate of Brownian motion on weighted manifolds,* *Applicable Analysis*, **71** (1999) no.1-4, 63-89.

[16] Grigor'yan A., *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds,* *Bull. Amer. Math. Soc.*, **36** (1999) 135-249.

[17] Grigor'yan A., *Heat kernels on weighted manifolds and applications,* Contem. Math. 398, (2006) 93-191.

[18] Grigor'yan A., *Heat kernel and Analysis on manifolds,* AMS-IP Studies in Advanced Mathematics 47, AMS - IP, 2009.

[19] Grigor'yan A., Faraji Sh., *On biparabolicity of Riemannian manifolds,* *Revista Matemática Iberoamericana*, **35** (2019) no.7, 2025-2034.

[20] Grigor'yan A., Hsu, Elton P., *Volume growth and escape rate of Brownian motion on a Cartan-Hadamard manifold,* *in:* "Sobolev Spaces in Mathematics II", ed. V. Maz'ya, International Mathematical Series 9, Springer, 2009. 209-225.

[21] Grigor'yan A., Kelbert M., *Range of fluctuation of Brownian motion on a complete Riemannian manifold,* *Ann. Prob.*, **26** (1998) 78-111.

[22] Grigor'yan A., Kelbert M., *On Hardy-Littlewood inequality for Brownian motion on Riemannian manifolds,* *J. London Math. Soc. (2)*, **62** (2000) 625-639.

[23] Grigor'yan A., Saloff-Coste L., *Hitting probabilities for Brownian motion on Riemannian manifolds,* *J. Math. Pures et Appl.*, **81** (2002) 115-142.

[24] Grigor'yan A., Saloff-Coste L., *Heat kernel on manifolds with ends,* *Ann. Inst. Fourier, Grenoble*, **59** (2009) 1917-1997.

[25] Grigor'yan A., Sun Y., *On non-negative solutions of the inequality* $\Delta u + u^\sigma \leq 0$ *on Riemannian manifolds,* *Comm. Pure Appl. Math.*, **67** (2014) 1336–1352.

[26] Grigor'yan A., Sun Y., Verbitsky I., *Superlinear elliptic inequalities on manifolds,* *J. Funct. Anal.*, **278** (2020) art.108444.

[27] Hoffman D., Spruck J., *Sobolev and isoperimetric inequalities for Riemannian submanifolds,* *Comm. Pure Appl. Math.*, **27** (1974) 715–727. See also *"A correction to: Sobolev and isoperimetric inequalities for Riemannian submanifolds",* *Comm. Pure Appl. Math.*, **28** (1975) no.6, 765–766.

[28] Hsu E.P., Qin G., *Volume growth and escape rate of Brownian motion on a complete Riemannian manifold,* *Ann. Prob.*, **38** (2010) no.4, 1570–1582.

[29] Karp L., *Subharmonic functions, harmonic mappings and isometric immersions,* *in:* "Seminar on Differential Geometry", ed. S.T.Yau, Ann. Math. Stud. 102, Princeton, 1982.

[30] Karp L., Li P., *The heat equation on complete Riemannian manifolds,* unpublished manuscript 1983.

[31] Kuz'menko Yu.T., Molchanov S.A., *Counterexamples to Liouville-type theorems,* (in Russian) *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, (1979) no.6, 39-43. Engl. transl.: *Moscow Univ. Math. Bull.*, **34** (1979) 35-39.

[32] Li P., Yau S.-T., *On the parabolic kernel of the Schrödinger operator,* *Acta Math.*, **156** (1986) no.3-4, 153-201.

[33] Mitidieri E., Pohozaev S. I., *Absence of global positive solutions of quasilinear elliptic inequalities,* *Dokl. Akad. Nauk.*, **359** (1998) no.4, 456–460.

[34] Sun Y., *Uniqueness result on nonnegative solutions of a large class of differential inequalities on Riemannian manifolds,* *Pacific J. Math.*, **280** (2016) no.1, 241–254.

[35] Sun Y., *The absence of global positive solutions to semilinear parabolic differential inequalities in exterior domain,* *Proc. AMS*, **145** (2017) no.8, 3455–3464.

[36] Takeda M., *On a martingale method for symmetric diffusion process and its applications,* *Osaka J. Math*, **26** (1989) 605-623.

[37] Varopoulos N.Th., *Potential theory and diffusion of Riemannian manifolds,* *in:* "Conference on Harmonic Analysis in honor of Antoni Zygmund. Vol I, II.", Wadsworth Math. Ser., Wadsworth, Belmont, Calif., 1983. 821-837.

[38] Yau S.-T., *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry,* *Indiana Math. J.*, **25** (1976) 659-670.

Department of Mathematics, University of Bielefeld, Bielefeld, 33501 Germany
E-mail address: grigor@math.uni-bielefeld.de