Sharp propagation rate for solutions of Leibenson's equation on Riemannian manifolds

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Abstract

We consider on arbitrary Riemannian manifolds the Leibenson equation $\partial_t u = \Delta_p u^q$. This equation is also known as doubly nonlinear evolution equation, and it comes from hydrodynamics where it describes filtration of a turbulent compressible liquid in porous medium. It was proved by the authors in [15] that if $q(p-1) > 1$ then solutions to this equation have finite propagation speed. In this paper obtain a sharp estimate of the propagation rate of solutions, although under additional restrictions on p, q.

Contents

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1 Introduction

We are concerned here with a non-linear evolution equation

$$\partial_t u = \Delta_p u^q \tag{1.1}$$

where $p > 1$, $q > 0$, $u = u(x,t)$ is an unknown non-negative function, and Δ_p is the p-Laplacian

$$\Delta_p v = \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right).$$

For the physical meaning of (1.1) see [24, 25, 15].

The equation (1.1) is referred to as a *Leibenson* equation or a *doubly non-linear parabolic* equation. In the case $q = 1$, it becomes an evolutionary *p-Laplace* equation $\partial_t u = \Delta_p u$, and if in addition $p = 2$ then it amounts to the classical heat equation $\partial_t u = \Delta u$.

Barenblatt [3] constructed spherically symmetric self-similar solutions of (1.1) in $\mathbb{R}^n$, that are nowadays called *Barenblatt solutions.* If

$$q(p-1) > 1 \tag{1.2}$$

then the Barenblatt solution $u(x,t)$ has the property that

$$u(x,t) = 0 \ \text{ whenever } \ |x| > ct^{1/\beta},$$

where

$$\beta = p + n[q(p-1) - 1] \tag{1.3}$$

and c is a large enough constant (see also Proposition 5.1); thus, $u(\cdot,t)$ has a compact support for any $t > 0$. One says in this case that u has a *finite propagation speed,* and the *propagation rate* is given by $ct^{1/\beta}$.

On the other hand, if $q(p-1) \le 1$, then the Barenblatt solution is positive for all $x \in \mathbb{R}^n$ and $t > 0$, which means an *infinite propagation speed.*

In [15] the authors proved that, under condition (1.2), solutions of (1.1) have finite propagation speed also on an arbitrary Riemannian manifold (in the case $q = 1$ this was also proved in [7]). However, the estimate of the rate of propagation in [15] was not optimal.

The purpose of this paper is to obtain better estimates for propagation rate for solutions of (1.1) on Riemannian manifolds although under the additional restrictions

$$p > 2, \quad \frac{1}{p-1} < q \le 1. \tag{1.4}$$

Moreover, if in addition

$$q < \frac{2}{p-1}, \tag{1.5}$$

then our estimate of the propagation rate is sharp for a large class of manifolds (including $\mathbb{R}^n$).

From now on let M be a geodesically complete Riemannian manifold. We understand solutions of (1.1) in $M \times \mathbb{R}_+$ in a certain weak sense (see Section 2 for the definition). The main result of the present paper is as follows (cf. Theorem 4.1).

Theorem 1.1. *Assume that (1.4) is satisfied and let u be a bounded non-negative solution to (1.1) in $M \times \mathbb{R}_+$ with an initial function $u_0 = u(\cdot,0)$. Let σ be a real such that*

$$\sigma \ge 1 \text{ and } \sigma > q(p-1) - 1. \tag{1.6}$$

If u_0 vanishes in a geodesic ball B_0 in M of radius R then

$$u = 0 \quad \text{in } \frac{1}{2}B_0 \times [0, t_0],$$

where

$$t_0 = \eta\mu(B_0)^{\frac{q(p-1)-1}{\sigma}} R^p ||u_0||_{L^\sigma(M)}^{-[q(p-1)-1]}, \tag{1.7}$$

and the constant $\eta > 0$ depends on the intrinsic geometry of B_0.

Hence, the solution u has a finite propagation speed inside ball B_0, and the rate of propagation is determined by t_0 that depends on the intrinsic geometry of B_0 via the constant η.

Let us mention for comparison that a similar result was obtain in [15] but with a different value of t_0:

$$t_0 = \eta R^p ||u_0||_{L^\infty(M)}^{-[q(p-1)-1]}, \tag{1.8}$$

(the same value of t_0 was obtained also in [7] in the case $q = 1$). Clearly, (1.8) matches (1.7) with $\sigma = \infty$, and (1.7) gives a larger value of t_0 for $\sigma < \infty$ as it takes into account the volume $\mu(B_0)$.

The value of t_0 from (1.8) leads to the following estimate of the propagation rate: if $K = \text{supp } u_0$ is compact, then

$$\text{supp } u(\cdot, t) \subset K_{ct^{1/p}}$$

while in $\mathbb{R}^n$ the sharp estimate is

$$\text{supp } u(\cdot, t) \subset K_{ct^{1/\beta}} \tag{1.9}$$

where $\beta > p$ is given by (1.3). The value of t_0 from (1.7) leads in $\mathbb{R}^n$ to the sharp result (1.9) provided p and q satisfy (1.4) and (1.5), which allows to choose $\sigma = 1$ in (1.7).

Of course, Theorem 1.1 allows us to obtain a sharp propagation rate also on a larger class of Riemannian manifolds.

Corollary 1.2. *Let M satisfy a relative Faber-Krahn inequality (see Section 3 for definition). Assume that (1.4) is satisfied and let u be a bounded non-negative solution in $M \times \mathbb{R}_+$ with the initial condition $u(\cdot, 0) = u_0$; set $K = \operatorname{supp} u_0$. Assume that, for some $x_0 \in K$, $\alpha > 0$ and all large enough r,*

$$\mu(B(x_0, r)) \geq cr^\alpha, \tag{1.10}$$

where $c > 0$. Then, for all $t > 0$,

$$\operatorname{supp} u(\cdot, t) \subset K_{Ct^{1/\beta}},$$

where

$$\beta = p + \alpha\frac{q(p-1) - 1}{\sigma} \tag{1.11}$$

with σ as in (1.6) and the constant C depends on $\|u_0\|_{L^\sigma}, p, q, \alpha, c$.

For example, this result applies on all manifolds of non-negative Ricci curvature as the relative Faber-Krahn inequality is satisfied on such manifolds (see [5, 12, 31]).

In $\mathbb{R}^n$ we have (1.10) with $\alpha = n$. Comparing the values of β in (1.3) and (1.11) we see that Corollary 1.2 gives a sharp propagation rate in $\mathbb{R}^n$ provided $\sigma = 1$. By (1.6), we can take $\sigma = 1$ if $q(p-1) - 1 < 1$, which is equivalent to (1.5).

In Proposition 5.1 we show that the propagation rate of Corollary 1.2 is sharp also in a class spherically symmetric (model) manifolds under the above restrictions on p and q.

Let us discuss the differences in methods of the proof of finite propagation speed in [15] and the present paper and how they yield different rates of propagation. Even though, in both papers, the finite propagation speed follows from a certain non-linear mean value inequality for solutions, these mean value inequalities are different and their proofs are carried out in entirely different ways.

Let us first discuss the mean value inequality of the present paper (cf. Lemma 3.2), which says the following. Assume that (1.4) holds. Let u be a non-negative bounded subsolution of (1.1) in a cylinder

$$Q = B \times [0, t]$$

where B is a geodesic ball in M of radius R. Assume that $u(\cdot, 0) = 0$ in B. Then, for the cylinder

$$Q' = \frac{1}{2} B \times [0, t]$$

we have

$$\|u\|_{L^\infty(Q')} \leq \left(\frac{C_B}{\mu(B) R^p} \int_Q u^\sigma \right)^{1/\lambda}, \tag{1.12}$$

where $\lambda > 0, \sigma = \lambda + q(p-1) - 1$, and C_B depends on the intrinsic geometry of the ball B, namely, on a *Faber-Krahn inequality* in B (see Section 5).

The mean value inequality (1.12) allows to get the recursive estimate

$$J_{k+1} \leq C_B 2^{k/\lambda} \left(\frac{t}{R^p} \right)^{1/\lambda} J_k^{\frac{\sigma}{\lambda}}, \tag{1.13}$$

for the integrals $J_k = \int_{Q_k} u^\sigma$, where Q_k is a certain sequence of shrinking cylinders interpolating between Q and Q'. Iterating (1.13) and using that $\sigma > \lambda$, we obtain then a super-exponential decay of J_k provided $t \leq t_0$ (where t_0 given by (1.7)), which leads to the proof of Theorem 1.1.

In contrast to (1.12), the mean value inequality of [15] says that, under the above assumptions,

$$\|u\|_{L^\infty(Q')} \leq \left(\frac{C_B}{\mu(B) R^p} ||u||_{L^\infty(Q)}^{q(p-1)-1} \int_Q u^\lambda \right)^{1/\lambda}, \tag{1.14}$$

where again $\lambda > 0$. However, one obtains from (1.14) only the recursive estimate (1.13) for $J_k = ||u||_{L^\infty(Q_k)}$, which in the end leads to (1.8) and hence, to the non-optimal propagation rate.

Let us also make some comments on the differences in the proofs of the mean value inequalities (1.12) and (1.14).

The mean value inequality (1.14) was proved by the authors in [15] using a modification of the Moser iteration method [28]. In the present paper we use a different approach based on the following observation, which is interesting in its own right: if u is a non-negative subsolution of (1.1), then the function

$$(u^a - \theta)_+^{1/a} \tag{1.15}$$

is also a subsolution of (1.1), provided $\theta \geq 0$ and

$$a := \frac{q(p-1) - 1}{p - 2} \in (0, 1] \tag{1.16}$$

(cf. Lemma 2.6). In particular, the condition $a \in (0, 1]$ in (1.16) is satisfied provided (1.4) holds. The proof of (1.12) employs then a modification of the classical De Giorgi iteration argument [6]. Namely, we consider a shrinking sequence of cylinders $\{Q_k\}_{k=0}^{\infty}$ interpolating between $Q_0 = Q$ and $Q_\infty = Q'$, and a sequence of truncated functions

$$u_k = \left(u^a - \left(1 - 2^{-k}\right)\theta\right)_+^{1/a}, \quad k \geq 0,$$

for some fixed $\theta > 0$, where a is given by (1.16). Using a *Caccioppoli type inequality* (Lemma 2.8) and the Faber-Krahn inequality, we prove that, for $J_k = \int_{Q_k} u_k^\sigma$,

$$J_{k+1} \leq \frac{CA^k}{\left(\mu(B)\theta^{\frac{\lambda}{a}}R^p\right)^\nu} J_k^{1+\nu}, \tag{1.17}$$

where A, C are some positive constants and the exponent $\nu > 0$ comes from the Faber-Krahn inequality in B (see Lemma 3.1 for details). Iterating (1.17), we then show that if

$$\theta \geq \left(\frac{CJ_0}{\mu(B)R^p}\right)^{\frac{a}{\lambda}}, \tag{1.18}$$

then $J_k \to 0$ for $k \to \infty$, which implies

$$\int_{Q'} \left[(u^a - \theta)_+^{1/a}\right]^\sigma = 0,$$

and hence $u^a \leq \theta$ in Q'. Choosing θ minimal from (1.18), we conclude (1.12).

Note that if $q = 1$ then $a = 1$ by (1.16). In this case, the fact that $(u-\theta)_+$ is a subsolution, was known before, and it was used to obtain similar mean value inequalities for subsolutions of the p-Laplacian in [9, 11] in $\mathbb{R}^n$ and in [7] on manifolds.

For mean value inequalities in various other settings see also [1, 14, 17]. Related results from the theory of the p-Laplace equation can be found, for instance, in [8, 10, 20, 21]. See also [2, 27, 30, 32] for other results about the asymptotic behaviour of solutions of (1.1).

The structure of the paper is as follows.

In Section 2, we define the notion of a weak solution of the Leibenson equation (1.1). In this section we prove in Lemma 2.6 that the truncated function $(u^a - \theta)_+^{1/a}$ is again a subsolution.

In Section 3 we prove the central technical result of this paper $-$ the mean value inequality for subsolutions (Lemma 3.2).

In Section 4 we prove our main results about finite propagation speed.

In Section 5 (Appendix) we construct the exact solutions of (1.1) on the model manifolds (generalizing the Barenblatt solutions) that show sharpness of our estimates of propagation rate.

2 Weak subsolutions

2.1 Definition and basic properties

We consider in what follows the following evolution equation on a Riemannian manifold M:

$$\partial_t u = \Delta_p u^q. \tag{2.1}$$

By a *subsolution* of (2.1) we mean a non-negative function u satisfying

$$\partial_t u \leq \Delta_p u^q \tag{2.2}$$

in a certain weak sense as explained below.

We assume throughout that

$$p > 1 \quad \text{and} \quad q > 0.$$

Set

$$\delta = (p-1)q - 1.$$

Let μ denote the Riemannian measure on M. For simplicity of notation, we frequently omit in integrations the notation of measure. All integration in M is done with respect to $d\mu$, and in $M \times \mathbb{R}$ – with respect to $d\mu dt$, unless otherwise specified.

Let Ω be an open subset of M and I be an interval in $[0, \infty)$.

Definition 2.1. We say that a non-negative function $u = u(x,t)$ is a *weak subsolution* of (2.1) in $\Omega \times I$, if

$$u \in L^{\infty}_{loc}\left(I; L^1(\Omega)\right) \text{ and } u^q \in L^p_{loc}\left(I; W^{1,p}(\Omega)\right) \tag{2.3}$$

and (2.2) holds weakly in $\Omega \times I$, that is, for and all non-negative *test functions*

$$\psi \in W^{1,\infty}_{loc}\left(I; L^{\infty}(\Omega)\right) \cap L^p_{loc}\left(I; W^{1,p}_0(\Omega)\right), \tag{2.4}$$

and for all $t_1, t_2 \in I$ with $t_1 < t_2$, we have

$$\left[\int_\Omega u\psi\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega -u\partial_t\psi + |\nabla u^q|^{p-2}\langle \nabla u^q, \nabla \psi\rangle \leq 0. \tag{2.5}$$

For different notions of weak solutions see also [10, 33]. Existence and uniqueness results for the Cauchy problem with the above notion of weak solutions of (2.1) were obtained in the euclidean case, for example, in [18, 19, 23, 29] and on manifolds in [16].

If u is of the class (2.3), we define

$$\nabla u := \begin{cases} q^{-1}u^{1-q}\nabla(u^q), & u > 0, \\ 0, & u = 0. \end{cases}$$

Remark 2.2. Note that it follows from (2.3) and (2.4) that the integrals in (2.5) are finite. Indeed, we have by Hölder's inequality

$$\begin{aligned} \int_{t_1}^{t_2}\int_\Omega |\nabla u^q|^{p-2}\left|\langle \nabla u^q, \nabla\psi\rangle\right| &\leq \int_{t_1}^{t_2}\int_\Omega |\nabla u^q|^{p-1}|\nabla\psi| \\ &\leq \left(\int_{t_1}^{t_2}\int_\Omega \left(|\nabla u^q|\right)^p\right)^{\frac{p-1}{p}}\left(\int_{t_1}^{t_2}\int_\Omega |\nabla\psi|^p\right)^{\frac{1}{p}}. \end{aligned}$$

From now on in this section, let $I = [0, T)$, where $0 < T \leq \infty$.

Definition 2.3. Let $u = u(x,t)$ be a measurable function in $\Omega \times [0,T)$ and $u(\cdot, 0) = u_0$. Then we define, for $h \in (0,T)$,

$$u^h(\cdot, t) = \frac{1}{h}\int_0^t e^{(s-t)/h}u(\cdot, s)ds$$

and

$$u_h(\cdot, t) = e^{-t/h}u_0 + \frac{1}{h}\int_0^t e^{(s-t)/h}u(\cdot, s)ds.$$

The properties of u^h and u_h in the following Lemma are proved in Lemma 2.2 in [22] and in Lemma B.1 and Lemma B.2 in [4].

Lemma 2.4. *Let $p \geq 1$ and suppose that $u \in L^p(\Omega \times [0,T))$. Then*

$$||u^h||_{L^p(\Omega\times[0,T))} \leq ||u||_{L^p(\Omega\times[0,T))}$$

and

$$||u_h||_{L^p(\Omega\times[0,T))} \leq ||u||_{L^p(\Omega\times[0,T))} + h^{1/p}||u_0||_{L^p(\Omega)},$$

Moreover, $u^h \to u$ and $u_h \to u$ in $L^p(\Omega \times [0,T))$ as $h \to 0$ and

$$\partial_t u_h = \frac{1}{h}(u - u_h) \in L^p(\Omega \times [0,T)). \tag{2.6}$$

Lemma 2.5. *[15] Let Ω be an open subset of M and $u = u(x,t)$ be a non-negative bounded weak subsolution of (2.1) in $\Omega \times [0,T)$. Then*

$$\int_0^\tau \int_\Omega (\partial_t u_h)\psi + \langle [|\nabla u^q|^{p-2}\nabla u^q]^h, \nabla\psi\rangle \leq 0, \tag{2.7}$$

for all $\tau \in (0,T)$ and $\psi \in L^p\left([0,\tau]; W_0^{1,p}(\Omega)\right)$.

Lemma 2.6. *Let u be a non-negative bounded weak subsolution of (2.1) in $\Omega\times[0,T)$. Assume that either*

$$p > 2 \quad \text{and} \quad \frac{1}{p-1} < q \leq 1 \quad \text{or} \quad 1 < p < 2 \quad \text{and} \quad 1 \leq q < \frac{1}{p-1}. \tag{2.8}$$

For any $\theta \geq 0$, define

$$f(s) = (s^a - \theta)_+^{1/a},$$

where

$$a = \frac{q(p-1)-1}{p-2} = \frac{\delta}{p-2}. \tag{2.9}$$

Then $f(u)$ is also a weak subsolution of (2.1).

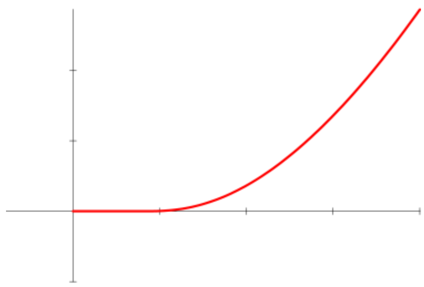

Figure 1: Function $f(s)$

Remark 2.7. For the p-Laplacian, that is when $q = 1$, we have $a = 1$. In this case, it is proved in [7] that $f(u) = (u - \theta)_+$ is a subsolution of (2.1).

Proof. On $\{s^a > \theta\}$ we have

$$f'(s) = \left(\frac{f(s)}{s}\right)^{1-a}. \tag{2.10}$$

Noticing that the condition (2.8) is equivalent to $0 < a \leq 1$, we obtain that f is locally Lipschitz in $[0,\infty)$ and in particular, f is continuously differentiable when $0 < a < 1$. Consider in $[0,\infty)$ also the function $\Phi(s) = \left(s^{\frac{a}{q}} - \theta\right)_+^{q/a}$. By (2.8), $q - a = \frac{1-q}{p-2} \geq 0$, so that using the same arguments as for f, Φ is also a locally Lipschitz function. Because $\Phi(0) = 0$, it follows that $f(u)^q(\cdot,t) = \Phi(u^q)(\cdot,t) \in W^{1,p}(\Omega)$ for all $t \in [0,T)$, which proves that $f(u)$ is in the class (2.3).

Hence, it remains to show that $f(u)$ satisfies (2.5), that is,

$$\left[\int_\Omega f(u)\psi\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega -f(u)\partial_t\psi + |\nabla f(u)^q|^{p-2}\langle\nabla f(u)^q, \nabla\psi\rangle \leq 0, \tag{2.11}$$

for all ψ in the class (2.4).

On $\{u^a > \theta\}$ we have

$$\nabla f(u)^q = \Phi'(u^q)\nabla u^q = \left(\frac{f(u)}{u}\right)^{q-a}\nabla u^q. \tag{2.12}$$

and thus,

$$|\nabla f(u)^q|^{p-2}\nabla f(u)^q = \left(\frac{f(u)}{u}\right)^{(q-a)(p-1)}|\nabla u^q|^{p-2}\nabla u^q.$$

Since $(q-a)(p-1) = 1-a$ the inequality (2.11) is therefore equivalent to

$$\left[\int_\Omega f(u)\psi\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_\Omega -f(u)\partial_t\psi + f'(u)|\nabla u^q|^{p-2}\langle\nabla u^q, \nabla\psi\rangle \leq 0. \tag{2.13}$$

Clearly, the fact that $0 < a \leq 1$ implies on $\{s^a > \theta\}$,

$$\begin{aligned} f''(s) &= (1-a)\left(f(s)^{1-2a}s^{2a-2} - s^{a-2}f(s)^{1-a}\right) \\ &= (1-a)f(s)^{1-a}s^{a-2}\left(\left(\frac{s}{f(s)}\right)^a - 1\right) \geq 0. \end{aligned}$$

Let us consider, for $\nu < \frac{1}{4}(t_2 - t_1)$, the function

$$\theta_\nu(t) = \begin{cases} 0, & t < t_1, \\ \frac{1}{\nu}(t - t_1), & t_1 \leq t < t_1 + \nu, \\ 1, & t_1 + \nu \leq t < t_2 - \nu, \\ \frac{1}{\nu}(t_2 - t), & t_2 - \nu \leq t < t_2, \\ 0, & t \geq t_2 \end{cases} \tag{2.14}$$

(cf. [26]). In order to prove (2.13), we want to apply (2.7) with test function

$$\widetilde{\psi}_k = f_k'(u)\psi\theta_\nu,$$

where ψ is a bounded function of the class (2.4) and f_k is a sequence of $C^2([0,\infty))$ functions such that

$$f_k \to f \text{ and } f_k' \to f' \text{ as } k \to \infty$$

and, for all k,

$$f_k'' \geq 0 \text{ and } f_k''(s) = 0 \text{ on } \{f(s) = 0\} = \{s^a \leq \theta\}.$$

For that, let us first show that for all k, $\widetilde{\psi}_k(\cdot,t) \in W_0^{1,p}(\Omega)$ for all fixed t. Indeed, we have $\widetilde{\psi}_k(\cdot,t) \in L^p(\Omega)$ since $\psi(\cdot,t) \in L^p(\Omega)$ and on the other hand,

$$\nabla\widetilde{\psi}_k = f_k'(u)\theta_\nu\nabla\psi + f_k''(u)\psi\theta_\nu\nabla u.$$

Using $\nabla\psi \in L^p(\Omega)$ and

$$f_k''(u)\psi\nabla u = q^{-1} f_k''(u)\psi\theta_\nu u^{1-q}\nabla u^q \in L^p(\Omega),$$

where the latter holds because f_k'' is bounded on bounded subsets of $[0,\infty)$, $f_k''(u) = 0$ on $\{u = 0\} \subset \{f = 0\}$ and $\nabla u^q \in L^p$, we get $\widetilde{\psi}_k \in W_0^{1,p}(\Omega)$.

Hence, applying (2.7) with $\widetilde{\psi} = f_k'(u)\psi\theta_\nu$, we deduce

$$\int_Q (\partial_t u_h) f_k'(u)\psi\theta_\nu + \langle [|\nabla u^q|^{p-2}\nabla u^q]^h, \nabla(f_k'(u)\psi)\rangle\theta_\nu \leq 0,$$

where $Q = [t_1, t_2] \times \Omega$. Let us write

$$\int_Q \partial_t u_h f_k'(u)\psi\theta_\nu = \int_Q \partial_t u_h f_k'(u_h)\psi\theta_\nu + \int_Q \partial_t u_h (f_k'(u) - f_k'(u_h))\psi\theta_\nu.$$

By (2.6), we see that

$$\int_Q \partial_t u_h (f_k'(u) - f_k'(u_h))\psi\theta_\nu = \frac{1}{h}\int_Q (u - u_h)(f_k'(u) - f_k'(u_h))\psi\theta_\nu \geq 0,$$

because $s \mapsto f_k'(s)$ is non-decreasing.

Whence, we obtain

$$\int_Q \partial_t u_h f_k'(u_h)\psi\theta_\nu + \langle [|\nabla u^q|^{p-2}\nabla u^q]^h, \nabla(f_k'(u)\psi)\rangle\theta_\nu \leq 0. \tag{2.15}$$

By using

$$\int_Q \partial_t u_h f_k'(u_h)\psi\theta_\nu = \int_Q \partial_t (f_k(u_h))\psi\theta_\nu = \left[\int_\Omega f_k(u_h)\psi\theta_\nu\right]_{t_1}^{t_2} - \int_Q f_k(u_h)\partial_t\psi\theta_\nu - \int_Q f_k(u_h)\psi\partial_t\theta_\nu,$$

we get, since $\theta_\nu(t_1) = \theta_\nu(t_2) = 0$,

$$-\int_Q f_k(u_h)\psi\partial_t\theta_\nu + \int_Q \langle [|\nabla u^q|^{p-2}\nabla u^q]^h, \nabla(f_k'(u)\psi)\rangle\theta_\nu - f_k(u_h)\partial_t\psi\theta_\nu \leq 0. \tag{2.16}$$

We now want to let $h \to 0$ in (2.16) and apply Lemma 2.4 and then let $\nu \to 0$. Note that $|\nabla u^q|^{p-1} \in L^{\frac{p}{p-1}}(Q)$, so that by Lemma 2.4, for $h \to 0$,

$$[|\nabla u^q|^{p-2}\nabla u^q]^h \to |\nabla u^q|^{p-2}\nabla u^q \quad \textit{in } L^{\frac{p}{p-1}}(Q).$$

Together with $|\nabla(f_k'(u)\psi)|\theta_\nu \in L^p(Q)$, we obtain

$$\lim_{h\to 0}\int_Q \langle [|\nabla u^q|^{p-2}\nabla u^q]^h, \nabla(f_k'(u)\psi)\rangle\theta_\nu = \int_Q \langle [|\nabla u^q|^{p-2}\nabla u^q], \nabla(f_k'(u)\psi)\rangle\theta_\nu.$$

For the convergence of the remaining terms in (2.16), we will use the boundedness of u. Note that by assumption $u \in L^1(Q)$ whence Lemma 2.4 implies that $u_h \to u$ in $L^1(Q)$. Since the function $s \mapsto f_k(s)$ is Lipschitz on any bounded subset of $[0,\infty)$, we get $f_k(u_h) \to f_k(u)$ in $L^1(Q)$ and thus,

$$\lim_{h\to 0}\int_Q f_k(u_h)\partial_t\psi\theta_\nu = \int_Q f_k(u)\partial_t\psi\theta_\nu.$$

The convergence

$$\lim_{h\to 0}\int_Q f_k(u_h)\psi\partial_t\theta_\nu = \int_Q f_k(u)\psi\partial_t\theta_\nu$$

follows by the same arguments. Hence,

$$-\int_Q f_k(u)\psi\partial_t\theta_\nu + \int_Q \langle[|\nabla u^q|^{p-2}\nabla u^q], \nabla(f_k'(u)\psi)\rangle\theta_\nu - f_k(u)\partial_t\psi\theta_\nu \le 0.$$

Sending now $\nu \to 0$, we deduce

$$\left[\int_\Omega f_k(u)\psi\right]_{t_1}^{t_2} + \int_Q \langle[|\nabla u^q|^{p-2}\nabla u^q], \nabla(f_k'(u)\psi)\rangle - f_k(u)\partial_t\psi \le 0.$$

Using that

$$\nabla(f_k'(u)\psi) = f_k'(u)\nabla\psi + q^{-1}f_k''(u)\psi u^{1-q}\nabla u^q,$$

we get

$$\begin{aligned}\int_Q |\nabla u^q|^{p-2}\langle\nabla u^q, \nabla(f_k'(u)\psi)\rangle &= \int_\Omega |\nabla u^q|^{p-2}\left(\langle\nabla u^q, f_k'(u)\nabla\psi\rangle + q^{-1}\langle\nabla u^q, f_k''(u)\psi u^{1-q}\nabla u^q\rangle\right)\\ &= \int_Q f_k'(u)|\nabla u^q|^{p-2}\langle\nabla u^q, \nabla\psi\rangle + q^{-1}|\nabla u^q|^p f_k''(u)\psi u^{1-q}.\end{aligned}$$

Noticing that

$$q^{-1}\int_Q |\nabla u^q|^p f_k''(u)\psi u^{1-q} \ge 0,$$

we obtain

$$\left[\int_\Omega f_k(u)\psi\right]_{t_1}^{t_2} + \int_Q f_k'(u)|\nabla u^q|^{p-2}\langle\nabla u^q, \nabla\psi\rangle - f_k(u)\partial_t\psi \le 0.$$

Using that $f_k' \to f' \in C([0,\infty))$ implies $f_k' \to f'$ in L^∞ on bounded sets, we get that

$$\lim_{k\to\infty}\int_Q f_k'(u)|\nabla u^q|^{p-2}\langle\nabla u^q, \nabla\psi\rangle = \int_Q f'(u)|\nabla u^q|^{p-2}\langle\nabla u^q, \nabla\psi\rangle.$$

Since $f(u) \le u \in L^1$ and $f_k \in C^2$, there is a function g so that $|f_k(u)| \le g(u) \in L^1$, whence

$$\lim_{k\to\infty}\left[\int_\Omega f_k(u)\psi\right]_{t_1}^{t_2} = \left[\int_\Omega f(u)\psi\right]_{t_1}^{t_2}$$

and

$$\lim_{k\to\infty}\int_Q f_k(u)\partial_t\psi = \int_Q f(u)\partial_t\psi$$

by the dominated convergence theorem. This proves (2.13) and finishes the proof. ■

Lemma 2.8. *[15] Let $v = v(x,t)$ be a non-negative bounded subsolution to (2.1) in a cylinder $\Omega\times[0,T)$. Let $\eta(x,t)$ be a locally Lipschitz non-negative bounded function in $\Omega\times[0,T)$ such that $\eta(\cdot,t)$ has compact support in Ω for all $t\in[0,T)$. Fix some real σ such that*

$$\sigma \ge \max(p, pq) \tag{2.17}$$

and set

$$\lambda = \sigma - \delta \quad \text{and} \quad \alpha = \frac{\sigma}{p}. \tag{2.18}$$

Choose $0 \le t_1 < t_2 < T$ and set $Q = \Omega\times[t_1,t_2]$. Then

$$\left[\int_\Omega v^\lambda\eta^p\right]_{t_1}^{t_2} + c_1\int_Q |\nabla(v^\alpha\eta)|^p \le \int_Q \left[pv^\lambda\eta^{p-1}\partial_t\eta + c_2 v^\sigma|\nabla\eta|^p\right], \tag{2.19}$$

where c_1, c_2 are positive constants depending on p, q, λ.

In particular, if η does not depend on t, then

$$\left[\int_\Omega v^\lambda \eta^p\right]_{t_1}^{t_2} + c_1 \int_Q |\nabla (v^\alpha \eta)|^p \le c_2 \int_Q v^\sigma |\nabla \eta|^p . \tag{2.20}$$

Let us recall for later that

$$v^\alpha \eta \in L^p_{loc}\left([0,T]; W_0^{1,p}(\Omega)\right). \tag{2.21}$$

Indeed, using $\alpha \ge q$, we get that the function $\Phi(s) = s^{\frac{\alpha}{q}}$ is Lipschitz on any bounded interval in $[0,\infty)$. Thus, $v^\alpha = \Phi(v^q) \in W^{1,p}(\Omega)$ and

$$|\nabla v^\alpha| = \left|\Phi'(v^q)\nabla v^q\right| \le C\, |\nabla v^q| ,$$

whence

$$\int_Q |\nabla (v^\alpha \eta)|^p \le C' \int_Q |\nabla v^\alpha|^p \,\eta^p + v^{\alpha p}|\nabla \eta|^p = C' \int_Q |\nabla v^q|^p \,\eta^p + v^\sigma |\nabla \eta|^p,$$

which is finite since

$$\int_Q v^\sigma |\nabla \eta|^p \le \text{const } ||v||_{L^\infty}^{\sigma - pq} \int_Q v^{pq}$$

and proves (2.21).

2.2 Norm decay of subsolutions

Lemma 2.9. *Let M be geodesically complete and $v = v(x,t)$ be a bounded non-negative subsolution to* (2.1) *in $M \times [0,T)$. If $\lambda \ge 1$, including $\lambda = \infty$, then the function*

$$t \mapsto \|v(\cdot,t)\|_{L^\lambda(M)}$$

is monotone decreasing in $[0,T)$.

Proof. Let f_k be a sequence of non-negative locally Lipschitz functions in $[0,\infty)$ such that for all $k \ge 0$, $f_k(0) = 0$ and $f_k' \ge 0$.

We want to apply (2.7) with test function $\psi_k = f_k(v^q)\theta_\nu$, where $\theta_\nu(t)$ is defined by (2.14). Indeed, since f_k is Lipschitz on bounded subsets of $[0,\infty)$, $f_k(0) = 0$ and $v \in L^p(M \times [0,\tau])$, we have

$$f_k(v^q)\theta_\nu \in L^p(M \times [0,\tau]).$$

Therefore, using that

$$v^q(\cdot,t) \in W^{1,p}(M) = W_0^{1,p}(M)$$

by the completeness of M, we see that

$$f_k(v^q) \in (\cdot,t) \in W_0^{1,p}(M)$$

and

$$\nabla(f_k(v^q)) = f_k'(v^q)\nabla v^q. \tag{2.22}$$

Hence, applying (2.7) with this test function, we get

$$\int_Q \partial_t v_h f_k(v^q)\theta_\nu + \langle [|\nabla v^q|^{p-2}\nabla v^q]^h, \nabla(f_k(v^q))\rangle \theta_\nu \le 0,$$

where $Q = M \times [t_1, t_2]$. Let us write

$$\int_Q \partial_t v_h f_k(v^q)\theta_\nu = \int_Q \partial_t v_h f_k(v_h^q)\theta_\nu + \int_Q \partial_t v_h (f_k(v^q) - f_k(v_h^q))\theta_\nu.$$

By (2.6), we deduce

$$\int_Q \partial_t v_h (f_k(v^q) - f_k(v_h^q))\theta_\nu = \frac{1}{h}\int_Q (v - v_h)(f_k(v^q) - f_k(v_h^q))\theta_\nu \geq 0,$$

since $s \mapsto f_k(s^q)$ is non-decreasing. Whence, we obtain

$$\int_Q \partial_t v_h f_k(v_h^q)\theta_\nu + \langle [|\nabla v^q|^{p-2}\nabla v^q]^h, \nabla(f_k(v^q))\rangle\theta_\nu \leq 0. \tag{2.23}$$

Setting

$$\varphi_k(u) = \int_0^u f_k(s^q)ds, \tag{2.24}$$

we get

$$\int_Q \partial_t v_h f_k(v_h^q)\theta_\nu = \int_Q \partial_t \varphi_k(v_h)\theta_\nu = \left[\int_M \varphi_k(v_h)\theta_\nu\right]_{t_1}^{t_2} - \int_Q \varphi_k(v_h)\partial_t\theta_\nu.$$

Since $\theta_\nu(t_1) = \theta_\nu(t_2) = 0$, we obtain

$$-\int_Q \varphi_k(v_h)\partial_t\theta_\nu + \int_Q \langle [|\nabla v^q|^{p-2}\nabla v^q]^h, \nabla(f_k(v^q))\rangle\theta_\nu \leq 0. \tag{2.25}$$

We now want to let $h \to 0$ in (2.25) and apply Lemma 2.4. Note that

$$|\nabla v^q|^{p-1} \in L^{\frac{p}{p-1}}(Q),$$

so that by Lemma 2.4, for $h \to 0$,

$$[|\nabla v^q|^{p-2}\nabla v^q]^h \to |\nabla v^q|^{p-2}\nabla v^q \quad in\ L^{\frac{p}{p-1}}(Q).$$

Together with $|\nabla(f_k(v^q))|\theta_\nu \in L^p(Q)$, we obtain

$$\lim_{h\to 0}\int_Q \langle [|\nabla v^q|^{p-2}\nabla v^q]^h, \nabla(f_k(v^q))\rangle\theta_\nu = \int_Q \langle |\nabla v^q|^{p-2}\nabla v^q, \nabla(f_k(v^q))\rangle\theta_\nu.$$

For the convergence of the remaining term in (2.25) we have, since $v \in L^1(Q)$,

$$\int_Q |\varphi_k(v_h) - \varphi_k(v)| = \int_Q \left|\int_v^{v_h} f_k(s^q)ds\right| \leq C\int_Q |v_h - v| \to 0 \quad \text{for } h \to 0$$

and thus,

$$\lim_{h\to 0}\int_Q \varphi_k(v_h)\partial_t\theta_\nu = \int_Q \varphi_k(v)\partial_t\theta_\nu.$$

Hence, we obtain from (2.25),

$$-\int_Q \varphi_k(v)\partial_t\theta_\nu + \int_Q \langle |\nabla v^q|^{p-2}\nabla v^q, \nabla(f_k(v^q))\rangle\theta_\nu \leq 0.$$

By (2.22), we have

$$\int_Q \langle |\nabla v^q|^{p-2}\nabla v^q, \nabla(f_k(v^q))\rangle\theta_\nu = \int_Q |\nabla v^q|^p f_k'(v^q)\theta_\nu \geq 0,$$

so that by sending $\nu \to 0$ we get

$$\left[\int_M \varphi_k(v)\right]_{t_1}^{t_2} \le 0.$$

Choosing f_k such that for all $s > 0$, $f_k(s) \to s^{\frac{\lambda-1}{q}}$ for $k \to \infty$, we obtain from (2.24), $\varphi_k(v) \to v^\lambda$ as $k \to \infty$. Also noticing that $\varphi_k(v) \le Cv$, we conclude

$$\left[\int_M v^\lambda\right]_{t_1}^{t_2} \le 0,$$

which finishes the proof. ■

3 Mean value inequality

Let M be a connected Riemannian manifold of dimension n. Let d be the geodesic distance on M. For any $x \in M$ and $r > 0$, denote by $B(x,r)$ the geodesic ball of radius r centered at x, that is,

$$B(x,r) = \{y \in M : d(x,y) < r\}.$$

3.1 Faber-Krahn inequality

Let the geodesic ball B be precompact. Then the following *Faber-Krahn inequality* in B of order $p \ge 1$ holds: if $w \in W_0^{1,p}(B)$ is non-negative and

$$D = \{w > 0\}$$

then

$$\int_B |\nabla w|^p \ge \frac{1}{r^p}\left(\iota(B)\frac{\mu(B)}{\mu(D)}\right)^\nu \int_B w^p, \tag{3.26}$$

where $\nu > 0$ and $\iota(B)$ is a positive constant that depends on the geometry of B. In fact, the value of ν is independent of B and can be chosen as follows:

$$\nu = \begin{cases} \dfrac{p}{n}, & \text{if } n > p, \\ \text{any number} \in (0,1), & \text{if } n \le p. \end{cases} \tag{3.27}$$

Choosing $\iota(B)$ to be an optimal constant in (3.26) and denoting by $r(B)$ the radius of a ball B, we obtain that the function

$$B \mapsto \frac{(\iota(B)\mu(B))^\nu}{r(B)^p} \tag{3.28}$$

is monotone decreasing with respect to the partial order $\subset$ on balls.

We say that M satisfies a *relative Faber-Krahn inequality* of order p if (3.26) holds with $\iota(B) \ge \text{const} > 0$ for all geodesic balls $B \subset M$. This holds for example, if M is a complete manifold with non-negative Ricci curvature (see [5, 12, 31]).

3.2 Comparison in two cylinders

We assume here that

$$p > 2 \quad \text{and} \quad \frac{1}{p-1} < q \le 1 \quad \text{or} \quad 1 < p < 2 \quad \text{and} \quad 1 \le q < \frac{1}{p-1}. \tag{3.29}$$

Let a be defined by (2.9), that is,

$$a=\frac{q(p-1)-1}{p-2}=\frac{\delta}{p-2}. \tag{3.30}$$

Observe that under condition (3.29) we have $a\in(0,1]$.

Lemma 3.1. *Consider two balls $B_0=B(x_0,r_0)$ and $B_1=B(x_0,r_1)$ with $0<r_1<r_0$, and two cylinders*

$$Q_i=B_i\times[0,T].$$

Assume that B_0 is precompact. Let v_0 be non-negative bounded subsolution in Q_0 such that

$$v_0(\cdot,0)=0. \tag{3.31}$$

Set, for some $\theta>0$,

$$v_1=(v_0^a-\theta)_+^{1/a},$$

where a as in (3.30). Let λ and σ be reals satisfying (2.17) and (2.18). Set also

$$J_i=\int_{Q_i}v_i^\sigma d\mu dt.$$

Then

$$J_1\le\frac{Cr_0^p}{\left(\iota(B_0)\mu(B_0)\theta^{\frac{\lambda}{a}}(r_0-r_1)^p\right)^\nu(r_0-r_1)^p}J_0^{1+\nu}. \tag{3.32}$$

where ν is the Faber-Krahn exponent, $\iota(B_0)$ is the Faber-Krahn constant in B_0 and C depends on p, q and λ.

Proof. From Lemma 2.6 we know that v_1 is also a subsolution. Let $\eta(x,t)=\eta(x)$ be a bump function of B_1 in $B_{1/2}=B\left(x_0,\frac{r_0+r_1}{2}\right)$. Recall that by (2.21),

$$v_1^\alpha\eta\in L^p\left([0,T];W_0^{1,p}(B)\right),$$

where α is defined by (2.18), that is $\alpha=\frac{\sigma}{p}$. Hence, applying the Faber-Krahn inequality (3.26) in ball B_0 for any $t\in[0,T]$ we get that

$$\int_{B_1}v_1^\sigma\le\int_{B_0}(v_1^\alpha\eta)^p\le r_0^p\left(\frac{\mu(D_t)}{\iota(B_0)\mu(B_0)}\right)^\nu\int_{B_0}|\nabla(v_1^\alpha\eta)|^p, \tag{3.33}$$

where we used that $\alpha p=\sigma$ and $\eta=1$ on B_1 and

$$D_t=\{v_1^\alpha\eta(\cdot,t)>0\}=\{v_1>0\}\cap\{\eta>0\}=\left\{v_0(\cdot,t)>\theta^{1/a}\right\}\cap B_{1/2}.$$

We have $\eta_t=0$ and $|\nabla\eta|\le\frac{2}{r_0-r_1}$. From (2.20) we therefore obtain

$$c_1\int_0^T\int_{B_0}|\nabla(v_1^\alpha\eta)|^p\le\int_0^T\int_{B_0}v_1^\sigma|\nabla\eta|^p\le\frac{c_3}{(r_0-r_1)^p}J_0, \tag{3.34}$$

where $c_3=c_2 2^p$ and we used that $v_1\le v_0$.

Let us now apply Lemma 2.8 to function v_0 in $B_0\times[0,t]$ where $t\in[0,T]$. This time we take $\eta(x,t)=\eta(x)$ as a bump function of $B_{1/2}=B\left(x_0,\frac{r_0+r_1}{2}\right)$ in B_0. From (2.20) we obtain

$$\left[\int_{B_0}v_0^\lambda\eta^p\right]_0^t\le c_2\int_0^t\int_{B_0}|\nabla\eta|^p v_0^\sigma\le\frac{c_3}{(r_0-r_1)^p}\int_0^t\int_{B_0}v_0^\sigma\le\frac{c_3}{(r_0-r_1)^p}J_0.$$

Hence, by (3.31),

$$\int_{B_{1/2}} v_0^{\lambda}(\cdot, t) \leq \frac{c_3}{(r_0 - r_1)^p} J_0.$$

Thus, we deduce

$$\mu(D_t) \leq \frac{1}{\theta^{\lambda/a}} \int_{B_{1/2}} v_0^{\lambda}(\cdot, t) \leq \frac{c_3}{\theta^{\lambda/a}(r_0 - r_1)^p} J_0.$$

Combining this with (3.33) and (3.34) we obtain

$$\begin{aligned} J_1 = \int_0^T \int_{B_1} v_1^{\sigma} &\leq \left(\frac{p}{2}\right)^p r_0^p \left(\frac{c_3 J_0}{\iota(B_0)\mu(B_0)\theta^{\lambda/a}(r_0 - r_1)^p}\right)^{\nu} \frac{c_3}{c_1 (r_0 - r_1)^p} J_0 \\ &= \left(\frac{p}{2}\right)^p \frac{r_0^p c_3^{1+\nu}}{\left(\iota(B_0)\mu(B_0)\theta^{\lambda/a}(r_0 - r_1)^p\right)^{\nu} c_1 (r_0 - r_1)^p} J_0^{1+\nu} \end{aligned}$$

which implies (3.32) and finishes the proof. ■

3.3 Iterations and the mean value theorem

Lemma 3.2. *Suppose that (3.29) is satisfied. Let the ball $B = B(x_0, r)$ be precompact. Let u be a non-negative bounded subsolution in*

$$Q = B \times [0, t]$$

such that

$$u(\cdot, 0) = 0 \text{ in } B.$$

Let σ and λ be reals such that

$$\sigma > 0 \quad \text{and} \quad \lambda = \sigma - \delta > 0. \tag{3.35}$$

Then, for the cylinder

$$Q' = \frac{1}{2} B \times [0, t],$$

we have

$$\|u\|_{L^{\infty}(Q')} \leq \left(\frac{C}{\iota(B)\mu(B) r^p} \int_Q u^{\sigma}\right)^{1/\lambda}, \tag{3.36}$$

where $\iota(B)$ is the Faber-Krahn constant in B, and the constant C depends on p, q and λ.

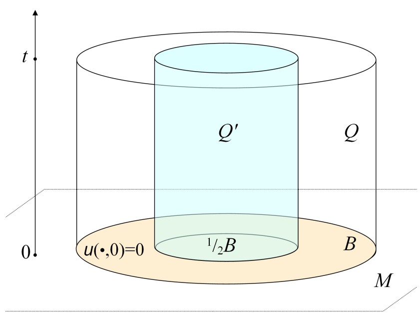

Figure 2: Cylinders Q and Q'

Proof. Let us first prove (3.36) for σ large enough as in Lemmas 2.8 and 3.1. Choose some $\theta > 0$ to be specified later and define a sequence of functions $\{u_k\}$ by

$$u_0 = u, \quad u_k = \left(u_{k-1}^a - 2^{-k}\theta\right)_+^{1/a} \text{ for } k \geq 1.$$

The function $f_\theta(s) = (s^a - \theta)_+^{1/a}$ has the property that $f_{\theta_1} \circ f_{\theta_2} = f_{\theta_1+\theta_2}$. Hence, we obtain

$$u_k = \left(u^a - \frac{1}{2}\theta - \ldots - \frac{1}{2^k}\theta\right)_+^{1/a} = \left(u^a - \left(1 - 2^{-k}\right)\theta\right)_+^{1/a}.$$

Consider a sequence $r_k = \left(\frac{1}{2} + 2^{-k-1}\right) r$, and set

$$B_k = B(x_0, r_k), \qquad Q_k = B_k \times [0, t]$$

so that

$$B_0 = B, \quad Q_0 = Q \quad \text{and} \quad Q_\infty := \lim_{k\to\infty} Q_k = Q'.$$

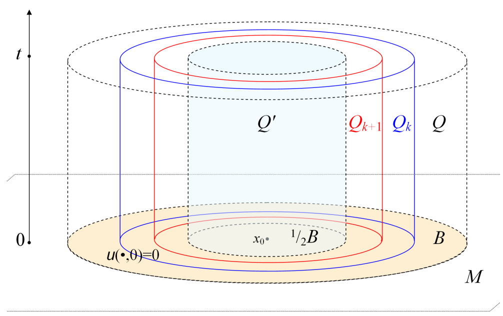

Figure 3: Cylinders Q_k

Setting $J_k = \int_{Q_k} u_k^\sigma$ we obtain by Lemma 3.1 that

$$J_{k+1} \leq \frac{C r_k^p}{\left(\iota(B_k)\mu(B_k)\left(2^{-(k+1)}\theta\right)^{\frac{\lambda}{a}}(r_k - r_{k+1})^p\right)^\nu (r_k - r_{k+1})^p} J_k^{1+\nu}.$$

Observe that, by monotonicity of the function (3.28), we have

$$\frac{r_k^p}{(\iota(B_k)\mu(B_k))^\nu} \leq \frac{r^p}{(\iota(B)\mu(B))^\nu}.$$

Since $r_k - r_{k+1} = 2^{-(k+2)}r$, we obtain

$$J_{k+1} \leq \frac{C 2^{(k+1)\frac{\lambda\nu}{a}} r^p}{\left(\iota(B)\mu(B)\theta^{\frac{\lambda}{a}}\left(2^{-(k+2)}r\right)^p\right)^\nu \left(2^{-(k+2)}r\right)^p} J_k^{1+\nu}$$
$$= \frac{C 2^{(k+1)\frac{\lambda\nu}{a} + (k+2)(p\nu+p)}}{\left(\iota(B)\mu(B)\theta^{\frac{\lambda}{a}} r^p\right)^\nu} J_k^{1+\nu} = \frac{A^k}{\Theta} J_k^{1+\nu},$$

where

$$A = 2^{\frac{\lambda\nu}{a} + (p\nu+p)} \quad \text{and} \quad \Theta = C^{-1}\left(\iota(B)\mu(B)\theta^{\frac{\lambda}{a}} r^p\right)^\nu.$$

Now let us apply Lemma 5.2 with $\omega = \nu$: if

$$\Theta \geq A^{1/\nu} J_0^\nu, \tag{3.37}$$

then, for all $k \geq 0$,

$$J_k \leq A^{-k/\nu} J_0.$$

In terms of θ the condition (3.37) is equivalent

$$C^{-1} \left(\iota(B)\mu(B)\theta^{\frac{\lambda}{a}} r^p \right)^\nu \geq A^{1/\nu} J_0^\nu$$

that is,

$$\theta \geq \left(\frac{CJ_0}{\iota(B)\mu(B)r^p} \right)^{\frac{a}{\lambda}},$$

where A is absorbed into a new constant C. Hence, we choose θ as follows:

$$\theta = \left(\frac{CJ_0}{\iota(B)\mu(B)r^p} \right)^{\frac{a}{\lambda}},$$

and for this choice of θ we have $J_k \to 0$, which implies $u^a \leq \theta$ in Q_∞. Hence, we obtain

$$\begin{aligned} \|u\|_{L^\infty(Q_\infty)} &\leq \left(\frac{CJ_0}{\iota(B)\mu(B)r^p} \right)^{1/\lambda} \\ &= \left(\frac{C}{\iota(B)\mu(B)r^p} \int_Q u^\sigma \right)^{1/\lambda}, \end{aligned} \tag{3.38}$$

which was to be proved.

Now we prove (3.36) for any σ so that (3.35) is satisfied. Let σ_0 be such that (3.36) is already known for $\sigma = \sigma_0$ and let $\sigma < \sigma_0$. Denote

$$\lambda_0 = \sigma_0 - \delta \quad \text{and} \quad \lambda = \sigma - \delta$$

so that $\lambda < \lambda_0$.

For simplicity of notation, for any set $E \subset M$, denote $E^t = E \times [0, t]$.

By the first part of the proof, we have, for any precompact ball B of radius r,

$$\|u\|^{\lambda_0}_{L^\infty(\frac{1}{2}B^t)} \leq \frac{C}{\chi(B)r^p} \int_{B^t} u^{\sigma_0},$$

where $\chi(B) = \iota(B)\mu(B)$. Consider for $k \geq 0$, a sequence

$$r_k = \left(1 - \frac{1}{2^{k+1}}\right) r,$$

so that $r_0 = \frac{1}{2}r$ and $r_k \uparrow r$ as $r \to \infty$, and set $B_k = B(x_0, r_k)$. Denoting also $B = B(x_0, r)$, we see that

$$\frac{1}{2}B \subset B_k \subset B \quad \text{and} \quad B_k \uparrow B$$

as $k \to \infty$. Set also $\rho_k = r_{k+1} - r_k = \frac{1}{2^{k+2}} r$.

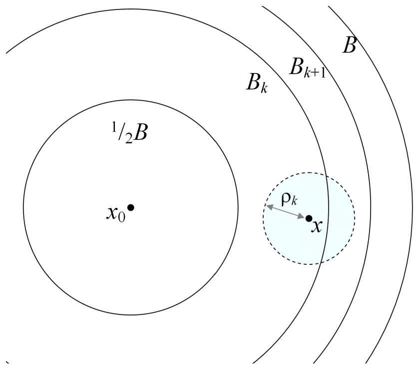

Figure 4: Balls B_k and $B(x, \rho_k)$

For any point $x \in B_k$, applying Theorem 3.2 in the ball $B(x, \rho_k)$, we obtain

$$\begin{aligned}\|u\|^{\lambda_0}_{L^\infty(B^t(x,\frac{1}{2}\rho_k))} &\le \frac{C}{\chi\left(B(x,\rho_k)\right)\rho_k^p}\int_{B^t(x,\rho_k)} u^{\sigma_0} \\ &\le \frac{C}{\chi\left(B(x,\rho_k)\right)\rho_k^p}\|u\|^{\sigma_0-\sigma}_{L^\infty(B^t(x,\rho_k))}\int_{B^t(x,\rho_k)} u^{\sigma}.\end{aligned}$$

Since $B(x, \rho_k) \subset B_{k+1} \subset B$, we have by the monotonicity of (3.28)

$$\frac{\chi(B(x,\rho_k))}{\rho_k^{p/\nu}} \ge \frac{\chi(B)}{r^{p/\nu}}$$

whence

$$\frac{1}{\chi(B(x,\rho_k))} \le \frac{(r/\rho_k)^{p/\nu}}{\chi(B)} = \frac{2^{(k+2)p/\nu}}{\chi(B)}.$$

Hence, we obtain

$$\|u\|^{\lambda_0}_{L^\infty(B^t(x,\frac{1}{2}\rho_k))} \le \frac{C2^{kp(\nu^{-1}+1)}}{\chi(B)\,r^p}\|u\|^{\lambda_0-\lambda}_{L^\infty(B^t_{k+1})}\int_{B^t} u^{\sigma}.$$

Covering B_k by a sequence of balls $B(x, \frac{1}{2}\rho_k)$ with $x \in B_k$, we obtain

$$\|u\|^{\lambda_0}_{L^\infty(B^t_k)} \le \frac{C2^{kp(\nu^{-1}+1)}}{\chi(B)\,r^p}\|u\|^{\lambda_0-\lambda}_{L^\infty(B^t_{k+1})}\int_{B^t} u^{\sigma}. \tag{3.39}$$

Setting $J_k = \|u\|^{-(\lambda_0-\lambda)}_{L^\infty(B^t_k)}$, we rewrite (3.39) as follows:

$$J_{k+1} \le \frac{A^k}{\Theta} J_k^{\frac{\lambda_0}{\lambda_0-\lambda}} = \frac{A^k}{\Theta} J_k^{1+\omega},$$

where $A = 2^{p(\nu^{-1}+1)}$,

$$\Theta^{-1} = \frac{C}{\chi(B)\,r^p}\int_{B^t} u^{\sigma}$$

and $\omega = \frac{\lambda_0}{\lambda_0-\lambda} - 1 = \frac{\lambda}{\lambda_0-\lambda}$. Applying Lemma 5.2, we obtain

$$J_k \le \left(\frac{J_0}{\left(A^{-1/\omega}\Theta\right)^{1/\omega}}\right)^{(1+\omega)^k}\left(A^{-1/\omega}\Theta\right)^{1/\omega},$$

that is,

$$J_0 \geq \left(A^{-1/\omega}\Theta\right)^{1/\omega} \left(\left(A^{1/\omega}\Theta^{-1}\right)^{1/\omega} J_k\right)^{\frac{1}{(1+\omega)^k}}.$$

Since $J_k \geq \|u\|_{L^\infty(B^t)}^{-(\lambda_0-\lambda)} =: \text{const} > 0$, we see that

$$\liminf_{k\to\infty} \left(\left(A^{1/\omega}\Theta^{-1}\right)^{1/\omega} J_k\right)^{\frac{1}{(1+\omega)^k}} \geq 1,$$

whence

$$J_0 \geq \left(A^{-1/\omega}\Theta\right)^{1/\omega}.$$

It follows that $J_0^{-1} \leq A^{1/\omega^2}\Theta^{-1/\omega}$, that is,

$$\|u\|_{L^\infty(B_0^t)}^{\lambda_0-\lambda} \leq A^{1/\omega^2} \left(\frac{C}{\chi(B)\, r^p} \int_{B^t} u^\sigma\right)^{1/\omega},$$

and thus,

$$\|u\|_{L^\infty(\frac{1}{2}B\times[0,t])} \leq \left(\frac{C}{\iota(B)\mu(B)r^p} \int_{B\times[0,t]} u^\sigma\right)^{1/\lambda},$$

which was to be proved. ■

4 Finite propagation speed

In this section we assume that M is geodesically complete and

$$p > 2 \quad \text{and} \quad \frac{1}{p-1} < q \leq 1.$$

In particular, this implies that

$$\delta = q(p-1) - 1 > 0.$$

4.1 Propagation inside a ball

The next result contains Theorem 1.1 from the Introduction.

Theorem 4.1. *Let u be a bounded non-negative subsolution in $M\times[0,T]$. Let $B_0 = B(x_0, R)$ be a ball such that*

$$u_0 = 0 \text{ in } B_0.$$

Let σ be a real such that

$$\sigma \geq 1 \text{ and } \sigma > \delta. \tag{4.40}$$

Set

$$t_0 = \eta\iota(B_0)\mu(B_0)^{\frac{\delta}{\sigma}} R^p ||u_0||_{L^\sigma(M)}^{-\delta} \wedge T, \tag{4.41}$$

where $\eta = \eta(p, q, \nu, \sigma) > 0$ is sufficiently small. Then

$$u = 0 \quad \text{in} \quad \frac{1}{2}B_0 \times [0, t_0].$$

Remark 4.2. Although $\sigma = \infty$ is formally not included in this statement, (4.41) is true also for $\sigma = \infty$, that is, with

$$t_0 = \eta\iota(B_0)R^p||u_0||_{L^\infty(M)}^{-\delta} \wedge T,$$

where $\eta = \eta(p, q, \nu) > 0$ (see [15]).

Proof. Set $r = \frac{1}{2}R$ and fix for a while a point $x \in \frac{1}{2}B_0$ so that $B := B(x, r) \subset B_0$. Fix also some $t \in (0, T]$ and set

$$Q_k = 2^{-k}B \times [0, t] \quad \text{and} \quad J_k = \int_{Q_k} u^\sigma.$$

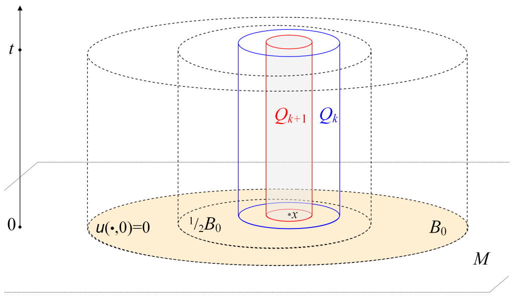

Figure 5: Cylinders Q_k

Since $\sigma > \delta$, we have $\lambda = \sigma - \delta > 0$. By Theorem 3.2, we obtain

$$\|u\|_{L^\infty(Q_{k+1})} \leq \left(\frac{C}{\iota(2^{-k}B)\mu\left(2^{-k}B\right)(2^{-k}r)^p} \int_{Q_k} u^\sigma \right)^{1/\lambda}.$$

It follows that

$$\begin{aligned} J_{k+1} = \int_{Q_{k+1}} u^\sigma &\leq \mu\left(2^{-(k+1)}B\right) t \, \|u\|^\sigma_{L^\infty(Q_{k+1})} \\ &\leq \mu(B_0)t \left(\frac{C}{\iota(2^{-k}B)\mu\left(2^{-k}B\right)(2^{-k}r)^p} \int_{Q_k} u^\sigma \right)^{\sigma/\lambda}. \end{aligned}$$

Since by the monotonicity of the function (3.28)

$$\frac{\iota(2^{-k}B)\mu\left(2^{-k}B\right)}{(2^{-k}r)^{p/\nu}} \geq \frac{\iota(B_0)\mu(B_0)}{R^{p/\nu}}$$

and $r = \frac{1}{2}R$, we obtain

$$J_{k+1} \leq \mu(B_0)t \left(\frac{C2^{kp\left(\nu^{-1}+1\right)}}{\iota(B_0)\mu(B_0)R^p} J_k \right)^{\sigma/\lambda} = \frac{A^k}{\Theta} J_k^{1+\omega},$$

where

$$\omega = \frac{\sigma}{\lambda} - 1 = \frac{\delta}{\lambda}, \quad A = 2^{p\left(\nu^{-1}+1\right)\sigma/\lambda} \quad \text{and} \quad \Theta = \frac{\left(\iota(B_0)R^p\right)^{1+\omega}\mu(B_0)^\omega}{Ct}.$$

By Lemma 5.2 we obtain

$$J_k \leq \left(\frac{A^{1/\omega} J_0^\omega}{\Theta} \right)^{\frac{(1+\omega)^k}{\omega}} \left(A^{-1/\omega}\Theta\right)^{1/\omega}. \tag{4.42}$$

We have

$$\frac{A^{1/\omega}J_0^\omega}{\Theta}=\frac{CA^{1/\omega}t\left(\int_{B\times[0,t]}u^\sigma\right)^\omega}{\left(\iota(B_0)R^p\right)^{1+\omega}\mu(B_0)^\omega}.$$

Since $\sigma\geq 1$, we have by Lemma 2.9

$$\int_{B\times[0,t]}u^\sigma\leq t\int_M u_0^\sigma$$

and

$$\frac{A^{1/\omega}J_0^\omega}{\Theta}\leq\frac{CA^{1/\omega}t^{1+\omega}\left(\int_M u_0^\sigma\right)^\omega}{\left(\iota(B_0)R^p\right)^{1+\omega}\mu(B_0)^\omega}.$$

We would like to have

$$\frac{A^{1/\omega}J_0^\omega}{\Theta}\leq\frac{1}{2} \tag{4.43}$$

For that it suffices to have

$$t^{1+\omega}\leq\frac{1}{2}C^{-1}A^{-1/\omega}\left(\iota(B_0)R^p\right)^{1+\omega}\mu(B_0)^\omega\left(\int_M u_0^\sigma\right)^{-\omega},$$

that is,

$$t\leq\eta\iota(B_0)R^p\mu(B_0)^{\frac{\omega}{1+\omega}}\left(\int_M u_0^\sigma\right)^{-\frac{\omega}{1+\omega}} \tag{4.44}$$

where $\eta=\left(\frac{1}{2}C^{-1}A^{-1/\omega}\right)^{\frac{1}{1+\omega}}$. Since $\omega=\frac{\delta}{\lambda}=\frac{\delta}{\sigma-\delta}$ and $\frac{\omega}{1+\omega}=\frac{\delta}{\sigma}$ we see that (4.44) is satisfied for $t=t_0$ where t_0 is given by (4.41).

Hence, it follows from (4.42) and (4.43) that

$$J_k\leq 2^{-\frac{(1+\omega)^k}{\omega}}K,$$

where $K=\left(A^{-1/\omega}\Theta\right)^{1/\omega}$ depends on B_0, R and t but does not depend on x or k; that is, for any $x\in\frac{1}{2}B_0$, for $t=t_0$ and, for any $k\geq 0$, we have

$$\int_{B(x,2^{-k}r)\times[0,t]}u^\sigma\leq 2^{-\frac{(1+\omega)^k}{\omega}}K. \tag{4.45}$$

Let $D=D(B_0)$ be such that any ball in B_0 of any radius $\rho\leq\frac{1}{2}R$ can be covered by D balls of radii $\rho/2$. Let us cover the ball $\frac{1}{2}B_0$ by a finite sequence of balls $\left\{B\left(x_i,2^{-k}r\right)\right\}_{i=1}^N$ with $x_i\in\frac{1}{2}B_0$. Then the number N is estimated as follows: $N\leq D^k$. It follows from (4.45) that

$$\int_{\frac{1}{2}B_0\times[0,t]}u^\sigma\leq\sum_{i=1}^N\int_{B\left(x_i,2^{-k}r\right)\times[0,t]}u^\sigma\leq D^k2^{-\frac{(1+\omega)^k}{\omega}}K. \tag{4.46}$$

Since the right hand side here $\to 0$ as $k\to\infty$, we conclude that

$$\int_{\frac{1}{2}B_0\times[0,t]}u^\sigma=0$$

that is, $u=0$ in $\frac{1}{2}B_0\times[0,t]$,which finishes the proof. ■

4.2 Propagation of support

Let $u(x,t)$ be a non-negative bounded subsolution in $M \times \mathbb{R}_+$ with the initial function $u_0 = u(\cdot, 0)$. Assume that the support

$$K = \operatorname{supp} u_0$$

of u_0 is compact. For any $r > 0$, denote by K_r a closed r-neighborhood of K.

Corollary 4.3. *Suppose that there exists a point $x_0 \in K$ and a continuous monotone increasing function $\varphi(r)$ converging to $+\infty$ such that for all large enough r,*

$$\eta\iota(B(x_0,r))\mu(B(x_0,r))^{\frac{\delta}{\sigma}} r^p \left(\int_M u_0^\sigma\right)^{-\frac{\delta}{\sigma}} \geq \varphi(r). \tag{4.47}$$

Then there exists a continuous monotone increasing function $\rho : (0,\infty) \to \mathbb{R}_+$ such that $\operatorname{supp} u(\cdot,t) \subset K_{\rho(t)}$ *for all $t \in (0,\infty)$.*

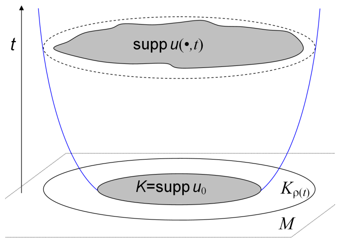

Figure 6: The support of $u(\cdot,t)$

Here $\rho(t)$ may depend on u. The function $\rho(t)$ is called a *propagation rate* or *propagation function* of u.

Proof. As a continuous monotone increasing function converging to $+\infty$, φ has an inverse function $\rho = \varphi^{-1}$ defined on $(0,\infty)$ that is also continuous and monotone increasing.

Let us show that $r = \rho(t)$ implies

$$\operatorname{supp} u(\cdot,t) \subset K_r,$$

that is,

$$u(\cdot,t) = 0 \text{ in } M \setminus K_r.$$

Let us fix a point $x \in K_{2r} \setminus K_r$. We have $d(x,K) \geq r$ and thus $B(x,r) \cap K = \emptyset$. By (4.47), $r = \rho(t)$ implies that for all large enough r,

$$t \leq \varphi(r) \leq \eta\iota(B(x,r))\mu(B(x,r))^{\frac{\delta}{\sigma}} r^p \left(\int_M u_0^\sigma\right)^{-\frac{\delta}{\sigma}}.$$

Since $u(\cdot,0) = 0$ in $B(x,r)$, we conclude by Theorem 4.1 that

$$u(\cdot,t) = 0 \text{ in } B(x,r/2).$$

Since this is true for any $x \in K_{2r} \setminus K_r$, we obtain that

$$u(\cdot,t) = 0 \text{ in } K_{2r} \setminus K_r. \tag{4.48}$$

Let us show that in this case also

$$u(\cdot,t) = 0 \ \text{ in } M \setminus K_r. \tag{4.49}$$

Fix some $s >> 2r$ and let $\eta(x)$ be a bump function of $K_s \setminus K_{2r}$ in $K_{2s} \setminus K_r$; that is, η is the following function of $|x| := d(x,K)$:

$$\eta(x) = \begin{cases} \left(\frac{|x|}{r} - 1\right)_+, & |x| \le 2r, \\ 1, & |x| \in [2r, s], \\ 2\left(1 - \frac{|x|}{2s}\right)_+, & |x| \ge s. \end{cases}$$

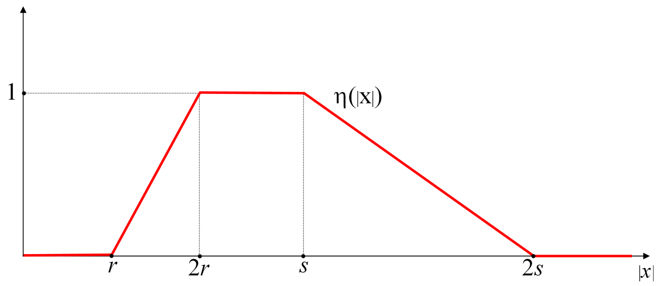

Figure 7: Function η

Applying the inequality (2.20) of Lemma 2.8 with large enough σ, we obtain

$$\left[\int_M u^\lambda \eta^p\right]_0^t \le c_2 \int_0^t \int_M u^\sigma |\nabla \eta|^p. \tag{4.50}$$

Since $u(\cdot,0) = 0$ on $\operatorname{supp}\eta$ and $\eta = 1$ on $K_s \setminus K_{2r}$, the left hand side here is bounded below by

$$\int_{K_s \setminus K_{2r}} u^\lambda(\cdot,t).$$

Since $\eta = 0$ in K_r, $u(\cdot,\tau) = 0$ in $K_{2r} \setminus K_r$ for all $\tau \le t$ (by (4.48)), and $\nabla\eta = 0$ in $K_s \setminus K_{2r}$, the right hand side in (4.50) is equal to

$$c_2 \int_0^t \int_{M \setminus K_s} u^\sigma |\nabla \eta|^p.$$

Since $|\nabla\eta| \le \frac{1}{s}$ in $M \setminus K_s$, we obtain that

$$\int_{K_s \setminus K_{2r}} u^\lambda(\cdot,t) \le c_2 \int_0^t \int_{M \setminus K_s} u^\sigma |\nabla\eta|^p \le \frac{c_2}{s^p} \int_0^t \int_{M \setminus K_s} u^\sigma.$$

The right hand side goes to 0 as $s \to \infty$, which implies that $u(\cdot,t) = 0$ in $M \setminus K_{2r}$, thus proving (4.49). ■

4.3 Curvature and propagation rate

Corollary 4.4. *Let M satisfy the relative Faber-Krahn inequality. Fix a reference point $x_0 \in K$ and assume that, for some $\alpha > 0$ and all large enough r,*

$$\mu(B(x_0,r)) \ge c r^\alpha. \tag{4.51}$$

Then u has a propagation function

$$\rho(t) = Ct^{1/\beta}$$

for large t, where

$$\beta = p + \alpha\frac{\delta}{\sigma}$$

with σ as in (4.40) and C depends on $\|u_0\|_{L^\sigma(M)}, p, q, n, \alpha$ and c.

Proof. Let compute the function $\rho(t)$ from Corollary 4.3. By assumption we have that the Faber-Krahn constant $\iota(B)$ has a uniform positive lower bound for all geodesic balls $B \subset M$. Using (4.51) and treating $\left(\int_M u_0^\sigma\right)^{-\frac{\delta}{\sigma}}$ as constant, we see that the function φ from (4.47) can be taken in this case as follows:

$$\varphi(r) = cr^{p+\alpha\frac{\delta}{\sigma}} = cr^\beta.$$

Finally, we conclude that

$$\rho(t) = \varphi^{-1}(t) = Ct^{1/\beta}$$

for large enough t, which was to be proved. ■

Remark 4.5. Under the hypothesis $\alpha \in (0, n]$ the model manifold constructed in Proposition 5.1 satisfies the volume doubling property and the Poincaré inequality, and in particular, also the relative Faber-Krahn inequality (see Proposition 4.10 in [13]).

Remark 4.6. In $\mathbb{R}^n$ we have (4.51) with $\alpha = n$. If $\sigma = 1$, we obtain the sharp propagation rate $1/\beta$, where $\beta = p + n\delta$. By (4.40), we can take $\sigma = 1$ provided $\delta < 1$, that is, when $q < \frac{2}{p-1}$. Hence, in the range

$$p > 2, \quad \frac{1}{p-1} < q \leq \min\left(\frac{2}{p-1}, 1\right) \tag{4.52}$$

(see Fig. 8), we get a sharp propagation rate. In this range of p, q we not only get a sharp propagation rate in $\mathbb{R}^n$, but by Proposition 5.1 also in the class of model manifolds satisfying the relative Faber-Krahn inequality and (4.51) with any $\alpha \in (0, n]$.

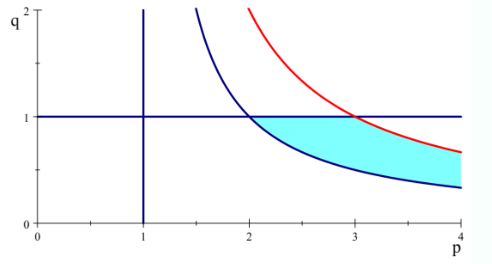

Figure 8: Range of p, q

Corollary 4.7. *Suppose that M satisfies the following isoperimetric inequality: for any pre-compact open set $\Omega \subset M$ with smooth boundary,*

$$\mu'(\partial\Omega) \geq c\mu(\Omega)^{\frac{\alpha-1}{\alpha}}, \tag{4.53}$$

for some $c > 0$ and where $\alpha \geq n$ and $\alpha > p$. Also, assume that for some $x_0 \in K$ and all large enough r,

$$\mu(B(x_0, r)) \leq Cr^\alpha, \tag{4.54}$$

where $C > 0$. Then u has a propagation function

$$\rho(t) = C' t^{1/\beta}$$

for large t, where

$$\beta = \alpha \frac{\delta}{\sigma} + p \tag{4.55}$$

with σ as in (4.40) and C' depends on $\|u_0\|_{L^\sigma(M)}, p, q, \alpha, c$ and C.

Note that the inequality (4.53) implies that for all $x \in M$ and $r > 0$,

$$\mu\left(B(x,r)\right) \geq \text{const}\, r^{\alpha}. \tag{4.56}$$

Proof. The isoperimetric inequality (4.53) implies the following *Sobolev inequality*: for all geodesic balls $B \subset M$ and all non-negative $w \in W_0^{1,p}(B)$,

$$\left(\int_B w^{\frac{\alpha p}{\alpha - p}}\right)^{\frac{\alpha - p}{\alpha}} \leq \text{const} \int_B |\nabla w|^p.$$

From that we obtain

$$\iota(B) \geq c \frac{r(B)^{\frac{p}{\nu}}}{\mu(B)}$$

where $\nu = \frac{p}{\alpha}$ (see Section 3 in [15]). Hence, applying condition (4.54), we deduce for all large enough r,

$$\iota(B(x_0, r)) \geq c r^{\frac{p}{\nu} - \alpha} = c.$$

Substituting this into (4.47) we obtain from (4.56) that φ can be taken as follows:

$$\varphi(r) = c r^{\alpha \frac{\delta}{\sigma} + p}.$$

Thus, we conclude $\rho(t) = C' t^{1/\beta}$, where β is given by (4.55). This completes the proof. ∎

5 Appendix

5.1 Radial solution on polynomial models

Let M be a *model manifold*, that is $M = (0, +\infty) \times \mathbb{S}^{n-1}$ as topological spaces and M is equipped with the Riemannian metric ds^2 given by

$$ds^2 = dr^2 + \psi^2(r) d\theta^2,$$

where $\psi(r)$ is a smooth positive function on $(0, +\infty)$ and $d\theta^2$ is the standard Riemannian metric on $\mathbb{S}^{n-1}$. We define $S(r) = \psi^{n-1}(r)$, which is called the *profile* of the model manifold.

We search for solutions u of (1.1) on M with finite propagation speed. We always assume that

$$p > 1 \quad \text{and } q(p-1) > 1.$$

Let $u(x,t) = u(r,t)$, that is, function u depends only on the polar radius r and time t. Assume also that $\partial_r u \leq 0$, then

$$\Delta_p u = -\frac{1}{S} \partial_r \left(S \left(-\partial_r u\right)^{p-1} \right)$$

so that (1.1) becomes

$$\partial_t u = -\frac{1}{S} \partial_r \left(S \left(-\partial_r u^q\right)^{p-1} \right). \tag{5.1}$$

Proposition 5.1. *Assume that, for some* $\alpha \in (0, n]$ *and all* $r \geq r_0$,

$$S(r) = Cr^{\alpha-1}.$$

Then the following function is a non-negative solution of (1.1) in $M \setminus B_{r_0} \times \mathbb{R}_+$:

$$u(x,t) = \frac{1}{t^{\alpha/\beta}} \left(C - \kappa \left(\frac{r}{t^{1/\beta}} \right)^{\frac{p}{p-1}} \right)_+^{1/\gamma} \tag{5.2}$$

where $C > 0$ *and*

$$\beta = p + \alpha \left[q(p-1) - 1 \right], \quad \gamma = q - \frac{1}{p-1}, \quad \kappa = \gamma \frac{p-1}{pq\beta^{\frac{1}{p-1}}}.$$

Note that the volume of the central balls on this manifold is of the order r^α, and the propagation rate of the above solution is $Ct^{1/\beta}$, which matches our main results in the case when we can take $\sigma = 1$.

Proof. By (5.1) the equation (1.1) for u becomes for $r > r_0$,

$$\partial_t u = -\frac{1}{r^{\alpha-1}} \partial_r \left(r^{\alpha-1} \left(-\partial_r u^q \right)^{p-1} \right). \tag{5.3}$$

We search for a solution of the form

$$u(x,t) = t^a f(rt^b) \text{ for large } r,$$

where f is a decreasing function. Let us require in addition that the solution $u(\cdot,t)$ has bounded L^1-norm. One can show that for that we need to require that $a = \alpha b$. Using the variable $s = rt^b$, we obtain that (5.3) is equivalent to

$$\frac{bt^{a-1}}{s^{\alpha-1}} \left(s^\alpha f(s) \right)' = -\frac{q^{p-1} t^{(aq+b)(p-1)}}{s^{\alpha-1}} t^b \partial_s \left(s^{\alpha-1} \left(-f(s)^{q-1} f'(s) \right)^{p-1} \right).$$

We also require that

$$(aq+b)(p-1) + b = a - 1,$$

which together with $a = b\alpha$ yields

$$b = -\frac{1}{\alpha\left(q(p-1)-1\right)+p} < 0.$$

Under the above choice of a and b, the powers of t and s in the above equation cancel out, and we obtain since $b < 0$,

$$f^{(q-1)-\frac{1}{p-1}} f' = -\frac{\left(|b|\, s\right)^{\frac{1}{p-1}}}{q}. \tag{5.4}$$

Note that $\gamma := q - \frac{1}{p-1} > 0$. Integration of (5.4) yields

$$f(s) = \left(C - \kappa s^{\frac{p}{p-1}} \right)^{1/\gamma}$$

where

$$\kappa = \gamma \frac{p-1}{p} \frac{|b|^{\frac{1}{p-1}}}{q} = \frac{q(p-1)-1}{p} \frac{|b|^{\frac{1}{p-1}}}{q}$$

and C is a positive constant. ∎

5.2 An auxiliary lemma

Lemma 5.2. *[15] Let a sequence $\{J_k\}_{k=0}^{\infty}$ of non-negative reals satisfy*

$$J_{k+1} \leq \frac{A^k}{\Theta} J_k^{1+\omega} \quad \text{for all } k \geq 0.$$

where $A, \Theta, \omega > 0$. Then, for all $k \geq 0$,

$$J_k \leq \left(\left(A^{1/\omega} \Theta^{-1} \right)^{1/\omega} J_0 \right)^{(1+\omega)^k} \left(A^{-k-1/\omega} \Theta \right)^{1/\omega}.$$

In particular, if $\Theta \geq A^{1/\omega} J_0^{\omega}$, then $J_k \leq A^{-k/\omega} J_0$ for all $k \geq 0$.

References

[1] S. Andres and M. T. Barlow. Energy inequalities for cutoff functions and some applications. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2015(699):183–215, 2015.

[2] D. Andreucci and A. F. Tedeev. Asymptotic properties of solutions to the Cauchy problem for degenerate parabolic equations with inhomogeneous density on manifolds. *Milan Journal of Mathematics*, 89(2):295–327, 2021.

[3] G. I. Barenblatt. On self-similar motions of a compressible fluid in a porous medium. *Akad. Nauk SSSR. Prikl. Mat. Meh*, 16(6):679–698, 1952.

[4] V. Bögelein, F. Duzaar, and P. Marcellini. Parabolic systems with p, q-growth: a variational approach. *Archive for Rational Mechanics and Analysis*, 210(1):219–267, 2013.

[5] P. Buser. A note on the isoperimetric constant. *Ann. Sci. Ecole Norm. Sup.*, 15:213–230, 1982.

[6] E. De Giorgi. Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino*, 3:25–43, 1957.

[7] S. Dekkers. Finite propagation speed for solutions of the parabolic p-laplace equation on manifolds. *Communications in Analysis and Geometry*, 13(4):741–768, 2005.

[8] J. I. Díaz and M. Herrero. Estimates on the support of the solutions of some nonlinear elliptic and parabolic problems. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 89(3-4):249–258, 1981.

[9] E. DiBenedetto. *Degenerate parabolic equations*. Springer Science & Business Media, 1993.

[10] E. DiBenedetto, U. P. Gianazza, and V. Vespri. *Harnack's inequality for degenerate and singular parabolic equations*. Springer Science & Business Media, 2011.

[11] E. DiBenedetto and M. A. Herrero. On the Cauchy problem and initial traces for a degenerate parabolic equation. *Transactions of the American Mathematical Society*, 314(1):187–224, 1989.

[12] A. Grigor'yan. The heat equation on non-compact Riemannian manifolds. *Math. USSR Sb.*, 72:47–77, 1992.

[13] A. Grigor'yan and L. Saloff-Coste. Stability results for harnack inequalities. In *Annales de l'institut Fourier*, volume 55, pages 825–890, 2005.

[14] A. Grigoryan. Estimates of heat kernels on Riemannian manifolds. *London Math. Soc. Lecture Note Ser*, 273:140–225, 1999.

[15] A. Grigoryan and P. Sürig. Finite propagation speed for Leibensons equation on Riemannian manifolds. *to appear in Comm. Anal. Geom.*, 2023.

[16] G. Grillo, M. Muratori, and F. Punzo. The porous medium equation with measure data on negatively curved Riemannian manifolds. *Journal of the European Mathematical Society*, 20(11):2769–2812, 2018.

[17] C. E. Gutiérrez and R. L. Wheeden. Mean value and Harnack inequalities for degenerate parabolic equations. In *Colloquium Mathematicum*, volume 1, pages 157–194, 1990.

[18] K. Ishige. On the existence of solutions of the cauchy problem for a doubly nonlinear parabolic equation. *SIAM Journal on Mathematical Analysis*, 27(5):1235–1260, 1996.

[19] A. V. Ivanov. Regularity for doubly nonlinear parabolic equations. *Journal of Mathematical Sciences*, 83(1):22–37, 1997.

[20] A. S. Kalashnikov. Some problems of the qualitative theory of non-linear degenerate second-order parabolic equations. *Russian Mathematical Surveys*, 42(2):169, 1987.

[21] S. Kamin and J. L. Vázquez. Fundamental solutions and asymptotic behaviour for the p-Laplacian equation. *Revista Matemática Iberoamericana*, 4(2):339–354, 1988.

[22] J. Kinnunen and P. Lindqvist. Pointwise behaviour of semicontinuous supersolutions to a quasilinear parabolic equation. *Annali di Matematica Pura ed Applicata*, 185(3):411–435, 2006.

[23] O. Ladyzhenskaya, V. Solonnikov, and N. Uraltseva. Linear and quasilinear equations of parabolic type, transl. math. *Monographs, Amer. Math. Soc*, 23, 1968.

[24] L. Leibenson. General problem of the movement of a compressible fluid in a porous medium. izv akad. nauk sssr. *Geography and Geophysics*, 9:7–10, 1945.

[25] L. Leibenson. Turbulent movement of gas in a porous medium. *Izv. Akad. Nauk SSSR Ser. Geograf. Geofiz*, 9:3–6, 1945.

[26] Q. Li. Weak harnack estimates for supersolutions to doubly degenerate parabolic equations. *Nonlinear Analysis*, 170:88–122, 2018.

[27] J. J. Manfredi and V. Vespri. Large time behavior of solutions to a class of doubly nonlinear parabolic equations. *Electronic Journal of Differential Equations.*, 1994(1994)(02):1–17, Mar. 1994.

[28] J. Moser. Harnack inequality for parabolic differential equations. *Comm. Pure Appl. Math.*, 17:101–134, 1964.

[29] P.-A. Raviart. Sur la résolution de certaines équations paraboliques non linéaires. *Journal of Functional Analysis*, 5(2):299–328, 1970.

[30] J. Saá. Large time behaviour of the doubly nonlinear porous medium equation. *Journal of mathematical analysis and applications*, 155(2):345–363, 1991.

[31] L. Saloff-Coste. *Aspects of Sobolev-type inequalities*. LMS Lecture Notes Series, vol. 289. Cambridge Univ. Press, 2002.

[32] D. Stan and J. L. Vázquez. Asymptotic behaviour of the doubly nonlinear diffusion equation $u_t = \Delta_p u^m$ on bounded domains. *Nonlinear Analysis: Theory, Methods & Applications*, 77:1–32, 2013.

[33] S. Sturm. Existence of weak solutions of doubly nonlinear parabolic equations. *Journal of Mathematical Analysis and Applications*, 455(1):842–863, 2017.

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