On the existence of positive solutions of semi-linear elliptic inequalities on Riemannian manifolds

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April 19, 2009

1 Introduction and main results

Let M be a smooth connected Riemannian manifold and consider the differential inequality on M

$$\operatorname{div}(A(x)\nabla u) + V(x)\,u^{\sigma} \leq 0, \tag{1.1}$$

where ∇ and div are respectively the Riemannian gradient and divergence, $u = u(x)$ is an unknown positive function on M, $\sigma > 1$ is a given constant, V is a given positive measurable function on M, and A is a given measurable tensor field on M such that $A(x)$ is a non-negative definite symmetric operator in the tangent space T_xM. The inequality (1.1) is understood in a weak sense to be explained below.

We are concerned with the question when (1.1) has no positive solution u on M. This question in the setting of Euclidean spaces has a long history, starting with the pioneering work of Gidas and Spruck [3]. We refer the reader to [10] for the survey of this problem. Let us cite only one result in this direction, which already exhibits the phenomenon that the answer depends on the interplay of all the data, including the geometry of M and the value of σ. Indeed, it is known that the following inequality in $\mathbb{R}^n$, $n > 2$,

$$\Delta u + u^{\sigma} \leq 0$$

has no positive solution if and only if $\sigma \leq \frac{n}{n-2}$. If $n \leq 2$ then there is no positive solution for any σ.

The previously developed methods for investigation of the above question include such advanced tools as Harnack inequalities and estimates of fundamental solutions. Here we adopt another approach that originates from [8] and that uses only very

[*]Supported by SFB 701 of the German Research Council
[†]Partially supported by SFB 701 of the German Research Council

basic tools as capacities and volumes. This enables us to replace a traditional assumption on $A(x)$ to be positive definite, by the non-negative definiteness.

In the rest of this section we state the main results: the capacity tests and the volume test. The former are proved in Section 2, and the latter in Section 4. In Sections 3 and 5 we give examples, showing the sharpness of the above tests.

Let us explain in what sense we understand (1.1). Let μ be the Riemannian measure on M. All the spaces $L^p(M)$ will be considered with respect to μ. Recall that $W^1(M)$ is the Sobolev space defined by

$$W^1(M) = \left\{ f \in L^2(M) : |\nabla f| \in L^2(M) \right\},$$

where ∇f is the weak gradient of f. Let $W_c^1(M)$ be a subspace of $W^1(M)$ that consists of functions with compact supports.

Similarly, define a local Sobolev space $W_{loc}^1(M)$ by

$$W_{loc}^1(M) = \left\{ f \in L_{loc}^2(M) : |\nabla f| \in L_{loc}^2(M) \right\}.$$

Definition. A function u on M is called a *positive (weak) solution* of the inequality (1.1) on M if u is a positive function from $W_{loc}^1(M)$, such that $\frac{1}{u} \in L_{loc}^\infty(M)$, and, for any non-negative function $\psi \in W_c^1(M)$, the following inequality holds:

$$-\int_M (A(x)\nabla u, \nabla\psi)\, d\mu + \int_M V(x)\, u^\sigma \psi d\mu \le 0 \tag{1.2}$$

where $(\cdot,\cdot)$ is the inner product in T_xM given by the Riemannian metric.

To ensure the finiteness of the integrals in (1.2), we assume henceforth that the function $x \mapsto \|A(x)\|$ is locally bounded, where $\|A(x)\|$ is the norm of the operator $A(x)$ in T_xM, that is, the maximal eigenvalue of the operator $A(x)$. Indeed, since $K := \operatorname{supp}\psi$ is compact, we have

$$\begin{aligned} \int_M |(A(x)\nabla u, \nabla\psi)|\, d\mu &= \int_K |(A(x)\nabla u, \nabla\psi)|\, d\mu \\ &\le \operatorname*{esssup}_{x\in K} \|A(x)\|\, \|\nabla u\|_{L^2(K)} \|\nabla\psi\|_{L^2(K)} \\ &< \infty. \end{aligned}$$

The second integral in (1.2) is then finite due to (1.2).

Consider the following quadratic form in the tangent space T_xM:

$$(\xi,\eta)_A := (A(x)\xi,\eta),$$

and the corresponding semi-norm

$$|\xi|_A := (A(x)\xi,\xi)^{1/2}.$$

In particular, for any function $f \in W_{loc}^1(M)$, we have

$$|\nabla f|_A = (A(x)\nabla f, \nabla f)^{1/2}.$$

Definition. Let A and V be as above, and fix two constants $p > 0$ and $q \geq 0$. For any precompact set $K \subset M$, define the capacity $\mathrm{cap}_{p,q}(K)$ as follows:

$$\mathrm{cap}_{p,q}(K) := \inf_{\varphi \in \mathcal{T}(K)} \int_M |\nabla \varphi|_A^p V^{-q} d\mu,$$

where $T(K)$ is the class of test function, defined by

$$\mathcal{T}(K) = \left\{ \varphi \in Lip_c(M) : 0 \leq \varphi \leq 1, \text{ and } \varphi \equiv 1 \text{ in a neighborhood of } \overline{K} \right\},$$

and $Lip_c(M)$ is the class of Lipschitz functions on M with compact supports.

If $q = 0$ then we write

$$\mathrm{cap}_p(K) \equiv \mathrm{cap}_{p,0}(K) = \inf_{\varphi \in \mathcal{T}(K)} \int_M |\nabla \varphi|_A^p \, d\mu,$$

so that $\mathrm{cap}_p(K)$ is independent of $V(x)$. It is well-known that if $\mathrm{cap}_2(K) = 0$ for any compact set (or for some compact set with non-empty interior) then any positive solution to the inequality

$$\mathrm{div}\,(A(x)\nabla u) \leq 0 \tag{1.3}$$

must be constant (see, for example, [5]). Since any positive solution to (1.1) satisfies (1.3), we obtain in this setting that $u \equiv \mathrm{const}$, which implies by (1.1) that $u \equiv 0$. Hence, the condition $\mathrm{cap}_2(K) = 0$ implies the absence of a positive solution of (1.1) for any potential $V(x)$ and any σ.

We state now more subtle conditions in terms of higher capacities that take into account also σ and $V(x)$. Fix some $\sigma > 1$ in (1.1) and set

$$p = \frac{2\sigma}{\sigma - 1}, \quad q = \frac{1}{\sigma - 1}. \tag{1.4}$$

Theorem 1.1 *If, for some compact set $K \subset M$ with non-empty interior, the following condition is satisfied*

$$\mathrm{cap}_{p-2\varepsilon, q-\varepsilon}(K) = o\left(\varepsilon^{p/2}\right) \text{ as } \varepsilon \to 0+, \tag{1.5}$$

then (1.1) has no positive solution on M.

Theorem 1.2 *If, for some compact set $K \subset M$ with non-empty interior and some $\varepsilon \in (0, q]$, the following condition is satisfied*

$$\mathrm{cap}_{p-2\varepsilon, q-\varepsilon}(K) = 0, \tag{1.6}$$

then (1.1) has no positive solution on M.

Let $d(x, y)$ be the geodesic distance on M and $B(x, r)$ be the open geodesic ball of radius r centered at $x \in M$. Assume further that M is geodesically complete, which is equivalent to the relative compactness of all geodesic balls in M. For any $\varepsilon \geq 0$, consider a measure ν_ε on M defined by

$$d\nu_\varepsilon = \|A\|^{p/2-\varepsilon} \, V^{-(q-\varepsilon)} d\mu,$$

where p, q are the same as in (1.4).

Theorem 1.3 *Let M be a geodesically complete Riemannian manifold and assume that, for some $x_0 \in M$, $C > 0$, $\kappa < q$, the following inequality holds*

$$\nu_\varepsilon (B(x_0, r)) \le C r^{p+C\varepsilon} \log^\kappa r \tag{1.7}$$

for all large enough r and all small enough $\varepsilon > 0$. Then (1.1) has no positive solution.

As we will see in Section (5), the restriction $\kappa < q$ is sharp. More precisely, in the case $\kappa > q$ there is a counterexample with a positive solution. The borderline case $\kappa = q$ requires further investigation.

Note that

$$d\nu_0 = \|A\|^{p/2} V^{-q} d\mu$$

and

$$d\nu_\varepsilon = \left(\frac{V}{\|A\|}\right)^\varepsilon d\nu_0$$

(although the latter makes sense only if $\|A(x)\| > 0$ a.e.). Clearly, the condition (1.7) holds provided

$$\nu_0 (B(x_0, r)) \le C r^p (\log r)^\kappa \tag{1.8}$$

and

$$\frac{V}{\|A\|}(x) \le C (1 + d(x, x_0))^C, \tag{1.9}$$

for some $C > 0$.

Example. Let $A = \mathrm{id}$ and $V \equiv 1$ so that the inequality (1.1) becomes

$$\Delta u + u^\sigma \le 0. \tag{1.10}$$

The condition (1.7) is equivalent to

$$\mu (B(x_0, r)) \le C r^p \log^\kappa r, \tag{1.11}$$

where p and κ are related to σ as above, that is,

$$p = \frac{2\sigma}{\sigma - 1} \text{ and } \kappa < \frac{1}{\sigma - 1}.$$

Assume now that (1.11) is given with some non-negative p and κ, and determine for which $\sigma > 1$ the inequality (1.1) has no positive solutions. Set

$$p_\sigma = \frac{2\sigma}{\sigma - 1}, \quad \kappa_\sigma = \frac{1}{\sigma - 1}.$$

If either $p < p_\sigma$ or $p = p_\sigma$ and $\kappa < \kappa_\sigma$ then (1.11) implies

$$\mu (B(x_0, r)) \le C r^{p_\sigma} \log^{\kappa_\sigma - \varepsilon} r$$

for some $\varepsilon > 0$, whence the absence of positive solutions of (1.10) follows. In terms of σ, the above conditions are satisfied in any of the three cases:

1. $p \leq 2$, $\sigma > 1$, κ is any,
2. $p > 2$, $\sigma < \frac{p}{p-2}$, κ is any,
3. $p > 2$, $\sigma = \frac{p}{p-2}$, and $\kappa < \frac{p}{2} - 1$.

For example, if $M = \mathbb{R}^n$ then (1.11) holds with $p = n$ and $\kappa = 0$. If $n \leq 2$ then (1.10) has no positive solutions for any $\sigma > 1$ (in fact, for any real σ), and if $n > 2$ then (1.10) has no positive solutions for $\sigma \leq \frac{n}{n-2}$, as it was already mentioned above.

2 Proof of the capacity tests

Here we prove Theorems 1.1 and 1.2, using the approach of Kurta [8]. Let u be a positive solution of (1.1). We first obtain some estimates of u without using specific hypotheses of Theorems 1.1 and 1.2. Fix some function $\varphi \in Lip_c(M)$, such that $0 \leq \varphi \leq 1$, constants $0 < t \leq 1$, $s \geq 2$, and take in (1.2) the test function $\psi = u^{-t}\varphi^s$. Clearly, ψ has compact support and is bounded, due to the local boundedness of u^{-1}. We have

$$\nabla\psi = -tu^{-t-1}\varphi^s\nabla u + su^{-t}\varphi^{s-1}\nabla\varphi,$$

whence it is clear that $|\nabla\psi| \in L^2(M)$ and, consequently, $\psi \in W_c^1(M)$. We obtain from (1.2) that

$$t\int_M |\nabla u|_A^2 u^{-t-1}\varphi^s d\mu + \int_M u^{\sigma-t}\varphi^s V d\mu \leq s\int_M (\nabla u, \nabla\varphi)_A u^{-t}\varphi^{s-1}d\mu. \tag{2.1}$$

By the Cauchy-Schwarz inequality, we estimate the right hand side of (2.1) as follows:

$$\begin{aligned} s\int_M (\nabla u, \nabla\varphi)_A u^{-t}\varphi^{s-1}d\mu &= \int_M \left(\sqrt{t}u^{-\frac{t+1}{2}}\varphi^{s/2}\nabla u, \frac{s}{\sqrt{t}}u^{-\frac{t-1}{2}}\varphi^{s/2-1}\nabla\varphi\right)_A d\mu \\ &\leq \frac{t}{2}\int_M |\nabla u|_A^2 u^{-t-1}\varphi^s d\mu \\ &\quad +\frac{s^2}{2t}\int_M |\nabla\varphi|_A^2 u^{1-t}\varphi^{s-2}d\mu. \end{aligned}$$

Substituting into (2.1) and cancelling out the half of the first term in (2.1), we obtain

$$\frac{t}{2}\int_M |\nabla u|_A^2 u^{-t-1}\varphi^s d\mu + \int_M u^{\sigma-t}\varphi^s V d\mu \leq \frac{s^2}{2t}\int_M |\nabla\varphi|_A^2 u^{1-t}\varphi^{s-2}d\mu. \tag{2.2}$$

In what follows, assume that $0 < t < 1$ and set

$$\alpha = \frac{\sigma - t}{1-t}, \quad \beta = \frac{\sigma - t}{\sigma - 1}$$

so that α and β are Hölder conjugate. Applying the Young inequality in the form

$$\int fg\,d\mu \leq \int |f|^{\alpha}\,d\mu + \int |g|^{\beta}\,d\mu$$

we estimate the right hand side of (2.2) as follows:

$$\begin{aligned}
\frac{s^2}{2t}\int_M |\nabla\varphi|_A^2\,u^{1-t}\varphi^{s-2}d\mu &= \frac{1}{2}\int_M \left[u^{1-t}\varphi^{\frac{s}{\alpha}}V^{\frac{1}{\alpha}}\right]\left[\frac{s^2}{t}\,|\nabla\varphi|_A^2\,\varphi^{\frac{s}{\beta}-2}V^{-\frac{1}{\alpha}}\right]d\mu \\
&\leq \frac{1}{2}\int_M u^{\sigma-t}\varphi^s V\,d\mu \\
&\quad +\frac{1}{2}\left(\frac{s^2}{t}\right)^{\frac{\sigma-t}{\sigma-1}}\int_M |\nabla\varphi|_A^{2\frac{\sigma-t}{\sigma-1}}\,\varphi^{s-2\frac{\sigma-t}{\sigma-1}}V^{-\frac{1-t}{\sigma-1}}d\mu.
\end{aligned}$$

Now we substitute this estimate into (2.2), using also that

$$\left(\frac{s^2}{t}\right)^{\frac{\sigma-t}{\sigma-1}} \leq \left(\frac{s^2}{t}\right)^{\frac{\sigma}{\sigma-1}},$$

and

$$\varphi^{s-2\frac{\sigma-t}{\sigma-1}} \leq 1$$

provided

$$s > \frac{2\sigma}{\sigma-1},$$

which will be assumed in the sequel. Noticing that a half of the middle term in (2.2) cancels out and multiplying by 2, we obtain

$$t\int\limits_M |\nabla u|_A^2\,u^{-t-1}\varphi^s d\mu + \int\limits_M u^{\sigma-t}\varphi^s V\,d\mu \leq \left(\frac{s^2}{t}\right)^{\frac{\sigma}{\sigma-1}}\int\limits_M |\nabla\varphi|_A^{2\frac{\sigma-t}{\sigma-1}}\,V^{-\frac{1-t}{\sigma-1}}d\mu. \tag{2.3}$$

Proof of Theorem 1.1. Let K be a compact set from (1.5) and let φ be a test function from the class $\mathcal{I}(K)$. Applying (2.3) with this function φ and taking infimum in φ on the right hand side, we obtain

$$\begin{aligned}
\int_K u^{\sigma-t}V\,d\mu &\leq \left(\frac{s^2}{t}\right)^{\frac{\sigma}{\sigma-1}}\operatorname{cap}_{2\frac{\sigma-t}{\sigma-1},\frac{1-t}{\sigma-1}}(K) \\
&= c_{s,\sigma}\varepsilon^{-p/2}\operatorname{cap}_{p-2\varepsilon,q-\varepsilon}(K), \qquad (2.4)
\end{aligned}$$

where $\varepsilon = \frac{t}{\sigma-1}$. Letting $\varepsilon \to 0$ and using the hypothesis (1.5), we see that the right hand side here goes to 0, whence

$$\int_K u^{\sigma}V\,d\mu = 0,$$

which contradicts the positivity of u and V. ■

Proof of Theorem 1.2. Let K be a compact set from (1.6). If $0 < \varepsilon < q$ then set $t = \varepsilon(\sigma - 1)$ so that $0 < t < 1$. Then the right hand side of (2.4) vanishes due to (1.6), whence we again obtain the contradiction.

If $\varepsilon = q$ then $t = 1$ and (2.3), (2.4) do not apply. In this case the condition (1.6) becomes $\operatorname{cap}_2(K) = 0$, which implies that any positive solution of the inequality (1.3) is constant. Hence, (1.1) has no positive solution. Alternatively, we obtain from (2.2) with $s = 2$ that

$$\int_M u^{\sigma-1}\varphi^2 V d\mu \le 2\int_M |\nabla\varphi|_A^2 \, d\mu. \tag{2.5}$$

The hypothesis $\operatorname{cap}_2(K) = 0$ implies that the infimum of the right hand side of (2.5) over all $\varphi \in \mathcal{T}(K)$ is equal to 0, which finishes the proof. ■

The condition (1.6) of Theorem 1.2 can be replaced by the following assumption: there is a constant $C > 0$ such that for any compact set $K \subset M$,

$$\operatorname{cap}_{p-2\varepsilon,q-\varepsilon}(K) \le C. \tag{2.6}$$

Indeed, using certain properties of capacities (cf. [6, Lemma 2.5]), it is possible to show that (2.6) implies (1.6).

3 Examples to the capacity test

In this section, we set $M = \mathbb{R}^n$, $n > 2$, μ is the Lebesgue measure, $A(x) = (a_{ij}(x))$ where $a_{ij} \in L^\infty(\mathbb{R}^n)$, and $V(x) \equiv 1$. Set $\sigma = \frac{n}{n-2}$ which is the critical exponent for the problem (1.1). Let B_R be the Euclidean ball of radius R centered at the origin. Let us use the following expression for the Euclidean capacity (see [2], [9]): for any $s \in (1, n)$,

$$\inf_{\varphi\in\mathcal{I}(B_R)} \int_{\mathbb{R}^n} |\nabla\varphi|^s \, d\mu = (n-s)^{s-1} \frac{\omega_n}{(s-1)^s} R^{n-s},$$

where ω_n is the area of the unit sphere in $\mathbb{R}^n$. Since for the above value of σ we have $p = n$, and $\|A\|$ is uniformly bounded, we obtain for $s = p - 2\varepsilon = n - 2\varepsilon$ that

$$\begin{aligned}
\operatorname{cap}_{p-2\varepsilon,q-\varepsilon}(B_R) &\le C \inf_{\varphi\in\mathcal{I}(B_R)} \int_{\mathbb{R}^n} |\nabla\varphi|^{n-2\varepsilon} \, d\mu \\
&= C(2\varepsilon)^{n-2\varepsilon-1} \frac{\omega_n}{(n-2\varepsilon-1)^{n-2\varepsilon}} R^{2\varepsilon} \\
&= O(\varepsilon^n) \text{ as } \varepsilon \to 0.
\end{aligned}$$

The condition (1.5), that is,

$$\operatorname{cap}_{p-2\varepsilon,q-\varepsilon}(B_R) = o\left(\varepsilon^{n/2}\right),$$

is obviously satisfied, and we obtain by Theorem 1.1 that (1.1) has no positive solution. This result was previously known for positive definite matrices $A(x)$.

Let us show that one cannot set $\varepsilon = 0$ in Theorem 1.1; that is, the condition

$$\operatorname{cap}_{p,q}(K) = 0$$

does not necessarily imply the non-existence of positive solutions. Before we can state an example supporting this claim, let us cite the following theorem of Atkinson.

Proposition 3.1 (Atkinson [1]) *Let $\sigma > 1$ be a constant and $\beta(x)$ be a continuous function on $(x_0, +\infty)$ such that*

$$\int^{\infty} x\,|\beta(x)|\,dx < \infty. \tag{3.1}$$

Then there exists a positive solution $y(x)$ to the differential equation

$$y'' + \beta(x)\,y^{\sigma} = 0$$

in an interval $(x_1, +\infty)$ with a large enough x_1, such that

$$y(x) \to 1 \text{ and } y'(x) \to 0 \text{ as } x \to +\infty.$$

We will use the following generalization of Proposition 3.1.

Proposition 3.2 *Let $\alpha(x)$ be a positive C^1-function on $(x_0, +\infty)$ such that*

$$\int^{\infty} \frac{dx}{\alpha(x)} < \infty. \tag{3.2}$$

Define the function $\gamma(x)$ on $(x_0, +\infty)$ by

$$\gamma(x) = \int_x^{\infty} \frac{ds}{\alpha(s)}.$$

Let $\beta(x)$ be a continuous function on $(x_0, +\infty)$ such that

$$\int^{\infty} \gamma(x)^{\sigma}\,|\beta(x)|\,dx < \infty. \tag{3.3}$$

Then the differential equation

$$(\alpha(x)\,y')' + \beta(x)\,y^{\sigma} = 0 \tag{3.4}$$

has a positive solution $y(x)$ on an interval $(x_1, +\infty)$ for large enough x_1, such that

$$y(x) \sim \gamma(x) \text{ as } x \to +\infty.$$

Proof. Introducing an independent variable $z=\frac{1}{\gamma(x)}$ and a function $u(z)=y(x)\,z$, we obtain by the chain rule that

$$\frac{d^2u}{dz^2}+\widetilde{\beta}(z)\,u^{\alpha}=\alpha\gamma^3\frac{d}{dx}\left(\alpha\frac{dy}{dx}\right)+\frac{\widetilde{\beta}}{\gamma^{\sigma}}y^{\sigma},$$

so that (3.4) is equivalent to the equation

$$\frac{d^2u}{dz^2}+\widetilde{\beta}(z)\,u^{\alpha}=0$$

with $\widetilde{\beta}(z)=\alpha\gamma^{\sigma+3}\beta$. By Proposition 3.1, this equation has a positive solution in a neighborhood of $+\infty$ provided

$$\int^{\infty}z\left|\widetilde{\beta}(z)\right|dz<\infty. \tag{3.5}$$

By (3.2), $z\to\infty$ is equivalent to $x\to\infty$. Since $dz=-\frac{\gamma'}{\gamma^2}dx=\frac{1}{\alpha\gamma^2}dx$, the condition (3.5) becomes

$$\int^{\infty}\frac{1}{\gamma}\left|\beta(x)\right|\alpha\gamma^{\sigma+3}\frac{1}{\alpha\gamma^2}dx<\infty,$$

which coincides with (3.3). Finally, by Proposition 3.1, there is a solution $u(z)\sim 1$ as $z\to\infty$, which implies $y(x)\sim\gamma(x)$ as $x\to\infty$. ■

Our purpose here is to construct in $\mathbb{R}^n$ a positive solution $u(x)$ of the inequality

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(a(r)\frac{\partial u}{\partial x_i}\right)+u^{\sigma}\leq 0, \tag{3.6}$$

where $\sigma=\frac{n}{n-2}$, r is the polar radius in $\mathbb{R}^n$ and the function $a(r)$ is a positive constant for small r and

$$a(r)=\log^k r\ \text{ for large } r,$$

where k can be any constant such that

$$k>\frac{n-2}{n}. \tag{3.7}$$

Since $p=n$ and $V\equiv 1$, the corresponding capacity is given by

$$\operatorname{cap}_{p,q}(K)=\operatorname{cap}_n(K)=\inf_{\varphi\in\mathcal{T}(K)}\int_{\mathbb{R}^n}a^{n/2}(r)\left|\nabla\varphi\right|^n dx.$$

Evaluation of this capacity by the variational method shows that, for any ball B_R centered at the origin,

$$\operatorname{cap}_n(B_R)=c_n\left(\int_R^{\infty}\frac{dr}{\left(a^{n/2}(r)\,r^{n-1}\right)^{\frac{1}{n-1}}}\right)^{1-n},$$

where $c_n > 0$. Hence, $\mathrm{cap}_n(B_R) = 0$ if and only if

$$\frac{nk}{2(n-1)} \leq 1. \tag{3.8}$$

Clearly, there is k such that the both conditions (3.7) and (3.8) are satisfied. With this k, we obtain an example, where $\mathrm{cap}_{p,q}(K) = 0$ for any compact set K, whereas the inequality (3.6) has a positive solution.

We construct such a solution as a function of r only, so we write $u = u(r)$. Writing u' and a' for the derivative in r and using that $\frac{\partial x_i}{\partial r} = \frac{x_i}{r}$, one easily obtains

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a(r)\frac{\partial u}{\partial x_i}\right) &= au'' + a'u' + \frac{(n-1)a}{r}u' \\ &= r^{1-n}\left(a(r)r^{n-1}u'\right)'. \end{aligned}$$

Hence, (3.6) is equivalent to

$$\left(a(r)r^{n-1}u'\right)' + r^{n-1}u^{\alpha} \leq 0. \tag{3.9}$$

The condition (3.2) of Proposition (3.2) is obviously satisfied. The function $\gamma(r)$ is given for large r by

$$\gamma(r) = \int_r^{\infty} \frac{ds}{s^{n-1}a(s)} = \int_r^{\infty} \frac{ds}{s^{n-1}\log^k s} \simeq r^{-(n-2)}\log^{-k} r.$$

The condition (3.3) with $\beta(r) = r^{n-1}$ is satisfied provided

$$\int^{\infty} r^{-\sigma(n-2)}\log^{-\sigma k} r^{n-1} dr = \int^{\infty} \frac{dr}{r\log^{\alpha k} r} < \infty,$$

which is exactly the case when $k > \frac{1}{\sigma}$, which is the same as (3.7). By Proposition 3.2, there is a positive solution $u(r)$ to (3.9) in some interval $[r_0, +\infty)$ such that

$$u(r) \sim \gamma(r) \simeq r^{-(n-2)}\log^{-k} r \ \text{ as } r \to \infty,$$

in particular, $u(r) \to 0$ as $r \to \infty$. By increasing r_0 if necessary, we can assume that $u'(r_0) < 0$. For small values of r, namely for $r \leq \xi$ where ξ will be specified later on, the function $a(r)$ will be defined to be a constant, whose value will also be determined later.

So far consider the linear equation

$$v'' + \frac{n-1}{r}v' + \varepsilon v = 0$$

that has a solution $v(r)$ with the initial conditions

$$v(0) = 1, \quad v'(0) = 0.$$

This solution is positive and decreasing for $r < r_\varepsilon$ for some positive r_ε and vanishes at r_ε; moreover, $r_\varepsilon \to \infty$ as $\varepsilon \to 0$. Since $0 < v \le 1$ in $(0, r_\varepsilon)$, it follows that v is a positive solution in $(0, r_\varepsilon)$ of the inequality

$$v'' + \frac{n-1}{r} v' + \varepsilon v^\sigma \le 0. \tag{3.10}$$

Choose ε so small that $r_\varepsilon > r_0$ and

$$\frac{v'}{v}(r_0) > \frac{u'}{u}(r_0). \tag{3.11}$$

This is possible to achieve because for small enough ε the function $v(r)$ is nearly constant 1 up to $2r_0$ and $v'(r_0)$ can be made arbitrarily close to 0 (although negative), whereas $u'(r_0) < 0$ by construction.

Compare the functions $u(r)$ and $v(r)$ in the interval $[r_0, r_\varepsilon)$. Set

$$c = \inf_{r \in [r_0, r_\varepsilon)} \frac{u(r)}{v(r)}.$$

Since $u(r)/v(r) \to \infty$ as $r \to r_\varepsilon$, the value c is attained at some point, say $\xi \in [r_0, r_\varepsilon)$. We claim that $\xi > r_0$. Indeed, at $r = r_0$ we have by (3.11)

$$\left(\frac{u}{v}\right)'(r_0) = \frac{u'v - uv'}{v^2}(r_0) < 0$$

so that $u(r)/v(r)$ takes smaller values for some $r > r_0$. Hence, the minimum point ξ is contained in an open interval (r_0, r_ε), and at this point we have

$$\left(\frac{u}{v}\right)'(\xi) = 0.$$

It follows that

$$u(\xi) = cv(\xi) \quad \text{and} \quad u'(\xi) = cv'(\xi). \tag{3.12}$$

Now we extend/redefine the function $u(r)$ for $r < \xi$ by setting $u(r) = cv(r)$. It follows from (3.10) that u satisfies in $(0, \xi]$ the inequality

$$u'' + \frac{n-1}{r} u' + \frac{\varepsilon}{c^{\sigma-1}} u^\alpha \le 0.$$

Hence, setting $a(r) \equiv c^{\sigma-1}/\varepsilon$ in $[0, \xi]$, we obtain that u satisfies (3.6) for $r \le \xi$. Since u satisfies (3.6) also for $r \ge \xi$ and by (3.12) u is differentiable at ξ, we obtain that u is a weak solution of (3.6) in $\mathbb{R}^n$.

4 Proof of the volume test

Here we prove Theorem 1.3, using Theorem 1.1. Using the obvious inequality

$$|\nabla\varphi|_A \le \|A\|^{1/2} |\nabla\varphi|, \tag{4.1}$$

where $|\nabla\varphi|$ is the Riemannian length of the gradient $\nabla\varphi$, and setting in (2.3) $\varepsilon = \frac{t}{\sigma-1}$, we see that the integral in the right hand side of (2.3) can be estimated as follows:

$$\begin{aligned}\int_M |\nabla\varphi|_A^{p-2\varepsilon} V^{-(q-\varepsilon)} d\mu &\le \int_M |\nabla\varphi|^{p-2\varepsilon} \|A\|^{p/2-\varepsilon} V^{-(q-\varepsilon)} d\mu \\ &= \int_M |\nabla\varphi|^{p-2\varepsilon} d\nu_\varepsilon. \end{aligned} \tag{4.2}$$

Next we apply the following result: for any Radon measure ν on a complete Riemannian manifold, for any $s > 1$ and for any ball $B_r = B(x_0, r)$,

$$\inf_{\varphi\in\mathcal{T}(B_r,M)} \int_M |\nabla\varphi|^s d\nu \le C_s \left(\int_r^\infty \left(\frac{\rho}{\nu(B_\rho)}\right)^{\frac{1}{s-1}} d\rho\right)^{1-s} \tag{4.3}$$

(see [4], [5], [7], [9, section 2.2.2, Lemma 1]). The constant C_s is locally uniformly bounded in the interval $s \in (1, +\infty)$. The range of s that we are interested in is $s \approx p$ so that we can assume C_s is uniformly bounded from above independently of s.

Applying (4.3) with $\nu = \nu_\varepsilon$ and $s = p - 2\varepsilon$ and combining with (4.2), we obtain

$$\operatorname{cap}_{p-2\varepsilon,q-\varepsilon}(B_r) \le C \left(\int_r^\infty \left(\frac{\rho}{\nu_\varepsilon(B_\rho)}\right)^{\frac{1}{p-1-2\varepsilon}} d\rho\right)^{1+2\varepsilon-p}.$$

The condition (1.5) of Theorem 1.1 will be satisfied provided

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{p}{2(p-1)}} \int_r^\infty \left(\frac{\rho}{\nu_\varepsilon(B_\rho)}\right)^{\frac{1}{p-1-2\varepsilon}} d\rho = \infty.$$

In the view of the hypothesis (1.7), it suffices to show that

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{p}{2(p-1)}} \int_r^\infty \rho^{-\frac{p-1+C\varepsilon}{p-1-2\varepsilon}} (\log\rho)^{-\frac{\kappa}{p-1-2\varepsilon}} d\rho = \infty, \tag{4.4}$$

where r can be assumed large enough (but fixed). Making change $\rho = e^t$ and setting $\delta = \frac{(C+2)\varepsilon}{p-1-2\varepsilon}$ we obtain that the integral in (4.4) is equal to

$$\int_{\log r}^\infty \exp(-\delta t)\, t^{-\frac{\kappa}{p-1-2\varepsilon}} dt = \delta^{\frac{\kappa}{p-1-2\varepsilon}-1} \int_{\delta\log r}^\infty \exp(-\tau)\, \tau^{-\frac{\kappa}{p-1-2\varepsilon}} d\tau. \tag{4.5}$$

When $\varepsilon \to 0$, the right hand side of (4.5) is of the order

$$\text{const } \varepsilon^{\frac{\kappa}{p-1}-1}$$

where const is a positive constant. Hence, the expression under the limit in (4.4) is of the order

$$\varepsilon^{\frac{p}{2(p-1)}+\frac{\kappa}{p-1}-1}.$$

By the hypothesis, we have $\kappa < q = p/2 - 1$, which implies that the exponent of ε here is negative, which proves (4.4).

Remark. Assume that M is geodesically complete and consider the following measure

$$d\nu = \|A\| d\mu.$$

Clearly, we have

$$\mathrm{cap}_2(K) = \inf_{\varphi \in \mathcal{T}(K)} \int |\nabla \varphi|_A^2 \, d\mu \leq \inf_{\varphi \in \mathcal{T}(K)} \int_M |\nabla \varphi|^2 \, d\nu.$$

Using the estimate (4.3), we obtain that if

$$\int^{\infty} \frac{r dr}{\nu (B_r)} = \infty \tag{4.6}$$

then $\mathrm{cap}_2(K) = 0$. As it was remarked in Section 1, the latter implies that (1.1) has no positive solution regardless of V and σ. The condition (4.6) is satisfied if, for example,

$$\nu (B_r) \leq C r^2 \tag{4.7}$$

for all large r.

5 Examples to the volume test

Consider in this section the setting $M = \mathbb{R}^n$,

$$V(x) \simeq r^{-\alpha_1} \log^{-\alpha_2} r \quad \text{and} \quad \|A(x)\| \simeq r^{\beta_1} \log^{\beta_2} r \tag{5.1}$$

as $r := |x| \to \infty$, where α_i, β_i are real constants.

If $\beta_1 < 2 - n$ then it is easy to verify that the condition (4.7) of Remark 4 is satisfied and, hence, there is no positive solution to (1.1) for any σ and V.

Assume in the sequel that

$$\beta_1 + n - 2 > 0.$$

For functions $V(x)$ and $\|A(x)\|$ from (5.1), the condition (1.9) is obviously satisfied. Hence, the hypothesis (1.7) of Theorem 1.3 can be replaced by (1.8). Let us estimate $\nu_0 (B_R)$ where B_R is the ball of radius R centered at the origin. We have, for large R, that

$$\begin{aligned} \nu_0 (B_R) &\simeq \int_2^R \left(r^{\beta_1} \log^{\beta_2} r\right)^{p/2} \left(r^{\alpha_1} \log^{\alpha_2} r\right)^q r^{n-1} dr \\ &= \int_2^R r^{\frac{\alpha_1 + \beta_1 \sigma}{\sigma - 1} + n - 1} (\log r)^{\frac{\alpha_2 + \beta_2 \sigma}{\sigma - 1}} dr \\ &\leq C R^{\frac{\alpha_1 + \beta_1 \sigma}{\sigma - 1} + n} (\log R)^{\frac{\alpha_2 + \beta_2 \sigma}{\sigma - 1}}. \end{aligned}$$

The condition (1.8) is satisfied in the two cases (in all cases $\sigma > 1$):

1. either
$$\frac{\alpha_1+\beta_1\sigma}{\sigma-1}+n<\frac{2\sigma}{\sigma-1},$$

2. or
$$\frac{\alpha_1+\beta_1\sigma}{\sigma-1}+n=\frac{2\sigma}{\sigma-1} \text{ and } \frac{\alpha_2+\beta_2\sigma}{\sigma-1}<\frac{1}{\sigma-1}.$$

Solving these inequalities, we obtain that (1.8) is satisfied and, hence, (1.1) has no positive solutions, provided one of the following two cases takes place:

1. $\sigma<\sigma^*:=\frac{n-\alpha_1}{\beta_1+n-2}$.

2. $\sigma=\sigma^*$ and $\alpha_2+\beta_2\sigma<1$.

Assuming that $\sigma^*>1$, let us show that in the opposite case

$$\sigma=\sigma^*, \quad \alpha_2+\beta_2\sigma>1 \tag{5.2}$$

a positive solution to (1.1) does exist, which will show the sharpness of the volume test of Theorem 1.3.

The construction uses Proposition 3.2 and is similar to the example in Section 3. Assuming (5.2), we will construct a positive solution in $\mathbb{R}^n$ to the inequality

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(a\left(r\right)\frac{\partial u}{\partial x_i}\right)+V\left(r\right)u^{\sigma}\leq 0, \tag{5.3}$$

where $r=|x|$,

$$a\left(r\right)=r^{\beta_1}\log^{\beta_2}r \text{ and } V\left(r\right)=r^{-\alpha_1}\log^{-\alpha_2}r \text{ for large } r.$$

In the polar coordinates, the inequality (5.3) becomes

$$\left(a\left(r\right)r^{n-1}u'\right)'+r^{n-1}V\left(r\right)u^{\alpha}\leq 0. \tag{5.4}$$

The condition (3.2) of Proposition 3.2 becomes

$$\int^{\infty}\frac{dr}{r^{\beta_1+n-1}\log^{\beta_2}r}<\infty,$$

which is true due to $\beta_1+n-2>0$. Setting

$$\gamma\left(r\right)=\int_r^{\infty}\frac{ds}{a\left(s\right)s^{n-1}}=\int^{\infty}\frac{ds}{s^{\beta_1+n-1}\log^{\beta_2}s}\simeq r^{-(\beta_1+n-2)}\log^{-\beta_2}r,$$

we see that the condition (3.3) of Proposition 3.2 is equivalent to

$$\int^{\infty}\left(r^{-(\beta_1+n-2)}\log^{-\beta_2}r\right)^{\sigma}r^{n-1}\left(r^{-\alpha_1}\log^{-\alpha_2}r\right)dr<\infty,$$

which by $\sigma = \frac{n-\alpha_1}{\beta_1+n-2}$ is equivalent to

$$\int^{\infty} r^{-1} \log^{-(\alpha_2+\sigma\beta_2)} r dr < \infty.$$

The latter is obviously satisfied due to $\alpha_2 + \sigma\beta_2 > 1$. We conclude by Proposition 3.2 that (5.4) has a positive solution $u(r)$ in a neighborhood of $+\infty$, such that

$$u(r) \simeq r^{-(\beta_1+n-2)} \log^{-\beta_2} r \text{ as } r \to \infty.$$

Arguing further as in Section 3, one extends this function to be a solution of (5.3) on $\mathbb{R}^n$.

Similarly, one can show the existence of a positive solution of (5.3) in the case $\sigma > \sigma^*$.

References

[1] **Atkinson F.V.**, On second order nonlinear oscillations, *Pacif. J. Math.*, **5** (1955) no.1, 643–647.

[2] **Adams R.**, "Sobolev spaces", Academic Press, 1975.

[3] **Gidas B., Spruck J.**, Global and local behavior of positive solutions of nonlinear elliptic equations, *Comm. Pure Appl. Math.*, **34** (1981) 525–598.

[4] **Grigor'yan A.**, On the existence of positive fundamental solution of the Laplace equation on Riemannian manifolds, (in Russian) *Matem. Sbornik*, **128** (1985) no.3, 354-363. Engl. transl.: *Math. USSR Sb.*, **56** (1987) 349-358.

[5] **Grigor'yan A.**, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, *Bull. Amer. Math. Soc.*, **36** (1999) 135-249.

[6] **Grigor'yan A., Netrusov Yu., Yau S.-T.**, Eigenvalues of elliptic operators and geometric applications, *in:* "Eigenvalues of Laplacians and other geometric operators", Surveys in Differential Geometry IX, (2004) 147-218.

[7] **Holopainen I.**, Volume growth, Green's functions and parabolicity of ends, *Duke Math. J.*, **97** (1999) no.2, 319-346.

[8] **Kurta V.V.**, On the absence of positive solutions to semilinear elliptic equations, (in Russian) *Tr. Mat. Inst. Steklova*, **227** (1999) 162–169. Engl. transl.: *Proc. Steklov Inst. Math.*, **227** (1999) no.4, 155–162.

[9] **Maz'ya V.G.**, "Sobolev spaces", Springer, 1985.

[10] **Mitidieri E., Pokhozhaev S.I.**, Apriori estimates and the absence of solutions of nonlinear partial differential equations and inequalities, *Tr. Math. Inst. Steklova*, (in Russian) **234** (2001) 1-384. Engl. transl.: *Proc. Steklov Inst. Math.*, **234** (2001) no.3, 1–362.