Heat kernels on weighted manifolds and applications

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Contents

1 Introduction

A *weighted manifold* (called also a manifold with density) is a Riemannian manifold M endowed with a measure μ that has a smooth positive density with respect to the Riemannian measure. The space (M,μ) features the *weighted Laplace operator* Δ_μ, generalizing the Laplace-Beltrami operator, which is symmetric with respect to measure μ. It is possible to extend Δ_μ to a self-adjoint operator in $L^2(M,\mu)$, which allows to define the heat semigroup $e^{t\Delta_\mu}$. The heat semigroup has the integral kernel $p_t(x,y)$, which is called the *heat kernel* of (M,μ) and which is the subject of this survey.

The notion of a weighted Laplacian was introduced by I. Chavel and E. Feldman [33] and by E. B. Davies [55]. Many facts from the analysis on weighted manifolds are similar to those on Riemannian manifolds. However, in the former setting one has an added flexibility of changing the measure without changing the underlying Riemannian structure, which happens to be a powerful technical tool, as was earlier observed by E. B. Davies and B. Simon [60]. A natural setup for this approach would be a metric measure space with an energy form in the spirit of [74], but this would bring additional technical complications, caused by the singularity of the space.

We have selected here those results about heat kernels on weighted manifolds, which emphasize the role of the reference measure μ. The material presented here naturally splits into the following categories.

1. *The textbook material.* This includes already mentioned constructions of the Laplace operator and the heat kernel, criteria for stochastic completeness, comparison results of heat kernels (Sections 2-4), as well as the construction of the Brownian motion and the Feynman-Kac formula (Section 8).

2. *The heat kernel estimates obtained in the past 10-15 years.* These are the core results in this area, and many applications depend upon them. They include upper bounds of heat kernels via Faber-Krahn inequalities, Gaussian upper estimates (Section 5), Harnack inequalities, and two sided Li-Yau estimates (Section 6).

3. *Selected applications of heat kernel estimates.* These are estimates of the number of negative eigenvalues of Schrödinger operators including estimates of the stability index of a minimal surface (Section 7), as well as certain path properties of the Brownian motion and symmetric stable processes (Section 9).

4. *New estimates for the heat kernels of Schrödinger operators.* Our approach is based on the following well-known observation, which goes back to [173] and [116]: if a Schrödinger operator $H = -\Delta_\mu + \Phi(x)$ has a positive solution $h(x)$ then $\frac{1}{h} \circ H \circ h = -\Delta_{\widetilde{\mu}}$ where measure $\widetilde{\mu}$ is defined by $d\widetilde{\mu} = h^2 d\mu$. Hence, the question of obtaining bounds for the heat kernel $p_t^\Phi(x,y)$ of H amount to that of the heat kernel $\widetilde{p}_t(x,y)$ of $\Delta_{\widetilde{\mu}}$. The key results which enable one to estimate $\widetilde{p}_t$ have been proved recently in [110]. Using them, we obtain a number of new estimates for $p_t^\Phi(x,y)$ including the case of a potential $\Phi(x)$ in $\mathbb{R}^n$ decaying as $|x|^{-2}$ when $x \to \infty$ (Section 10).

The new results are presented with proofs, and for many surveyed results the proofs are outlined.

For other aspects of heat kernels, we refer the reader to the following articles and references therein:

- heat kernels and curvature – [78], [150], [157], [182], [215], [218], [219];
- heat kernels in presence of group structure – [4], [5], [6], [19], [20], [46], [136], [171], [179], [180], [184], [213];
- heat kernels on fractals and fractal-like spaces – [12], [13], [14], [97], [101], [114], [119], [141], [142];
- heat kernels of non-linear operators – [63], [65];
- heat kernels of higher order elliptic operators – [56], [145], [184];
- heat kernels of non-symmetric operators – [72], [144], [176];
- heat kernels of subelliptic operators – [17], [18], [134], [135];
- heat kernels in infinite dimensional spaces – [20], [67].

Notation. For positive functions $f(x)$ and $g(x)$ on a set X, we write

$$f(x) \simeq g(x) \quad \text{for } x \in X$$

if there is a positive constant C, such that

$$C^{-1} \leq \frac{f(x)}{g(x)} \leq C \quad \text{for } x \in X.$$

For example, $f(x) \simeq 1$ means that the function f is bounded between two positive constants.

We write

$$f(x) \asymp h(x,c,C) \quad \text{for } x \in X,$$

if there are positive constants c_1,c_2, C_1, C_2 such that

$$h(x,c_1,C_1) \leq f(x) \leq h(x,c_2,C_2) \quad \text{for } x \in X.$$

For example, $f(x) \asymp C \exp(-cx)$ means that

$$C_1 \exp(-c_1 x) \leq f(x) \leq C_2 \exp(-c_2 x).$$

We reserve the letters c and C for positive constants whose values are unimportant and can change at any occurrence, unless otherwise stated.

2 The Laplace operator

2.1 Differential operators on manifolds

Let M be a (connected) Riemannian manifold and g be the Riemannian metric on M. For any smooth function u on M, the gradient ∇u is a vector field on M, which in local coordinates $x^1, ..., x^n$ has the form

$$(\nabla u)^i = g^{ij} \frac{\partial u}{\partial x^j},$$

where summation is assumed over repeated indices. For any smooth vector field F on M, the divergence $\operatorname{div} F$ is a scalar function on M, which is given in local coordinates by

$$\operatorname{div} F = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} F^i \right).$$

Let ν be the Riemannian volume on M, that is,

$$d\nu = \sqrt{\det g}\, dx^1 ... dx^n.$$

By the divergence theorem, for any smooth function u and a smooth vector field F, such that either u or F has compact support,

$$\int_M u \operatorname{div} F \, d\nu = -\int_M \langle \nabla u, F \rangle \, d\nu, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle \equiv g(\cdot, \cdot)$. In particular, if $F = \nabla v$ for a function v then we obtain

$$\int_M u \operatorname{div} \nabla v \, d\nu = -\int_M \langle \nabla u, \nabla v \rangle \, d\nu, \tag{2.2}$$

provided one of the functions u, v has compact support. The operator

$$\Delta := \operatorname{div} \circ \nabla$$

is called the *Laplace* (or *Laplace-Beltrami*) operator of the Riemannian manifold M. From (2.2), we obtain the *Green formulas*

$$\int_M u\,\Delta v\,d\nu = -\int_M \langle\nabla u,\nabla v\rangle\,d\nu = \int_M v\,\Delta u\,d\nu. \tag{2.3}$$

Let now μ be another measure on M defined by

$$d\mu = h^2 d\nu$$

where h is a smooth positive function on M. A triple (M,g,μ) (which will be frequently abbreviated to (M,μ)) is called a *weighted manifold*. The associated divergence div_μ is defined by

$$\operatorname{div}_\mu F := \frac{1}{h^2}\operatorname{div}\left(h^2F\right) = \frac{1}{h^2\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(h^2\sqrt{\det g}F^i\right),$$

and the *Laplace operator* Δ_μ of (M,g,μ) is defined by

$$\Delta_\mu := \operatorname{div}_\mu\circ\nabla = \frac{1}{h^2}\operatorname{div}\left(h^2\nabla\right) = \Delta + 2\frac{\langle\nabla h,\nabla\rangle}{h}. \tag{2.4}$$

It is easy to see that the Green formulas holds with respect to the measure μ, that is,

$$\int_M u\,\Delta_\mu v\,d\mu = -\int_M \langle\nabla u,\nabla v\rangle\,d\mu = \int_M v\,\Delta_\mu u\,d\mu, \tag{2.5}$$

provided u or v belongs to $C_0^\infty(M)$.

Example 2.1 Let $a(x)$, $b(x)$ be smooth positive functions on a weighted manifold (M,g,μ), and consider new metric $\widetilde{g}$ and measure $\widetilde{\mu}$ defined by

$$\widetilde{g} = ag \quad\text{and}\quad d\widetilde{\mu} = bd\mu.$$

Let us show that the Laplace operator $\widetilde{\Delta}_{\widetilde{\mu}}$ of the weighted manifold $(M,\widetilde{g},\widetilde{\mu})$ is given by

$$\widetilde{\Delta}_{\widetilde{\mu}} = \frac{1}{b}\operatorname{div}_\mu(\frac{b}{a}\nabla).$$

In particular, if $a=b$ then

$$\widetilde{\Delta}_{\widetilde{\mu}} = \frac{1}{a}\Delta_\mu.$$

Indeed, using the obvious fact $\widetilde{\nabla} = \frac{1}{a}\nabla$ where $\widetilde{\nabla}$ is the gradient of $\widetilde{g}$, we obtain by (2.5), for all $u,v\in C_0^\infty(M)$,

$$\begin{aligned}\int u\widetilde{\Delta}_{\widetilde{\mu}}v\,d\widetilde{\mu} &= -\int\langle\widetilde{\nabla}u,\widetilde{\nabla}v\rangle_{\widetilde{g}}d\widetilde{\mu} = -\int\frac{1}{a^2}\langle\nabla u,\nabla v\rangle_{ag}\,bd\mu \\ &= -\int\langle\nabla u,\frac{b}{a}\nabla v\rangle_g\,d\mu = \int u\operatorname{div}_\mu(\frac{b}{a}\nabla v)\,d\mu = \int u\left(\frac{1}{b}\operatorname{div}_\mu(\frac{b}{a}\nabla v)\right)d\widetilde{\mu},\end{aligned}$$

whence the claim follows.

2.2 Laplacian as an operator in L^2

Initially the operator Δ_μ is defined on smooth functions, in particular, on the space $\mathcal{D} := C_0^\infty(M)$ of smooth compactly supported functions, but then it extends by duality to distributions from the space $\mathcal{D}'$. However, our aim is to consider Δ_μ as an operator in $L^2 = L^2(M,\mu)$. By (2.5), the operator Δ_μ in L^2 with the domain $\mathcal{D}$ is symmetric and non-positive definite. It is natural to ask whether the operator $\Delta_\mu|_{\mathcal{D}}$ has a self-adjoint extension in L^2 and whether it is essentially self-adjoint. Let $W^1 = W^1(M,\mu)$ be the space of all functions $f \in L^2$, whose distributional gradient ∇f is also in L^2. Then W^1 is a Hilbert space with the inner product

$$(u,v)_{W^1} = \int_M uv\, d\mu + \int_M \langle \nabla u, \nabla v\rangle\, d\mu.$$

Let W_0^1 be the closure of $\mathcal{D}$ in W^1, and define the space $W_0^2 = W_0^2(M,\mu)$ as follows:

$$W_0^2 = \left\{ f \in W_0^1 : \Delta_\mu f \in L^2 \right\},$$

where $\Delta_\mu f$ is understood in distributional sense. Since $\mathcal{D} \subset W_0^2$, the operator $\Delta_\mu|_{W_0^2}$ is an extension of $\Delta_\mu|_{\mathcal{D}}$. It easily follows from the definitions of W_0^1, W_0^2 and (2.5) that the Green formula

$$\int_M u\Delta_\mu v\, d\mu = -\int_M \langle \nabla u, \nabla v\rangle\, d\mu \tag{2.6}$$

holds for all $u \in W_0^1$ and $v \in W_0^2$.

Theorem 2.2 ([35], [54], [75], [185], [192]) *The operator $\Delta_\mu|_{W_0^2}$ is a self-adjoint non-positive definite operator in L^2. Moreover, the operator $\Delta_\mu|_{W_0^2}$ is a unique self-adjoint extension of $\Delta_\mu|_{\mathcal{D}}$ with the domain in W_0^1.*

If in addition the manifold M is geodesically complete then the operator $\Delta_\mu|_{\mathcal{D}}$ is essentially self-adjoint, that is, $\Delta_\mu|_{W_0^2}$ is a unique self-adjoint extension of $\Delta_\mu|_{\mathcal{D}}$.

Approach to the proof. The first part amounts to the fact that the space W_0^1 is Hilbert so that the quadratic form $\mathcal{E}(u,v) = \int_M \langle \nabla u, \nabla v\rangle d\mu$ with the domain W_0^1 is closed in L^2. Therefore, it has the generator, which is a self-adjoint operator with domain W_0^2 and, hence, is the Friedrichs extension of $\Delta_\mu|_{\mathcal{D}}$.

In the second part, the completeness is used to ensure that the cutoff functions in geodesic balls have compact supports. Using them as test functions, one proves that any L^2-function u satisfying the equation $\Delta_\mu u - u = 0$ is identically equal to zero (see [216]). Consequently, one obtains that $u \in L^2$ and $\Delta_\mu u \in L^2$ imply $u \in W_0^1$, whence the claim follows. ∎

The operator $\Delta_\mu|_{W_0^2}$ is called the *Dirichlet Laplace operator* of (M,μ). This term is motivated by the following observation. Let Ω be a non-empty relatively compact open set in M, and consider the Dirichlet eigenvalue problem

$$\begin{cases} \Delta_\mu u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

where $\lambda = \text{const}$. This problem can be considered in a weak sense as follows: find a non-zero function $u \in W_0^1(\Omega,\mu)$ such that, for all $v \in W_0^1(\Omega,\mu)$,

$$-\int_\Omega \langle \nabla u, \nabla v\rangle\, d\mu + \lambda \int_\Omega uv\, d\mu = 0.$$

It is easy to prove that u is a solution to this problem if and only if $u \in W_0^2(\Omega, \mu)$ and $\Delta_\mu u + \lambda u = 0$. Considering (Ω, μ) as a weighted manifold, we conclude that the eigenvalues of the weak Dirichlet problem in Ω are exactly the eigenvalues of the self-adjoint operator $-\Delta_\mu|_{W_0^2(\Omega,\mu)}$ in $L^2(\Omega, \mu)$.

Theorem 2.3 ([185]) *For any non-empty relatively compact open set $\Omega \subset M$, the spectrum of the operator $-\Delta_\mu|_{W_0^2(\Omega,\mu)}$ is discrete and consists of a sequence $\{\lambda_k(\Omega)\}_{k=1}^{\infty}$ of non-negative real numbers such that $\lambda_k(\Omega) \to +\infty$ as $k \to \infty$.*

If in addition $M \setminus \overline{\Omega}$ is non-empty then $\lambda_1(\Omega) > 0$.

Approach to the proof. The discreteness of the spectrum of $-\Delta_\mu$ follows from the compactness of the resolvent $(-\Delta_\mu + \mathrm{id})^{-1}$. For the latter, one uses the compact embedding theorem saying that the identical embedding $W_0^1(\Omega, \mu) \hookrightarrow L^2(\Omega, \mu)$ is a compact operator. In $\mathbb{R}^n$ it is known as the Rellich-Kondrashov theorem, and on manifolds it can be deduced from the Rellich-Kondrashov theorem patching Ω by small charts. The compact embedding theorem is also used to prove the second assertion of Theorem 2.3. ■

Assuming that the eigenvalues in the sequence $\{\lambda_k(\Omega)\}$ are counted with multiplicity, one has the Weyl's asymptotic formula:

$$\lambda_k(\Omega) \sim c_n \left(\frac{k}{\mu(\Omega)}\right)^{2/n} \quad \text{as } k \to \infty, \tag{2.8}$$

where $n = \dim M$ and the constant $c_n > 0$ is the same as in $\mathbb{R}^n$.

Let M be a compact manifold. Then we have $\lambda_1(M) = 0$ because function $f = \mathrm{const}$ is an eigenfunction. Since $\Delta_\mu f = 0$ implies $f = \mathrm{const}$ (we assume by default that any Riemannian manifold is connected), the multiplicity of the bottom eigenvalue is 1 and, hence, $\lambda_2(M) > 0$. Evaluating or estimating the eigenvalues of the Laplacian on compact manifolds is an interesting and well developed area. We do not touch it here and refer the reader to [30], [31], [39], [59], [133], [190] and the references therein.

Let now (M, μ) be an arbitrary weighted manifold, and let $\lambda_{\min}(M)$ denote the bottom of the spectrum of $-\Delta_\mu|_{W_0^2(M,\mu)}$.

Theorem 2.4 (The Rayleigh principle) *On any weighted manifold (M, μ),*

$$\lambda_{\min}(M) = \inf_{f \in \mathcal{T} \setminus \{0\}} \frac{\int_M |\nabla f|^2 d\mu}{\int f^2 d\mu}, \tag{2.9}$$

where $\mathcal{T}$ is any class of test functions such that $\mathcal{D} \subset \mathcal{T} \subset W_0^1$.

Proof. By the variational principle for the operator $-\Delta_\mu|_{W_0^2}$ and by the Green formula (2.6), we obtain

$$\lambda_{\min}(M) = \inf_{f \in W_0^2 \setminus \{0\}} \frac{-(\Delta_\mu f, f)}{\|f\|_{L^2}^2} = \inf_{f \in W_0^2 \setminus \{0\}} \frac{\int_M |\nabla f|^2 d\mu}{\|f\|_{L^2}^2}.$$

The proof is finished by the observations that $\mathcal{D} \subset W_0^2 \subset W_0^1$ and $\mathcal{D}$ is dense in W_0^1. ■

For any open set $\Omega \subset M$, let $\lambda_{\min}(\Omega)$ be the bottom of the spectrum of $-\Delta_\mu|_{W_0^2(\Omega,\mu)}$. We say that a sequence $\{\Omega_k\}_{k=1}^{\infty}$ of open sets $\Omega_k \subset M$ is *exhausting* if $\Omega_k \subset \Omega_{k+1}$ for all k and the union of all sets Ω_k is M.

Corollary 2.5 *The function $\Omega \mapsto \lambda_{\min}(\Omega)$ is decreasing on expansion of Ω. Also, for any exhausting sequence $\{\Omega_k\}$, $\lambda_{\min}(\Omega_k) \to \lambda_{\min}(M)$ as $k \to \infty$.*

Proof. Both claims follow immediately from (2.9) with $\mathcal{T} = \mathcal{D}$ since the functional space $\mathcal{D}(\Omega)$ increases on expansion of Ω and the union of all spaces $\mathcal{D}(\Omega)$ is $\mathcal{D}(M)$. ■

In particular, we have $\lambda_{\min}(\Omega) \geq \lambda_{\min}(M)$.

2.3 Some examples

Example 2.6 Consider in $\mathbb{R}$ measure $d\mu = e^{-x^2}dx$. The Laplace operator of $(\mathbb{R}, \mu)$ is given by

$$\Delta_\mu = e^{x^2} \frac{d}{dx}\left(e^{-x^2}\frac{d}{dx}\right) = \frac{d^2}{dx^2} - 2x\frac{d}{dx}.$$

Its eigenfunctions are the Hermite polynomials

$$h_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}, \tag{2.10}$$

where $k = 0, 1, 2, ...,$ because they satisfies the equation

$$h_k'' - 2xh_k' + 2kh_k = 0.$$

Since the sequence $\{h_k\}_{k=0}^{\infty}$ forms an orthogonal basis in $L^2(\mathbb{R}, \mu)$, we conclude that the spectrum of $-\Delta_\mu|_{W_0^2}$ is discrete and consists of the simple eigenvalues $\{2k\}_{k=0}^{\infty}$.

Example 2.7 For the Euclidean space $\mathbb{R}^n$ with the Lebesgue measure μ, the operator Δ_μ coincides with the classical Laplace operator

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{(\partial x^i)^2}.$$

The spectrum of $-\Delta|_{W_0^2}$ is $[0, +\infty)$ and there are no eigenvalues, which can be easily established by using the Fourier transform.

Example 2.8 For the hyperbolic space $\mathbb{H}^n$ with the Riemannian measure μ, the spectrum of $-\Delta_\mu|_{W_0^2}$ is $[\frac{(n-1)^2}{4}, \infty)$, again without eigenvalues (see [30]).

Example 2.9 For the sphere $\mathbb{S}^n$ with the Riemannian measure μ, the spectrum of $-\Delta_\mu|_{W_0^2}$ is discrete, and its distinct eigenvalues are given by

$$\lambda_k = k(k+n-1), \quad k = 0, 1, ...,$$

where the multiplicity of λ_0 is 1 and the multiplicity of λ_k, $k \geq 1$, is equal to

$$\frac{(k+n-2)!}{(n-1)!k!}(2k+n-1)$$

(see [22] and [30]).

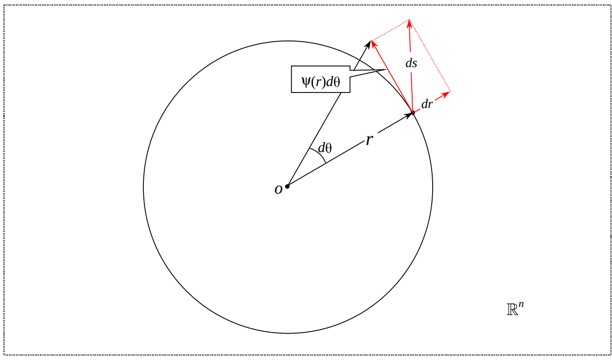


Figure 1: The Riemannian metric on a model manifold

2.4 Laplacian on model manifolds

Let g be a Riemannian metric on $\mathbb{R}^n$ such that g can be represented in the polar coordinates (r, θ) in $\mathbb{R}^n$ as follows

$$g = dr^2 + \psi^2(r) d\theta^2, \tag{2.11}$$

where $\psi(r)$ is a positive smooth function on $\mathbb{R}_+$. Here $\theta \in \mathbb{S}^{n-1}$ and $d\theta^2$ is the standard Riemannian metric on $\mathbb{S}^{n-1}$ (see Fig. 1).

manifold $M = (\mathbb{R}^n, g)$ is called a *Riemannian model.* Any positive smooth function ψ on $\mathbb{R}_+$ such that $\psi(0) = 0$, $\psi'(0) = 1$, and $\psi''(0) = 0$, determines such a manifold (see [80], [139]). For example, if $\psi(r) = r$ then M is $\mathbb{R}^n$ with the standard Euclidean metric. If $\psi(r) = \sinh r$ then M is isometric to the hyperbolic space $\mathbb{H}^n$. If $\psi(r) = \sin r$, where $0 < r < \pi$, then closing the annulus $\{(r, \theta) : 0 < r < \pi\}$ by adding two points $r = 0$ and $r = \pi$, we obtain $\mathbb{S}^n$. If $\psi(r) \equiv 1$ for $r > r_0$ then M can be viewed as the cylinder $\mathbb{R}_+ \times \mathbb{S}^{n-1}$, which is closed by gluing it to a compact manifold.

The Riemannian volume element of the model M is given by $d\nu = \psi^{n-1}(r)\, dr\, |d\theta|$ where $|d\theta|$ is the Riemannian volume element on $\mathbb{S}^{n-1}$. Fix a positive smooth function h on M, which depends only on the polar radius r, and define a new measure μ on M by $d\mu = h^2 d\nu$. The weighted manifold (M, μ) is called a *weighted model.*

Let o be the origin of the polar coordinates on M, and let B_r be the geodesic ball of radius r centered at o, that is

$$B_r = \{(\rho, \theta) \in M : 0 \le \rho < r\}.$$

Set $V(r) = \mu(B_r)$ and observe that

$$V(r) = \int_0^r S(t)\, dt$$

where $S(r) = \omega_n \psi(r)^{n-1} h^2(r)$ is the boundary area of the ball B_r and $\omega_n = \left|\mathbb{S}^{n-1}\right|$. We will refer to $V(r)$ as the *volume function* of the model and to $S(r)$ as the *boundary area function.*

The Laplace operator on (M, μ) is represented in the polar coordinates as follows

$$\Delta_\mu = \frac{\partial^2}{\partial r^2} + m(r) \frac{\partial}{\partial r} + \frac{1}{\psi^2(r)} \Delta_\theta, \tag{2.12}$$

where Δ_θ is the Laplace operator on $\mathbb{S}^{n-1}$ with the standard Riemannian metric, and the function $m(r)$ is defined by

$$m(r) := \frac{S'(r)}{S(r)} = (n-1) \frac{\psi'(r)}{\psi(r)} + 2 \frac{h'(r)}{h(r)}. \tag{2.13}$$

(see [35], [96]). If μ is the Riemannian measure of M then $m(r)$ is the mean curvature of the sphere $|x| = r$ in the radial direction. For example, $m(r) = \frac{n-1}{r}$ in $\mathbb{R}^n$, $m(r) = (n-1)\coth r$ in $\mathbb{H}^n$, and $m(r) = (n-1)\cot r$ on $\mathbb{S}^n$. For an arbitrary weighted model (M, μ), we will still refer to $m(r)$ as the *mean curvature function* of (M, μ).

Theorem 2.10 ([94]) *For any ball B_r on a weighted model (M, μ), we have*

$$\frac{1}{4F(r)} \leq \lambda_1(B_r) \leq \frac{1}{F(r)}, \tag{2.14}$$

where the function F is defined by

$$F(r) := \sup_{0<\xi<r} \left[V(\xi) \int_{\xi}^{r} \frac{dt}{S(t)} \right].$$

Approach to the proof. The proof is based on a certain relation between eigenvalues and capacities (see [158, Theorem 2.3.2/1]) and on the fact that the capacities of the balls B_r can be explicitly computed (see [82], [96]). ■

Theorem 2.10 will be used below in Section 5.5.

3 The heat kernel

In this section, (M, μ) is always a weighted manifold unless otherwise stated.

3.1 Heat semigroup

Any self-adjoint semi-bounded above operator H in $L^2 = L^2(M, \mu)$ defines the *heat semigroup* $\{e^{tH}\}_{t \geq 0}$, which is a family of positive definite bounded self-adjoint operators in L^2. We will normally deal with the heat semigroup

$$P_t := e^{t\Delta_\mu} \tag{3.1}$$

where the domain of Δ_μ is W_0^2, and refer to $\{P_t\}_{t \geq 0}$ as the heat semigroup of the weighted manifold (M, μ).

Theorem 3.1 ([35], [54], [185], [190], [192]) *For any $f \in L^2$, the function $(t, x) \mapsto P_t f(x)$ has a version $u(t, x)$ that is C^∞ smooth in $(t, x) \in \mathbb{R}_+ \times M$. The function $u(t, x)$ satisfies the heat equation*

$$\frac{\partial u}{\partial t} = \Delta_\mu u,$$

the initial condition

$$u(t, \cdot) \xrightarrow{L^2} f \quad \text{as } t \to 0+,$$

and the estimate

$$\operatorname{essinf} f \leq u(t, x) \leq \operatorname{esssup} f. \tag{3.2}$$

Approach to the proof. Using the functional calculus of self-adjoint operators, one easily shows that the function $P_t f(x)$ satisfies the heat equation in the L^2-sense, that is, as a path $t \mapsto P_t f$ in L^2. Then one applies the Weyl's lemma saying that a distributional solution to the heat equation is, in fact, a C^∞-smooth function (for example, this follows from Hörmander's condition for hypoellipticity – see [126, Theorem 22.2.1]). Alternatively, C^∞-smoothness can be proved using elliptic regularity and the Sobolev embedding theorem (see [54] and [190]). The proof of (3.2) is based on a version of the parabolic maximum principle. ■

From now on, we will denote by $P_t f$ its C^∞-version.

3.2 Heat kernel and fundamental solutions

A smooth function $u(t,x)$ on $\mathbb{R}_+ \times M$ is called a *fundamental solution* of the heat equation at a point $y \in M$ if the function $u(t,x)$ satisfies in $\mathbb{R}_+ \times M$ the heat equation

$$\frac{\partial u}{\partial t} = \Delta_\mu u \tag{3.3}$$

and the Dirac condition

$$u(t,\cdot) \to \delta_y \quad \text{as } t \to 0+,$$

The latter is understood in the distributional sense as follows: for any $\varphi \in \mathcal{D} := C_0^\infty(M)$,

$$\int_M u(t,x)\,\varphi(x)\,d\mu(x) \to \varphi(y) \quad \text{as } t \to 0+. \tag{3.4}$$

If in addition $u(t,x)$ is positive and, for all $t>0$,

$$\int_M u(t,x)\,d\mu(x) \le 1, \tag{3.5}$$

then $u(t,x)$ is called a *regular* fundamental solution at y. Observe that if a function $u(t,x)$ is positive and satisfies (3.5) then the Dirac condition (3.4) is equivalent to the following: for any open set U containing y,

$$\int_U u(t,x)\,d\mu(x) \to 1 \quad \text{as } t \to 0+.$$

Lemma 3.2 *Extend a regular fundamental solution $u(t,x)$ to $t \le 0$ by setting $u(t,x) \equiv 0$. Then $u(t,x)$ satisfies in $\mathbb{R} \times M$ the following equation in the distributional sense:*

$$\frac{\partial u}{\partial t} - \Delta_\mu u = \delta_{0,y}, \tag{3.6}$$

where $\delta_{0,y}$ is the Dirac function in $\mathbb{R} \times M$ with the pole at $(0,y)$. Also, $u(t,x)$ is C^∞-smooth in $\mathbb{R} \times M$ away from the pole $(0,y)$.

Sketch of proof. By (3.5), $u(t,x)$ is locally integrable in $\mathbb{R} \times M$ so that it indeed can be considered as a distribution in $\mathbb{R} \times M$. The equation (3.6) easily follows from (3.3) and (3.4). Since u satisfies the heat equation (3.3) in $\mathbb{R} \times M$ away from $(0,y)$, it follows by Weyl's lemma that $u(t,x)$ is C^∞-smooth in this domain. ■

A function $q_t(x,y)$ on $\mathbb{R}_+ \times M \times M$ is called a (regular) fundamental solution of the heat equation if, for any $y \in M$, the function $(t,x) \mapsto q_t(x,y)$ is a (regular) fundamental solution at y.

A function $p_t(x,y)$ defined on $\mathbb{R}_+ \times M \times M$ is called the *heat kernel* of the operator Δ_μ if, for all $f \in L^2$, $t>0$, $x \in M$,

$$P_t f(x) = \int_M p_t(x,y)\,f(y)\,d\mu(y). \tag{3.7}$$

In other words, the heat kernel p_t is the integral kernel of the operator $P_t = e^{t\Delta_\mu}$.

Theorem 3.3 ([22], [23], [30], [35], [54], [185], [190], [192]) *Any weighted manifold (M,μ) possesses a unique heat kernel $p_t(x,y)$, which is a C^∞ function on $\mathbb{R}_+ \times M \times M$. Furthermore, the heat kernel satisfies the following properties.*

- *$p_t(x,y)$ is a regular fundamental solution to the heat equation.*
- *Symmetry:*
$$p_t(x,y) = p_t(y,x). \tag{3.8}$$
- *The semigroup identity:*
$$p_{t+s}(x,y) = \int_M p_t(x,z)\, p_s(z,y)\, d\mu(z). \tag{3.9}$$

Approach to the proof. The difficult part is the existence of the heat kernel. The classical approach, which goes back to [163], [164] uses a parametrix of the heat equation to construct a fundamental solution $p_t(x,y)$ with the above properties. Then one proves that $p_t(x,y)$ is the integral kernel of the heat semigroup P_t.

An alternative approach is to construct first the heat kernel p_t^Ω in a relatively compact open set $\Omega \subset M$ using the eigenfunction expansion formula (3.14) (cf. Example 3.4 below). Then one shows that p_t^Ω increases as Ω expands, which allows to construct the heat kernel p_t as the limit of p_t^Ω when Ω exhausts M (cf. Theorem 3.5 below). The finiteness of the limit follows from $\int_\Omega p_t^\Omega(x,y)\, d\mu(y) \le 1$.

Yet another approach uses the Sobolev embedding theorem in relatively compact charts on M to show that, for any compact set $K \subset M$, there exists a function $F_K(t)$ such that, for all $f \in L^2$ and $t > 0$,
$$\sup_K |P_t f| \le F_K(t)\, \|f\|_{L^2} \tag{3.10}$$
(this approach can also be used in the proof of Theorem 3.1). Then the existence of the heat kernel follows by the Riesz representation theorem. Note that in all the proofs the existence of the heat kernel comes from certain *local* properties of manifolds inherited from $\mathbb{R}^n$.

The symmetry (3.8) of the heat kernel follows from the self-adjointness of operator P_t, and the semigroup identity (3.9) follows from $P_{t+s} = P_t P_s$. ■

Since $p_t \ge 0$ and
$$\int_M p_t(x,y)\, d\mu(y) \le 1, \tag{3.11}$$
the identity (3.7) allows to extend the operator P_t to all bounded or non-negative Borel functions f on M.

It follows from (3.8) and (3.9) that
$$p_{2t}(x,x) = \int_M p_t(x,y)^2\, d\mu(y) = \|p_t(x,\cdot)\|_{L^2}^2, \tag{3.12}$$
which in particular implies that $p_t(x,\cdot) \in L^2$.

Example 3.4 If $\{E_\lambda\}_{\lambda \ge 0}$ is the spectral resolution of the operator $-\Delta_\mu|_{W_0^2}$ in $L^2(M,\mu)$ then we obtain from (3.1)
$$P_t = \int_0^\infty e^{-\lambda t} dE_\lambda. \tag{3.13}$$
Assume that the spectrum of $-\Delta_\mu|_{W_0^2}$ is discrete and consists of eigenvalues $\{\lambda_k(M)\}_{k=1}^\infty$ (counted with multiplicity), and let $\{\varphi_k\}_{k=1}^\infty$ be the corresponding sequence of eigenfunctions forming an orthonormal basis in L^2. Then (3.13) implies the following eigenfunction expansion formula for the heat kernel:
$$p_t(x,y) = \sum_{k=1}^\infty e^{-\lambda_k(M)t} \varphi_k(x)\, \varphi_k(y). \tag{3.14}$$

As a consequence, we obtain

$$\lim_{t\to\infty}\frac{\log p_t(x,y)}{t}=-\lambda_1(M). \tag{3.15}$$

Another useful consequence of (3.14) is a *trace formula*:

$$\int_M p_t(x,x)\,d\mu(x)=\sum_{k=1}^{\infty}e^{-\lambda_k(M)t}. \tag{3.16}$$

For any open subset $\Omega\subset M$, denote by $p_t^\Omega(x,y)$ the heat kernel of the weighted manifold (Ω,μ). The next result is based on the parabolic comparison principle.

Theorem 3.5 ([30], [66]) *The heat kernel p_t^Ω increases on expansion of Ω. Moreover, if $\{\Omega_k\}_{k=1}^\infty$ is an exhausting sequence then $p^{\Omega_k}(x,y)\to p_t(x,y)$ as $k\to\infty$.*

Combining Theorem 3.5 with the parabolic comparison principle, one obtains the following alternative characterization of the heat kernel.

Corollary 3.6 ([30], [66]) *For any $y\in M$, the heat kernel $p_t(x,y)$ is the* ***minimal*** *positive fundamental solution at y.*

Hence, $p_t(x,y)$ is also the minimal regular fundamental solution at y.

Corollary 3.7 *On any weighted manifold (M,μ), we have, for all $x,y\in M$,*

$$\lim_{t\to\infty}\frac{\log p_t(x,y)}{t}=-\lambda_{\min}(M). \tag{3.17}$$

Proof. Set $\lambda=\lambda_{\min}(M)$. Since the spectrum of P_t is bounded by $e^{-\lambda t}$, we obtain that $\|P_t\|\le e^{-\lambda t}$ and hence, for any $f\in L^2$,

$$\|P_tf\|_{L^2}\le e^{-\lambda t}\|f\|_{L^2}. \tag{3.18}$$

Setting $f=p_s(x,\cdot)$, for some $s>0$, $x\in M$, and noticing that by (3.8) and (3.9) $P_tf=p_{t+s}(x,\cdot)$, we obtain by (3.12) and (3.18)

$$p_{2(t+s)}(x,x)=\|p_{t+s}(x,\cdot)\|_{L^2}^2\le e^{-2\lambda t}\|p_s(x,\cdot)\|_{L^2}^2=e^{-2\lambda t}p_{2s}(x,x), \tag{3.19}$$

whence it follows that

$$\limsup_{t\to\infty}\frac{\log p_t(x,x)}{t}\le-\lambda.$$

Observing that

$$p_t(x,y)=\int_M p_{\frac t2}(x,z)\,p_{\frac t2}(z,y)\,d\mu(z)\le\|p_{\frac t2}(x,\cdot)\|_{L^2}\|p_{\frac t2}(y,\cdot)\|_{L^2}=\sqrt{p_t(x,x)\,p_t(y,y)},$$

we obtain

$$\limsup_{t\to\infty}\frac{\log p_t(x,y)}{t}\le-\lambda.$$

To prove the opposite inequality, take any relatively compact open set $\Omega\subset M$ and recall that by Theorem 2.3 the spectrum of the Dirichlet Laplace operator in (Ω,μ) is discrete. By Example 3.4, the heat kernel p_t^Ω of (Ω,μ) satisfies (3.17). Since by Theorem 3.5 $p_t\ge p_t^\Omega$, we obtain

$$\liminf_{t\to\infty}\frac{\log p_t(x,y)}{t}\ge-\lambda_{\min}(\Omega).$$

Letting $\Omega \to M$ and noticing that, by Corollary 2.5, $\lambda_{\min}(\Omega) \to \lambda$, we finish the proof. ■

An interesting question of describing the large time behavior of

$$e^{-\lambda_{\min}(M)t} p_t(x,y)$$

was settled in [34], [176], [178].

The following two results use the local Euclidean methods to obtain short time asymptotics of the heat kernel.

Theorem 3.8 ([174], [206]) *On any weighted manifold (M, μ), we have*

$$\lim_{t \to 0} t \log p_t(x,y) = -\frac{1}{4} d^2(x,y), \tag{3.20}$$

where $d(x,y)$ is the geodesic distance between the points $x, y \in M$.

See [124] for an abstract version of (3.20).

Theorem 3.9 ([22], [79]) *On any weighted manifold (M, μ) of dimension n, there exists a smooth positive function $u(x,y)$ on $M \times M$ such that, for all $x \in M$,*

$$p_t(x,y) \sim \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{d^2(x,y)}{4t}\right) u(x,y) \quad \text{as } t \to 0, \tag{3.21}$$

provided y remains in a small enough neighborhood of x.

See [165] for a short time asymptotics of $p_t(x,y)$ for arbitrary x, y.

3.3 Stochastic completeness

A weighted manifold (M, μ) is called *stochastically complete* if, for all $y \in M$ and $t > 0$,

$$\int_M p_t(x,y)\, d\mu(x) = 1. \tag{3.22}$$

Corollary 3.10 *On a stochastically complete manifold, any regular fundamental solution $u(t,x)$ at a point $y \in M$ coincides with the heat kernel $p_t(x,y)$ and hence is unique.*

Proof. Indeed, by Corollary 3.6, $u(t,x) \geq p_t(x,y)$ whereas by (3.5) and (3.22)

$$\int_M u(t,x)\, d\mu(x) \leq \int_M p_t(x,y)\, d\mu(x).$$

Hence, we conclude $u(t,x) \equiv p_t(x,y)$. ■

Consider the classical the Cauchy problem to find a smooth function $u(t,x)$ on $\mathbb{R}_+ \times M$ such that

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta_\mu u & \text{on } \mathbb{R}_+ \times M, \\ u(0+, \cdot) = f & \text{on } M, \end{cases} \tag{3.23}$$

where $f \in C(M)$ is a given function. If in addition u is required to be a bounded function then we refer to this problem as a *bounded Cauchy problem.* It follows from Theorem 3.1 that, for any $f \in C_b(M)$, the function $u(t,x) = P_t f(x)$ is a bounded solution to the Cauchy problem (3.23).

It is well-known that in $\mathbb{R}^n$ a solution to the bounded Cauchy problem is unique, but in general this is not the case. For example, if (M, μ) is stochastically incomplete then the Cauchy problem (3.23) with $f \equiv 1$ has two distinct solutions: $u(t,x) \equiv 1$ and $u(t,x) = P_t f(x) < 1$.

Theorem 3.11 ([51], [96], [140]) *The following conditions are equivalent.*

(a) *A manifold (M,μ) is stochastically complete.*

(b) *Any non-negative bounded solution to the equation $\Delta_\mu v - v = 0$ on (M,μ) is identical* 0.

(c) *The bounded Cauchy problem on (M,μ) has unique solution.*

A manifold (M,μ) is called *parabolic* if any bounded subharmonic function (that is, a bounded function $v \in C^2(M)$ satisfying $\Delta_\mu v \geq 0$) is identical constant. For example, if M is compact then (M,μ) is parabolic because any subharmonic function satisfies the strong maximum principle. It is well-known that $\mathbb{R}^n$ is parabolic if and only if $n \leq 2$ (for general criteria of parabolicity see Section 9.1).

Corollary 3.12 *Any parabolic manifold is stochastically complete.*

Proof. Indeed, any non-negative bounded solution to the equation $\Delta_\mu v - v = 0$ is subharmonic, whence it follows that $v = \text{const}$. Applying the equation again, we conclude $v = 0$. ■

A convenient criterion of stochastic completeness is given in terms of the volume function of balls. Let $d(x,y)$ be the geodesic distance on M and let $B(x,r)$ denote the geodesic ball on M of radius r centered at x, that is

$$B(x,r) = \{y \in M : d(x,y) < r\}.$$

Recall that a manifold M is geodesically complete if and only if all geodesic balls are relatively compact sets.

Define the *volume function* $V(x,r)$ of (M,μ) by

$$V(x,r) := \mu(B(x,r)),$$

and notice that $V(x,r)$ is finite on any geodesically complete manifold.

Theorem 3.13 ([83], [96]) *Let M be a geodesically complete manifold. If, for some point $x \in M$,*

$$\int^\infty \frac{r\,dr}{\log V(x,r)} = \infty, \tag{3.24}$$

then (M,μ) is stochastically complete.

Approach to the proof. To show that any bounded solution $u(t,x)$ to the Cauchy problem (3.23) with the initial function $f = 0$ is identical zero, one uses a test function in the form $u\eta^2 e^\xi$ where η is a cutoff function and

$$\xi(t,x) := \frac{d^2(x,A)}{4(t-t_0)},$$

where A is a certain subset of M. A crucial observation is that $|\nabla d(\cdot,A)| \leq 1$ and hence ξ satisfies the inequality

$$\frac{\partial \xi}{\partial t} + |\nabla \xi|^2 \leq 0. \tag{3.25}$$

See [83], [96] for further details. ■

Remark 3.14 The condition (3.24) is, in particular, satisfied if, for large enough r,

$$V(x,r) \leq \exp\left(Cr^2\right). \tag{3.26}$$

The fact that (3.26) implies the stochastic completeness was also proved by various methods in [55], [138], [195], [199]. Historically, the first result in this direction is due to Gaffney [76] who proved the stochastic completeness assuming that $\log V(x,r) = o(r)$.

Note that the stochastic completeness is in general *not* stable under quasi-isometry[1] as was shown in [155].

Example 3.15 Let (M,μ) be a weighted model introduced in Section 2.4. Then (M,μ) is stochastically complete if and only if

$$\int^{\infty} \frac{V(r)}{S(r)} dr = \infty \tag{3.27}$$

(see [86] and [96]). For example, if $V(r) = \exp\left(r^{2+\varepsilon}\right)$ where $\varepsilon > 0$ then the integral in (3.27) converges and hence (M,μ) is stochastically incomplete (the first example of geodesically complete but stochastically incomplete manifold was produced in [8]). This also shows that the condition (3.24) for stochastic completeness is sharp.

Example 3.16 Let (M,μ) be $\mathbb{R}^n$ with the Lebesgue measure. It is well known that in this case the Gauss-Weierstrass function

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right) \tag{3.28}$$

is a regular fundamental solution to the heat equation. Since $\mathbb{R}^n$ is stochastically complete (which follows from (3.26), or (3.27), or Theorem 3.11), we conclude by Corollary 3.10 that the function (3.28) is the heat kernel in $\mathbb{R}^n$.

Example 3.17 Let M be a geodesically complete manifold with bounded below Ricci curvature, and let μ be its Riemannian measure. It follows from the Bishop-Gromov volume comparison theorem that

$$V(x,r) \leq \exp(Cr) \tag{3.29}$$

(see for example [27]) so that M is stochastically complete. In particular, $\mathbb{R}^n$ and $\mathbb{H}^n$ are stochastically complete. The stochastic completeness for Riemannian manifolds with bounded below Ricci curvature was first proved in [217] (see also [96], [127], [129] for certain extensions of this result). It was proved earlier in [8] that a *Cartan-Hadamard* manifold is stochastically complete provided its sectional curvature is bounded below.

Example 3.18 We say that a weighted manifold (M,μ) has *bounded geometry* if there exists a positive number r_0 (called a radius of bounded geometry) such that all geodesic balls $(B(x,r_0),\mu)$ are uniformly quasi-isometric to a Euclidean ball of radius r_0. We claim that a manifold of bounded geometry is stochastically complete. It is easy to see that for such a manifold there is a constant N such that any ball of radius r_0 can be covered by at most N balls of radius $r_0/2$. Then one proves by induction in k that any ball of radius $kr_0/2$ can be covered by at most N^{k-1} balls of radii $r_0/2$. It follows that all balls are relatively compact and the volume function $V(x,r)$ satisfies (3.29), whence the claim follows.

[1] A *quasi-isometry* is a diffeomorphism of two weighted manifolds, which changes the geodesic distances and the measures at most by a constant factor.

It is also worth mentioning that on manifolds of bounded geometry not only a regular fundamental solution is unique but also any positive fundamental solution is unique and hence coincides with the heat kernel (see [130], [143], [170]).

Example 3.19 Let (M,μ) be a geodesically complete weighted manifold with $\mu(M)<\infty$. Let us show that for all $x,y\in M$,

$$p_t(x,y)\to\frac{1}{\mu(M)}\quad\text{as } t\to\infty. \tag{3.30}$$

Clearly, 0 is an eigenvalue of the Dirichlet Laplace operator on (M,μ) with eigenfunction $\varphi(x)\equiv\frac{1}{\sqrt{\mu(M)}}$. Using the spectral decomposition (3.13) for the heat semigroup P_t and noticing that $e^{-\lambda t}\to\mathbf{1}_{\{\lambda=0\}}$ as $t\to\infty$, we obtain by the dominated convergence theorem that, for any $f\in L^2$,

$$P_tf\to\int_{\{0\}}dE_\lambda f=(f,\varphi)\,\varphi\quad\text{as } t\to\infty.$$

Choose $f=p_s(x,\cdot)$, for some $s>0$ and $x\in M$, and observe that by (3.8) and (3.9) $P_tf=p_{t+s}(x,\cdot)$, whereas by the stochastic completeness of (M,μ),

$$(f,\varphi)\,\varphi=\frac{1}{\mu(M)}\int_M p_s(x,\cdot)\,d\mu\equiv\frac{1}{\mu(M)}.$$

Combining the two previous equations, we obtain (3.30).

4 Relations between different heat kernels

4.1 Direct products

We say that a weighted manifold (M,μ) is the *direct product* of weighted manifolds (M',μ') and (M'',μ'') if M is the Riemannian product of M', M'' and $\mu=\mu'\otimes\mu''$. The corresponding Dirichlet Laplacians $\Delta_{\mu'}$ and $\Delta_{\mu''}$, obviously extended to $L^2(M,\mu)$, commute and $\Delta_\mu=\Delta_{\mu'}+\Delta_{\mu''}$. Therefore, $e^{t\Delta_\mu}=e^{t\Delta_{\mu'}}e^{t\Delta_{\mu''}}$ and hence the heat kernel p_t on M is a tensor product of the heat kernels p_t' and p_t'' on M' and M'', respectively; that is,

$$p_t(x,y)=p_t'\left(x',y'\right)p_t''\left(x'',y''\right) \tag{4.1}$$

where $x=(x',x'')\in M$ and $y=(y',y'')\in M$.

For example, to prove that (3.28) gives the heat kernel in $\mathbb{R}^n$ one can first show this in the case $n=1$ and then extends to an arbitrary n by induction using (4.1).

For another example, let (M',μ') be $\mathbb{R}^m$ with the Lebesgue measure and (M'',μ'') be a compact manifold. Then by (4.1), (3.28) and (3.30), the heat kernel on the direct product has the following asymptotics as $t\to\infty$:

$$p_t(x,y)\sim\mu''(M'')^{-1}\,(4\pi t)^{-m/2}\,. \tag{4.2}$$

4.2 Isometries

We say that a mapping $I:M\to M$ is an isometry of a weighted manifold (M,g,μ) if I is a diffeomorphism preserving g and μ. We claim that then I preserves the heat kernel, that is,

$$p_t(I(x),I(y))\equiv p_t(x,y)\,. \tag{4.3}$$

Define the operator J on functions by $Jf = f \circ I^{-1}$. It is clear that J maps L^2 to L^2 and $\mathcal{D}$ to $\mathcal{D}$, and it is follows from the definition of Δ_μ that $J \circ \Delta_\mu = \Delta_\mu \circ J$ on $\mathcal{D}$. By duality, this identity extends to $\mathcal{D}'$ and in particular to W_0^2, which means that the Dirichlet Laplacian commutes with J. Hence, so does also the operator P_t, whence (4.3) follows.

For example, if (M, g, μ) is a weighted model and o is its origin then the rotation group in $\mathbb{S}^{n-1}$ acts isometrically on M. We conclude that the heat kernel $p_t(o, x)$ does not depend on the polar angle θ of x and hence can be viewed as a function of t and r only where r is the polar radius of x.

4.3 Comparison of heat kernels

We start with the following elementary lemma.

Lemma 4.1 *Let (M, μ) and $(\widetilde{M}, \widetilde{\mu})$ be weighted models, $o, \widetilde{o}$ be their origins, $S(r), \widetilde{S}(r)$ be their boundary area functions, and $p_t, \widetilde{p}_t$ be their heat kernels, respectively. Assume that (M, μ) is stochastically complete and that $S(r) \equiv \widetilde{S}(r)$. Then $p_t(o, x) = \widetilde{p}_t(\widetilde{o}, \widetilde{x})$ whenever $|x| = |\widetilde{x}|$.*

Proof. The function $u(t, x) = p_t(o, x)$ is a regular fundamental solution on M at o, that is

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u, \\ \displaystyle\int_M u(t, x)\, d\mu(x) \leq 1, \\ \displaystyle\int_{B_R} u(t, x)\, d\mu(x) \to 1 \quad \text{as } t \to 0. \end{cases} \tag{4.4}$$

Since $u(t, x)$ depends only on t and $r = |x|$ (see Section 4.2) we can use the notation $u = u(t, r)$ and rewrite (4.4) as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{S'(r)}{S(r)} \dfrac{\partial u}{\partial r}, \\ \displaystyle\int_0^\infty u(t, r)\, S(r)\, dr \leq 1, \\ \displaystyle\int_0^R u(t, r)\, S(r)\, dr \to 1 \quad \text{as } t \to 0. \end{cases} \tag{4.5}$$

Since $S(r) = \widetilde{S}(r)$, these conditions are satisfied also with S replaced by $\widetilde{S}$, which means that $u(t, r)$ is a regular fundamental solution on $(\widetilde{M}, \widetilde{\mu})$ at $\widetilde{o}$. The stochastic completeness of (M, μ) implies that of $(\widetilde{M}, \widetilde{\mu})$ because the manifolds satisfy (3.27) simultaneously. By the stochastic completeness of $(\widetilde{M}, \widetilde{\mu})$, we conclude that $u(t, r)$ coincides with the heat kernel $\widetilde{p}_t(\widetilde{o}, \widetilde{x})$ for any $\widetilde{x} \in \widetilde{M}$ with $|\widetilde{x}| = r$, which was to be proved. ■

In fact, the statement of Lemma 4.1 is true without the assumption of stochastic completeness.

The idea behind the proof of Lemma 4.1 can be extended to obtain *inequalities* between the heat kernels. Let (M, μ) be a geodesically complete weighted manifold and let $o \in M$. Then the polar coordinates (r, θ) at o are defined in the domain $M \setminus \{o\} \setminus \mathrm{cut}(o)$ where $\mathrm{cut}(o)$ is the cut locus of o (see Fig. 2).

The Laplace operator of (M, μ) has the following representation in the domain of the polar coordinates (r, θ):

$$\Delta_\mu u = \frac{\partial^2 u}{\partial r^2} + m(r, \theta) \frac{\partial u}{\partial r} + \Delta_{S_r} u, \tag{4.6}$$

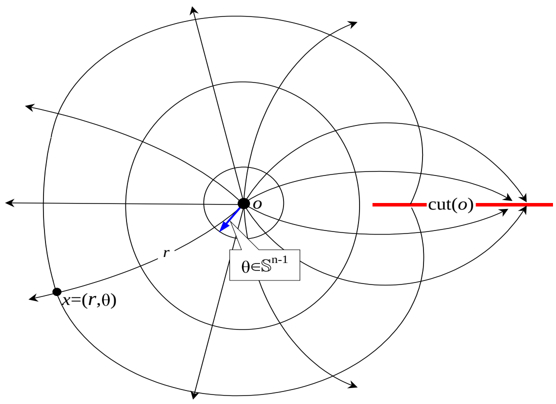


Figure 2: The polar coordinates in $M\setminus\{o\}\setminus\mathrm{cut}(o)$

where $S_r=\partial B(o,r)$ is the geodesic sphere, Δ_{S_r} is the Laplace operator on S_r, and $m(r,\theta)$ is the mean curvature of S_r at the point (r,θ) in the radial direction (see [35], [80], [96], [190]). Note that (2.12) is a particular case of (4.6). For any $x\in M$, we write $|x|=d(o,x)$.

Theorem 4.2 *Let (M,μ), o, $m(r,\theta)$ be as above, and let p_t be the heat kernel on (M,μ). Let $(\widetilde{M},\widetilde{\mu})$ be a weighted model with the origin $\widetilde{o}$, and let $\widetilde{m}(r)$ be its mean curvature function and $\widetilde{p}_t$ be the heat kernel on $(\widetilde{M},\widetilde{\mu})$.*

(a) ([35]) *If in the domain of the polar coordinates on (M,μ),*

$$m(r,\theta)\leq\widetilde{m}(r),\tag{4.7}$$

then

$$p_t(o,x)\geq\widetilde{p}_t(\widetilde{o},\widetilde{x})$$

for all $t>0$ and all $x\in M$, $\widetilde{x}\in\widetilde{M}$ such that $|x|=|\widetilde{x}|$.

(b) ([62]) *Assume in addition that $\mathrm{cut}(o)=\emptyset$. If, for all $r>0$ and $\theta\in\mathbb{S}^{n-1}$,*

$$m(r,\theta)\geq\widetilde{m}(r),$$

then

$$p_t(o,x)\leq\widetilde{p}_t(\widetilde{o},\widetilde{x}),$$

for all $t>0$ and all $x\in M$, $\widetilde{x}\in\widetilde{M}$ such that $|x|=|\widetilde{x}|$.

Approach to the proof. (a) One of the key points is that the function $u(t,x)=\widetilde{p}_t(\widetilde{o},x)$ on $\widetilde{M}$ is decreasing in $|x|$, that is $\frac{\partial u}{\partial r}\leq 0$. Transplanting this function on M by means of polar coordinates and using (4.6), (4.7), we obtain that

$$\Delta_\mu u=\frac{\partial^2u}{\partial r^2}+m(r,\theta)\frac{\partial u}{\partial r}\geq\frac{\partial^2u}{\partial r^2}+\widetilde{m}(r)\frac{\partial u}{\partial r}=\widetilde{\Delta}_{\widetilde{\mu}}u$$

whence

$$\frac{\partial u}{\partial t}-\Delta_\mu u\leq 0.\tag{4.8}$$

This inequality holds in the domain of polar coordinates, that is, away from $\mathrm{cut}(o)$. If $\mathrm{cut}(o)=\emptyset$ then, by the comparison principle for solutions to parabolic equations, one concludes that $u(t,x)\leq p_t(o,x)$. In the general case, one still shows that (4.8) holds on M in a weak sense, for which one approximates $M\setminus\mathrm{cut}(o)$ by domains with smooth boundaries.

(b) This part is proved in the same way, without complications with the cut locus. Note that the statement of part (b) is not true if $\mathrm{cut}(o)$ is non-empty. ■

4.4 Change of measure

Any real-valued function $\Phi \in L^2_{loc}(M,\mu)$ can be considered as a multiplication operator in $L^2(M,\mu)$ with the domain $\mathcal{D}$. Clearly, the sum $-\Delta_\mu + \Phi$ is a symmetric operator on $\mathcal{D}$, and one can ask if it admits a self-adjoint extension. We start with the following simple but useful observation.

Lemma 4.3 *Assume that a smooth positive function h on M satisfies the equation*

$$\Delta_\mu h - \Phi h = 0. \tag{4.9}$$

Let $\widetilde{\mu}$ be a measure on M defined by $d\widetilde{\mu} = h^2 d\mu$. Then

$$\Delta_{\widetilde{\mu}} = \frac{1}{h} \circ (\Delta_\mu - \Phi) \circ h, \tag{4.10}$$

where the both operators $\Delta_\mu - \Phi$ and $\Delta_{\widetilde{\mu}}$ are considered on the domain $\mathcal{D}(M)$.

Remark 4.4 The functions h and $\frac{1}{h}$ in (4.10) are considered as multiplication operators. In words, the identity (4.10) means that operators $\Delta_{\widetilde{\mu}}$ and $\Delta_\mu - \Phi$ are related by the *Doob transform.*

Proof. Observe that the mapping $f \mapsto hf$ provides an isometry of $L^2(M,\widetilde{\mu})$ and $L^2(M,\mu)$ which preserves $\mathcal{D}$. Using (4.9), we obtain that, for any $f \in \mathcal{D}$,

$$\begin{aligned}
\Delta_{\widetilde{\mu}} f &= \frac{1}{h^2}\operatorname{div}_\mu\left(h^2 \nabla f\right) = \Delta_\mu f + \frac{2\nabla h}{h}\nabla f \\
&= \frac{1}{h}\left(h\Delta_\mu f + 2\nabla h \nabla f + f\Delta_\mu h - \Phi f h\right) \\
&= \frac{1}{h}\left(\Delta_\mu (fh) - \Phi f h\right),
\end{aligned}$$

whence (4.10) follows. ∎

Corollary 4.5 *Under the hypothesis of* Lemma 4.3, *the operator $(-\Delta_\mu + \Phi)|_{\mathcal{D}}$ in $L^2(M,\mu)$ admits a self-adjoint extension. Furthermore, if M is geodesically complete then operator $(-\Delta_\mu + \Phi)|_{\mathcal{D}}$ is essentially self-adjoint in $L^2(M,\mu)$.*

Proof. By (4.10), the operators $\Delta_{\widetilde{\mu}}\big|_{\mathcal{D}}$ in $L^2(M,\widetilde{\mu})$ and $(\Delta_\mu - \Phi)|_{\mathcal{D}}$ in $L^2(M,\mu)$ are unitary equivalent. Using the self-adjoint extension of $\Delta_{\widetilde{\mu}}\big|_{\mathcal{D}}$ constructed in Theorem 2.2, we obtain a non-positive definite self-adjoint extension of the operator $(\Delta_\mu - \Phi)|_{\mathcal{D}}$. If M is geodesically complete then, by Theorem 2.2, the operator $\Delta_{\widetilde{\mu}}\big|_{\mathcal{D}}$ is essentially self-adjoint in $L^2(M,\widetilde{\mu})$, whence it follows that $(-\Delta_\mu + \Phi)|_{\mathcal{D}}$ is essentially self-adjoint in $L^2(M,\mu)$. ∎

Example 4.6 It follows easily from (4.10) that for any $f \in \mathcal{D}$,

$$(f, (-\Delta_\mu + \Phi) f)_{L^2(M,\mu)} = -\left(g, \Delta_{\widetilde{\mu}} g\right)_{L^2(M,\widetilde{\mu})},$$

where $g = f/h$. Applying the Green formula to both sides, we obtain the inequality

$$\int_M \left(|\nabla f|^2 + \Phi f^2\right) d\mu \geq 0. \tag{4.11}$$

For example, if $M = \mathbb{R}^n \setminus \{o\}$, μ is the Lebesgue measure, and $h(x) = |x|^\beta$ then set

$$\Phi := \frac{\Delta h}{h} = \frac{\beta^2 + (n-2)\beta}{|x|^2}.$$

The constant $\beta^2 + (n-2)\beta$ takes the minimal value $-\frac{(n-2)^2}{4}$ for $\beta = 1 - \frac{n}{2}$. Hence, using this β, we obtain from (4.11) the *Hardy inequality*:

$$\int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{f^2}{|x|^2} d\mu,$$

for any $f \in \mathcal{D}(\mathbb{R}^n \setminus \{o\})$.

Let now $M = \{x \in \mathbb{R}^n : |x| > 1\}$ and $h(x) = |x|^{1-\frac{n}{2}} \log^\gamma |x|$. It is not difficult to check that this function satisfies the equation

$$\Delta h + \left(\frac{(n-2)^2}{4|x|^2} + \frac{\gamma(1-\gamma)}{|x|^2 \log^2 |x|} \right) h = 0. \tag{4.12}$$

Taking $\gamma = \frac{1}{2}$, we obtain the following improvement of the Hardy inequality:

$$\int_{\mathbb{R}^n} |\nabla f|^2 \, d\mu \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{f^2}{|x|^2} d\mu + \frac{1}{4} \int_{\mathbb{R}^n} \frac{f^2}{|x|^2 \log^2 |x|} d\mu,$$

for any $f \in \mathcal{D}(|x| > 1)$. For further results in this direction see [10], [11].

Lemma 4.7 *Assume that a smooth positive function h on M satisfies the equation $\Delta_\mu h - \Phi h = 0$. Let H be the self-adjoint extension of the operator $(-\Delta_\mu + \Phi)|_{\mathcal{D}}$ in $L^2(M,\mu)$ constructed in* Corollary 4.5. *Then the corresponding heat semigroup $P_t^\Phi = e^{-tH}$ has the heat kernel p_t^Φ given by*

$$p_t^\Phi(x,y) = \widetilde{p}_t(x,y)\, h(x)\, h(y), \tag{4.13}$$

where $\widetilde{p}_t$ is the heat kernel of the weighted manifold $(M, \widetilde{\mu})$ with measure $\widetilde{\mu}$ defined by

$$d\widetilde{\mu} = h^2 d\mu. \tag{4.14}$$

Remark 4.8 It is useful to observe that $P_t^\Phi = e^{-\lambda t} P_t^{\Phi - \lambda}$, where $\lambda = \text{const}$. Hence, we obtain from (4.13) the identity

$$\widetilde{p}_t(x,y) = \frac{p_t^{\Phi-\lambda}(x,y)\, e^{-\lambda t}}{h(x)\, h(y)}. \tag{4.15}$$

Proof. Let $\widetilde{H}$ be the operator $-\Delta_{\widetilde{\mu}}$ with the domain $W_0^2(M, \widetilde{\mu})$. By Lemma 4.3 and Corollary 4.5, the operators H and $\widetilde{H}$ are related by the Doob transform $\widetilde{H} = \frac{1}{h} \circ H \circ h$ whence $H = h \circ \widetilde{H} \circ \frac{1}{h}$. Hence, the heat semigroup $P_t^\Phi = e^{-tH}$ in $L^2(M,\mu)$ is related to the heat semigroup $\widetilde{P}_t = e^{-t\widetilde{H}}$ in $L^2(M, \widetilde{\mu})$ by

$$P_t^\Phi = h \circ \widetilde{P}_t \circ \frac{1}{h}.$$

Therefore, for any $f \in \mathcal{D}$,

$$P_t^\Phi f(x) = h(x) \int_M \widetilde{p}_t(x,y)\, f(y) \frac{1}{h(y)} d\widetilde{\mu}(y) = h(x) \int_M \widetilde{p}_t(x,y)\, f(y)\, h(y)\, d\mu(y),$$

whence (4.13) follows. ∎

4.5 Some examples of heat kernels in $\mathbb{R}$

Let M be an open interval in $\mathbb{R}$, μ be the Lebesgue measure in M, and h be any positive smooth function on M that defines a new measure $\widetilde{\mu}$ by $d\widetilde{\mu} = h^2 d\mu$. Then we have

$$\Delta_{\widetilde{\mu}} = \frac{d^2}{dx^2} + 2\varphi \frac{d}{dx},$$

where $\varphi = \frac{h'}{h}$. On the other hand, a simple computation shows that $h'' - \Phi h = 0$ where

$$\Phi = \varphi^2 + \varphi'.$$

We conclude by (4.10) that

$$\Delta_{\widetilde{\mu}} = \frac{d^2}{dx^2} + 2\varphi \frac{d}{dx} = \frac{1}{h} \circ \left(\frac{d^2}{dx^2} - \left(\varphi^2 + \varphi' \right) \right) \circ h. \tag{4.16}$$

Let us consider now some concrete examples of the function h.

Example 4.9 Let $M = \mathbb{R}$ and

$$h(x) = e^{-\frac{1}{2}x^2}.$$

As follows from Example 2.6 and (3.14), the heat kernel $\widetilde{p}_t$ of $(\mathbb{R}, e^{-x^2} dx)$ admits the following representation

$$\widetilde{p}_t(x, y) = \sum_{k=0}^{\infty} e^{-2kt} \frac{h_k(x) h_k(y)}{\sqrt{\pi} 2^k k!},$$

where h_k are the Hermite polynomials defined by (2.10). This series can be evaluated (cf. [50, p.181]), which leads to the formula

$$\widetilde{p}_t(x, y) = \frac{1}{(2\pi \sinh 2t)^{1/2}} \exp\left(\frac{2xye^{-2t} - \left(x^2 + y^2\right) e^{-4t}}{1 - e^{-4t}} + t \right).$$

By (4.16) we obtain

$$\Delta_{\widetilde{\mu}} = \frac{d^2}{dx^2} - 2x \frac{d}{dx} = e^{\frac{1}{2}x^2} \circ \left(\frac{d^2}{dx^2} - x^2 + 1 \right) \circ e^{-\frac{1}{2}x^2}.$$

Denoting by $q_t(x, y)$ the heat kernel of the operator $\frac{d^2}{dx^2} - x^2$ in $(\mathbb{R}, dx)$, we obtain by (4.15)

$$\begin{aligned} q_t(x, y) &= \widetilde{p}_t(x, y) h(x) h(y) e^{-t} \\ &= \frac{1}{(2\pi \sinh 2t)^{1/2}} \exp\left(\frac{4xye^{-2t} - \left(x^2 + y^2\right)\left(1 + e^{-4t}\right)}{2\left(1 - e^{-4t}\right)} \right). \end{aligned} \tag{4.17}$$

The function q_t is a modification of the *Mehler kernel.*

Example 4.10 Let $M = \mathbb{R}$ and

$$h(x) = e^{\frac{1}{2}x^2}.$$

Then (4.16) yields

$$\Delta_{\widetilde{\mu}} = \frac{d^2}{dx^2} + 2x \frac{d}{dx} = e^{-\frac{1}{2}x^2} \circ \left(\frac{d^2}{dx^2} - x^2 - 1 \right) \circ e^{\frac{1}{2}x^2}.$$

If $\widetilde{p}_t$ is the heat kernel of $\Delta_{\widetilde{\mu}}$ in $(\mathbb{R}, e^{x^2}dx)$ and, as above, q_t is the heat kernel of $\frac{d^2}{dx^2} - x^2$ in $(\mathbb{R}, dx)$ then, by (4.15) and (4.17),

$$\widetilde{p}_t(x,y) = \frac{q_t(x,y)\, e^{-t}}{h(x)\, h(y)} = \frac{1}{(2\pi \sinh 2t)^{1/2}} \exp\left(\frac{2xye^{-2t} - x^2 - y^2}{1 - e^{-4t}} - t\right). \tag{4.18}$$

Note that, by Theorem 3.13, the weighted manifold $(\mathbb{R}, e^{x^2}dx)$ is stochastically complete. Therefore, another way of proving (4.18) would be to verify that the right hand side of (4.18) is a regular fundamental solution of the heat equation in $(\mathbb{R}, e^{x^2}dx)$, which implies then by Corollary 3.10 that, indeed, it is the heat kernel of $(\mathbb{R}, e^{x^2}dx)$.

Example 4.11 Let $M = \mathbb{R}_+$ and

$$h(r) = \sinh r.$$

Then we have $\varphi = \frac{h'}{h} = \coth r$ and $\Phi = \varphi^2 + \varphi' \equiv 1$. By (4.16) we obtain

$$\Delta_{\widetilde{\mu}} = \frac{d^2}{dr^2} + 2\coth r \frac{d}{dr} = \frac{1}{\sinh r} \circ \left(\frac{d^2}{dr^2} - 1\right) \circ \sinh r.$$

If $\widetilde{p}_t$ is the heat kernel of the operator $\Delta_{\widetilde{\mu}}$ in $\left(\mathbb{R}_+, \sinh^2 r dr\right)$ and $p_t(x,y)$ is the heat kernel of the operator $\frac{d^2}{dr^2}$ in $(\mathbb{R}_+, dr)$ then we obtain by (4.15)

$$\widetilde{p}_t(x,y) = \frac{p_t(x,y)\, e^{-t}}{\sinh x \sinh y} = \frac{e^{-t}}{(4\pi t)^{1/2} \sinh x \sinh y}\left(e^{-\frac{|x-y|^2}{4t}} - e^{-\frac{|x+y|^2}{4t}}\right).$$

4.6 Hyperbolic spaces

Let $M = \mathbb{H}^n$ and μ be the Riemannian measure. Then the heat kernel $p_t(x,y)$ depends only on $r = d(x,y)$ and t, and it admits the following representations (see [30], [54], [62], [106], [161]): if $n = 2m+1$ then

$$p_t(x,y) = \frac{(-1)^m}{(2\pi)^m} \frac{1}{(4\pi t)^{1/2}} \left(\frac{1}{\sinh r} \frac{\partial}{\partial r}\right)^m e^{-m^2 t - \frac{r^2}{4t}}, \tag{4.19}$$

and if $n = 2m+2$ then

$$p_t(x,y) = \frac{(-1)^m}{(2\pi)^m} \frac{\sqrt{2}}{(4\pi t)^{3/2}} e^{-\frac{(2m+1)^2}{4}t} \left(\frac{1}{\sinh r} \frac{\partial}{\partial r}\right)^m \int_r^\infty \frac{se^{-\frac{s^2}{4t}}}{(\cosh s - \cosh r)^{\frac{1}{2}}}\, ds.$$

It was shown in [58] that, for all $t > 0$ and $x, y \in \mathbb{H}^n$,

$$p_t(x,y) \simeq \frac{(1+r+t)^{\frac{n-3}{2}}(1+r)}{t^{n/2}} \exp\left(-\lambda t - \frac{r^2}{4t} - \sqrt{\lambda} r\right), \tag{4.20}$$

where $\lambda = \frac{(n-1)^2}{4}$ is the bottom of the spectrum of the Laplace operator on $\mathbb{H}^n$. In particular, if x, y are fixed then

$$p_t(x,y) \simeq \frac{1}{t^{3/2}} \exp(-\lambda t) \quad \text{as } t \to \infty.$$

By (4.19) one obtains a simple formula for the heat kernel on $\mathbb{H}^3$:

$$p_t(x,y) = \frac{1}{(4\pi t)^{3/2}} \frac{r}{\sinh r} \exp\left(-\frac{r^2}{4t} - t\right). \tag{4.21}$$

Of course, once the formula is known, one can prove it as follows: first show that the right hand side of (4.21) is a regular fundamental solution, and then apply Corollary 3.10 using the stochastic completeness of $\mathbb{H}^n$. We show here how one can obtain (4.21) from scratch. Fix a point $y = o$ in $\mathbb{H}^3$ and try to find $p_t(x, o)$. Consider the polar coordinates (r, θ) in $\mathbb{H}^3$ centered at o, and let $S(r)$ be the boundary area function of $\mathbb{H}^3$, that is,

$$S(r) = 4\pi \sinh^2 r.$$

By (2.12), the radial part of the Laplace operator Δ_μ in the coordinates (r, θ) is as follows:

$$\Delta_\mu^{\mathrm{rad}} = \frac{\partial^2}{\partial r^2} + \frac{S'(r)}{S(r)} \frac{\partial}{\partial r} = \frac{\partial^2}{\partial r^2} + 2 \coth r \frac{\partial}{\partial r}.$$

Let $h(x)$ be a smooth positive function on $\mathbb{H}^3$, which depends only on $r = |x|$ so that we write $h = h(r)$. Then $\widetilde{S}(r) = h^2(r) S(r)$ is the boundary area function of the weighted manifold $\left(\mathbb{H}^3, \widetilde{\mu}\right)$ where $d\widetilde{\mu} = h^2 d\mu$. Choose

$$h(r) = \frac{r}{\sinh r}$$

so that $\widetilde{S}(r) = 4\pi r^2$ is equal to the boundary area function in $\mathbb{R}^3$. By a miraculous coincidence, h happens to satisfy in $\mathbb{H}^3 \setminus \{o\}$ the equation

$$\Delta_\mu h + h = 0, \tag{4.22}$$

which is verified by a straightforward computation. Extending $h(r)$ to the origin by $h(o) = 1$, we obtain that o is a removable singularity for h and hence h satisfies (4.22) in $\mathbb{H}^3$. By (4.15) with $\Phi \equiv -1$, we conclude that

$$\widetilde{p}_t(x, y) = \frac{p_t(x, y) e^t}{h(x) h(y)},$$

where $\widetilde{p}_t$ is the heat kernel on $\left(\mathbb{H}^3, \widetilde{\mu}\right)$. Since the boundary area functions of $\left(\mathbb{H}^3, \widetilde{\mu}\right)$ and $\left(\mathbb{R}^3, dx\right)$ are the same, we conclude by Lemma 4.1 that their heat kernels at the origins are the same, whence

$$\widetilde{p}_t(x, o) = \frac{1}{(4\pi t)^{3/2}} \exp\left(-\frac{r^2}{4t}\right).$$

Finally, we obtain

$$p_t(x, o) = \widetilde{p}_t(x, o) e^{-t} h(x) h(o) = \frac{1}{(4\pi t)^{3/2}} \exp\left(-\frac{r^2}{4t} - t\right) \frac{r}{\sinh r},$$

which was to be proved.

5 Heat kernel estimates

Let (M, μ) be a weighted manifold. For any relatively compact open set $\Omega \subset M$, denote by $\lambda_k(\Omega)$ the k-th smallest eigenvalue of the (weak) Dirichlet problem for Δ_μ in Ω counted with multiplicity (cf. Theorem 2.3).

5.1 Uniform Faber-Krahn inequality

Let Λ be a function on $(0,+\infty)$. We say that a weighted manifold (M,μ) satisfies the (*uniform*) *Faber-Krahn inequality* with function Λ if, for any non-empty relatively compact open set $\Omega\subset M$,

$$\lambda_1(\Omega)\geq\Lambda(\mu(\Omega)). \tag{5.1}$$

For example, $\mathbb{R}^n$ satisfies the Faber-Krahn inequality with function $\Lambda(v)=cv^{-2/n}$. Also, any Cartan-Hadamard manifold of dimension n satisfies the Faber-Krahn inequality with the same function Λ (but possibly with a different constant c). If K is a k-dimensional compact manifold then the Riemannian product $M=\mathbb{R}^m\times K$ satisfies the Faber-Krahn inequality with function

$$\Lambda(v)=c\begin{cases}v^{-2/n}, & v\leq 1,\\ v^{-2/m}, & v\geq 1,\end{cases} \tag{5.2}$$

where $n=\dim M=k+m$ (see [45]). Any n-dimensional manifold with bounded geometry satisfies the Faber-Krahn inequality with the function

$$\Lambda(v)=c\begin{cases}v^{-2/n}, & v\leq 1,\\ v^{-2}, & v\geq 1\end{cases} \tag{5.3}$$

(see [91], [95]).

Theorem 5.1 ([89]) *Assume that (M,μ) satisfies the Faber-Krahn inequality (5.1) with a positive continuous decreasing function Λ such that*

$$\int_0\frac{dv}{v\Lambda(v)}<\infty. \tag{5.4}$$

Then, for all $t>0$,

$$\sup_{x\in M}p_t(x,x)\leq\frac{4}{\gamma(t/2)}, \tag{5.5}$$

where the function γ is defined by the identity

$$t=\int_0^{\gamma(t)}\frac{dv}{v\Lambda(v)}. \tag{5.6}$$

Approach to the proof. Assuming that (5.1) holds, one deduces the following *Nash type* inequality: for any non-zero function $u\in\mathcal{D}$,

$$\int_M|\nabla u|^2\,d\mu\geq(1-\varepsilon)\|u\|_{L^2}^2\Lambda\left(\frac{2}{\varepsilon}\frac{\|u\|_{L^1}^2}{\|u\|_{L^2}^2}\right), \tag{5.7}$$

for any $\varepsilon\in(0,1)$. Then one applies Nash's argument [172]: extending (5.7) to $u=p_t(x,\cdot)$ and noticing that

$$\int_M|\nabla u|^2\,d\mu=-\int_M u\Delta_\mu u\,d\mu=-\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 \tag{5.8}$$

and $\|u\|_{L^1}\leq 1$, one obtains from (5.7) and (5.8) a differential inequality for $\|u\|_{L^2}^2=p_{2t}(x,x)$, whence the result follows. ■

Any function $\gamma(t)$ defined on $\mathbb{R}_+$ by (5.6), satisfies the following properties:

$$\gamma\in C^1(\mathbb{R}_+),\quad\gamma'(t)>0,\quad\gamma(0)=0,\quad\gamma(\infty)=\infty,\quad\text{and}\quad\frac{\gamma'(t)}{\gamma(t)}\text{ is decreasing in }t. \tag{5.9}$$

Conversely, any function γ satisfying (5.9) determines the function Λ by

$$\Lambda\left(\gamma\left(t\right)\right)=\frac{\gamma'\left(t\right)}{\gamma\left(t\right)}, \tag{5.10}$$

which is a positive continuous decreasing function on $\mathbb{R}_+$ satisfying (5.4), (5.6).

Given $\delta>0$, let us say that a function γ on $\mathbb{R}_+$ is *δ-regular* if γ satisfies (5.9) and, in addition,

$$\frac{\gamma'(s)}{\gamma(s)}\geq\delta\frac{\gamma'(t)}{\gamma(t)},\quad \text{for all }\ 0<t\leq s\leq 2t.$$

Theorem 5.2 ([89]) *Let $\gamma\left(t\right)$ be a δ-regular function. If, for all $t>0$,*

$$\sup_{x\in M}p_t\left(x,x\right)\leq\frac{1}{\gamma\left(t\right)}, \tag{5.11}$$

then $\left(M,\mu\right)$ satisfies the Faber-Krahn inequality with function $c\Lambda$, where Λ defined by (5.10) and $c=c\left(\delta\right)>0$. Furthermore, for any non-empty relatively compact open set $\Omega\subset M$ and for any positive integer k, we have

$$\lambda_k\left(\Omega\right)\geq c\Lambda\left(\frac{\mu\left(\Omega\right)}{k}\right). \tag{5.12}$$

Approach to the proof. Applying the trace formula (3.16) in Ω, one obtains

$$\int_\Omega p_t^\Omega\left(x,x\right)d\mu\left(x\right)=\sum_{i=1}^\infty e^{-\lambda_i(\Omega)t}\geq ke^{-\lambda_k(\Omega)t},$$

which in combination with (5.11) already contains some lower bound for $\lambda_k\left(\Omega\right)$. Optimizing the choice of t, one obtains (5.12). ■

Theorems 5.1 and 5.2 imply that the heat kernel upper bound

$$\sup_{x\in M}p_t\left(x,x\right)\leq\frac{C}{\gamma\left(t\right)}, \tag{5.13}$$

is equivalent to the Faber-Krahn inequality

$$\lambda_1\left(\Omega\right)\geq c\Lambda\left(\mu\left(\Omega\right)\right),$$

provided the function γ is δ-regular and it is related to Λ by (5.6) or (5.10). For an alternative equivalent condition for (5.13) in terms of a log-Sobolev inequality see [54].

Combining Theorems 5.1 and 5.2, we obtain also the following result.

Corollary 5.3 ([89], [115]) *Under the hypotheses of* Theorem 5.1*, assume that the function $\gamma\left(t\right)$ is δ-regular. Then, for any non-empty relatively compact open set $\Omega\subset M$ and for any positive integer k, we have*

$$\lambda_k\left(\Omega\right)\geq c\Lambda\left(\frac{\mu\left(\Omega\right)}{k}\right).$$

Example 5.4 If Λ is given by (5.2) then (5.6) yields

$$\gamma\left(t\right)\simeq\left\{\begin{array}{ll} t^{n/2}, & t\leq 1,\\ t^{m/2}, & t\geq 1,\end{array}\right.$$

and we obtain by (5.5)

$$\sup_{x\in M} p_t(x,x) \le C \left\{ \begin{array}{ll} t^{-n/2}, & t \le 1, \\ t^{-m/2}, & t \ge 1. \end{array} \right. \tag{5.14}$$

In this case, Corollary 5.3 yields the estimate

$$\lambda_k(\Omega) \ge c\left(\frac{k}{\mu(\Omega)}\right)^{2/n}, \quad \text{if } k \ge \mu(\Omega),$$

which matches the Weyl's asymptotic formula (2.8) up to a constant factor.

Since on a manifold of bounded geometry the Faber-Krahn inequality holds with function (5.3), we obtain from (5.14) the following consequence.

Corollary 5.5 ([33], [42], [91], [212]) *If (M,μ) has bounded geometry then, for all $t \ge 1$,*

$$\sup_{x\in M} p_t(x,x) \le Ct^{-1/2}. \tag{5.15}$$

See also [37] for other estimates of heat kernels on manifolds of bounded geometry.

In the case when $\gamma(t) = ct^{n/2}$, there is a number of equivalent conditions for the heat kernel estimate (5.11).

Theorem 5.6 *The condition*

$$\sup_{x\in M} p_t(x,x) \le Ct^{-n/2}, \text{ for all } t > 0, \tag{5.16}$$

is equivalent to each of the following (all hold for any non-negative function $f \in \mathcal{D}$):

(*i*) ([211]) *The Sobolev inequality:*

$$\left(\int_M f^{\frac{2n}{n-2}} d\mu\right)^{\frac{n-2}{n}} \le C\int_M |\nabla f|^2 d\mu, \tag{5.17}$$

provided $n > 2$.

(*ii*) ([28], [43]) *The Nash inequality:*

$$\left(\int_M f d\mu\right)^{-4/n}\left(\int_M f^2 d\mu\right)^{1+2/n} \le C\int_M |\nabla f|^2 d\mu.$$

(*iii*) ([52], [53], [54]) *The one-parameter* log-*Sobolev inequality: for any $\varepsilon > 0$*

$$\int_M f^2 \log\frac{f}{\|f\|_{L^2}} d\mu \le \varepsilon\int_M |\nabla f|^2 d\mu + \left(C - \frac{n}{4}\log\varepsilon\right)\int_M f^2 d\mu. \tag{5.18}$$

(*iv*) ([29], [89]) *The Faber-Krahn inequality: for all non-empty relatively compact open sets $\Omega \subset M$,*

$$\lambda_1(\Omega) \ge c\mu(\Omega)^{-2/n}.$$

Of course, this theorem implies that all conditions $(i) - (iv)$ are equivalent each to other. It is curious to mention that the first proof of this fact was obtained exactly in this way, via the equivalence to the heat kernel estimate (5.16). Direct proofs can be found in [9] and [115].

The following theorem extends the result of Corollary 5.5.

Theorem 5.7 ([15]) *Let (M, μ) be a weighted manifold of bounded geometry. Assume that, for all points $x \in M$ and all $r > r_0$,*

$$V(x, r) \geq \mathcal{V}(r), \tag{5.19}$$

where $\mathcal{V}$ is a continuous increasing bijection from $(r_0, +\infty)$ to $(v_0, +\infty)$, and r_0, v_0 are positive reals. Then, for all $t > t_0 := r_0^2$,

$$\sup_{x \in M} p_t(x, x) \leq \frac{C}{\gamma(ct)}, \tag{5.20}$$

where function γ is defined by

$$t - t_0 = \int_{v_0}^{\gamma(t)} \mathcal{V}^{-1}(s) ds. \tag{5.21}$$

Approach to the proof. The proof consists of obtaining the Faber-Krahn inequality (5.1) with the function

$$\Lambda(v) = \frac{c}{\mathcal{V}^{-1}(v) v},$$

with subsequent application of Theorem 5.1. ■

Example 5.8 If $\mathcal{V}(r) \simeq r^{\alpha}$ then (5.21) yields for large t

$$\gamma(t) \simeq t^{\frac{\alpha}{\alpha+1}}$$

and hence

$$\sup_{x \in M} p_t(x, x) \leq C t^{-\frac{\alpha}{\alpha+1}}. \tag{5.22}$$

It is not difficult to show that on any manifold of bounded geometry the volume growth is at least linear that is, (5.19) holds with $\mathcal{V}(r) \simeq r$. Hence, (5.22) holds with $\alpha = 1$, which yields the estimate of Corollary 5.5.

Note that the estimate (5.22) with $\alpha = n$ is not sharp in $\mathbb{R}^n$ where one has $p_t(x, x) = ct^{-n/2}$. A manifold where the estimate (5.22) is sharp was constructed in [15] using fractal structures.

Example 5.9 If $\mathcal{V}(r) = e^r$ then by Theorem 5.7 we obtain a very moderate upper bound for the heat kernel:

$$\sup_{x \in M} p_t(x, x) \leq C \frac{\log t}{t},$$

and again the examples in [15] show that this estimate is nearly optimal.

We conclude this section with heat kernels on covering manifolds.

Theorem 5.10 ([47]) *Let M be a geodesically complete non-compact Riemannian manifold and μ be the Riemannian volume on M. Let M be a regular cover of a compact manifold. Fix $x_0 \in M$ and set*

$$V(r) = V(x_0, r). \tag{5.23}$$

Then the manifold (M, μ) satisfies the Faber-Krahn inequality with the function

$$\Lambda(v) := c \left(\frac{1}{V^{-1}(Cv)} \right)^2. \tag{5.24}$$

The following statement is an immediate consequence of Theorems 5.1 and 5.10.

Corollary 5.11 *Under the hypotheses of* Theorem 5.10, *the heat kernel on* (M,μ) *satisfies the estimate*

$$\sup_{x\in M} p_t(x,x) \le \frac{C}{\gamma(ct)},$$

where the function $\gamma(t)$ *is defined by the identity*

$$t = \int_0^{\gamma(t)} \frac{V^{-1}(v)^2\, dv}{v} \tag{5.25}$$

Example 5.12 If $V(r) \simeq r^n$ then (5.24) yields $\Lambda(v) \simeq v^{-2/n}$ and Corollary 5.11 yields

$$\sup_{x\in M} p_t(x,x) \le Ct^{-n/2}.$$

In this case, also a similar lower bound for $p_t(x,x)$ holds (see Theorem 5.30 below).

Example 5.13 Assume that $V(r) \ge \exp(cr)$ for large r (whereas for small r we always have $V(r) \simeq r^n$ with $n = \dim M$). Then (5.24) yields, for large v,

$$\Lambda(v) \ge \frac{c}{\log^2 v}.$$

By Corollary 5.11, we obtain $\gamma(t) \ge c\exp\left(ct^{1/3}\right)$ and

$$\sup_{x\in M} p_t(x,x) \le C\exp\left(-ct^{1/3}\right), \quad \text{for large } t.$$

It turns out that if in addition the deck transformation group of M is polycyclic then a matching lower bound holds (see [3], [46]) so that the term $t^{1/3}$ in the exponential is sharp.

5.2 Gaussian upper bounds

The following theorem shows that the Gaussian exponential term appears naturally in the heat kernel upper estimates on arbitrary manifolds.

Theorem 5.14 ([55], [76], [90], [95]) *Let* (M,μ) *be a weighted manifold and let* A *and* B *be two* μ*-measurable sets on* M*. Then*

$$\int_A\int_B p_t(x,y)d\mu(x)d\mu(y) \le \sqrt{\mu(A)\mu(B)}\exp\left(-\frac{d^2(A,B)}{4t}\right), \tag{5.26}$$

where $d(A,B)$ *is the geodesic distance between sets* A *and* B*.*

Let us consider the following weighted integral of the heat kernel:

$$E_D(t,x) := \int_M p_t^2(x,z)\exp\left(\frac{d^2(x,z)}{Dt}\right)d\mu(z), \tag{5.27}$$

where $D \in (0,+\infty]$. A priori, the value of $E_D(t,x)$ may be $+\infty$.

Lemma 5.15 ([89]) *For any* $D \in (0,+\infty]$ *and all* $x,y \in M$, $t>0$, *the following inequality holds*

$$p_t(x,y) \le \sqrt{E_D(t/2,x)E_D(t/2,y)}\exp\left(-\frac{d^2(x,y)}{2Dt}\right). \tag{5.28}$$

The proof of (5.28) is an easy application of the semigroup identity (3.9) and the Cauchy-Schwarz inequality.

Lemma 5.16 ([89]) *If $D \geq 2$ then for any $x \in M$, the function $E_D(t,x)$ is decreasing in t. In particular, if $E_D(t,x) < \infty$ for some $t = t_0$ then $E_D(t,x) < \infty$ for all $t > t_0$.*

Approach to the proof. The proof of the monotonicity of $E_D(t,x)$ amounts to verifying that its time derivative is non-positive. It is essential for the proof that $|\nabla d(x,\cdot)| \leq 1$, which implies that the function $\xi(t,x) = \frac{d^2(x,y)}{2Dt}$ satisfies (3.25). ∎

Theorem 5.17 ([92], [205]) *Assume that, for some $x \in M$ and for all $t > 0$,*

$$p_t(x,x) \leq \frac{C}{\gamma(t)}, \tag{5.29}$$

where $\gamma(t)$ is an increasing positive function on $\mathbb{R}_+$ satisfying the following condition:

$$\frac{\gamma(at)}{\gamma(t)} \leq A\frac{\gamma(as)}{\gamma(s)} \quad \textit{for all } 0 < t \leq s, \tag{5.30}$$

for some constants $a, A > 1$. Then, for any $D > 2$ and all $t > 0$,

$$E_D(t,x) \leq \frac{C}{\gamma(\varepsilon t)}, \tag{5.31}$$

for some $\varepsilon = \varepsilon(a,D) > 0$.

By putting together Theorem 5.17 and Lemma 5.15, we obtain the following result.

Corollary 5.18 *Assume that, for some points $x, y \in M$ and for all $t > 0$,*

$$p_t(x,x) \leq \frac{C}{\gamma_1(t)} \quad \textit{and} \quad p_t(y,y) \leq \frac{C}{\gamma_2(t)}, \tag{5.32}$$

where γ_1 and γ_2 are increasing positive function on $\mathbb{R}_+$ both satisfying (5.30). Then, for any $D > 2$ and all $t > 0$,

$$p_t(x,y) \leq \frac{C}{\sqrt{\gamma_1(\varepsilon t)\gamma_2(\varepsilon t)}} \exp\left(-\frac{d^2(x,y)}{2Dt}\right). \tag{5.33}$$

Corollary 5.19 *Let (M,μ) be a complete weighted manifold, Φ be a smooth function on M such that the equation $\Delta_\mu h - \Phi h = 0$ has a positive solution. Assume that, for some points $x, y \in M$ and for all $t > 0$,*

$$p_t^\Phi(x,x) \leq \frac{C}{\gamma_1(t)} \quad \textit{and} \quad p_t^\Phi(y,y) \leq \frac{C}{\gamma_2(t)}, \tag{5.34}$$

where γ_1 and γ_2 are increasing positive function on $\mathbb{R}_+$ both satisfying (5.30). Then, for any $D > 2$ and all $t > 0$,

$$p_t^\Phi(x,y) \leq \frac{C}{\sqrt{\gamma_1(\varepsilon t)\gamma_2(\varepsilon t)}} \exp\left(-\frac{d^2(x,y)}{2Dt}\right). \tag{5.35}$$

Proof. Let $\widetilde{\mu}$ be the measure on M such that $d\widetilde{\mu} = h^2 d\mu$, and let $\widetilde{p}_t$ be the heat kernel of $(M, \widetilde{\mu})$. By Lemma 4.7, we have

$$p_t^{\Phi}(x,y) = \widetilde{p}_t(x,y)\, h(x)\, h(y).$$

Hence, the hypotheses (5.34) imply

$$\widetilde{p}_t(x,x) \leq \frac{C}{\gamma_1(t)\, h^2(x)} \quad \text{and} \quad \widetilde{p}_t(y,y) \leq \frac{C}{\gamma_2(t)\, h^2(y)}.$$

By Corollary 5.18, we conclude

$$\widetilde{p}_t(x,y) \leq \frac{C}{\sqrt{\gamma_1(\varepsilon t)\gamma_2(\varepsilon t)}h(x)\, h(y)} \exp\left(-\frac{d^2(x,y)}{2Dt}\right),$$

whence (5.35) follows. ■

Corollary 5.20 ([85], [89]) *On any weighted manifold (M,μ) and for any $D>2$, $E_D(t,x)$ is finite for all $t>0$, $x\in M$.*

Sketch of proof. By Theorem 5.17 and Lemma 5.16, it suffices to prove that for any $x\in M$ there exist positive constants C and T such that

$$p_t(x,x) \leq Ct^{-n/2}, \quad \text{for all } 0<t<T, \tag{5.36}$$

where $n=\dim M$.

Fix a small relatively compact open set Ω containing the point x. By compactness argument, the weighted manifold (Ω,μ) satisfies the following Faber-Krahn inequality: for all open sets $U\subset\Omega$, such that $\mu(U)\leq \frac{1}{2}\mu(\Omega)$,

$$\lambda_1(U) \geq c\mu(U)^{-2/n},$$

where $c>0$ depends on Ω. Hence, by a slight modification of Theorem 5.1, we obtain that the heat kernel p_t^{Ω} of (Ω,μ) satisfies the estimate

$$p_t^{\Omega}(x,x) \leq Ct^{-n/2}, \quad \text{for all } t\in(0,T),$$

where C and T depend on Ω.

Consider the function

$$u(t,y) = p_t(x,y) - p_t^{\Omega}(x,y)$$

and extend it to $t\leq 0$ by setting $u(t,y)\equiv 0$. By Theorem 3.3 and Lemma 3.2, this function satisfies in $\mathbb{R}\times\Omega$ the equation $\frac{\partial u}{\partial t} = \Delta_\mu u$ and hence it is C^∞-smooth in $\mathbb{R}\times\Omega$. In particular, the function $t\mapsto u(t,x)$ is bounded on $[0,T]$, say $u(t,x)\leq C$. Then we obtain, for all $0<t<T$,

$$p_t(x,x) = p_t^{\Omega}(x,x) + u(t,x) \leq Ct^{-n/2} + C,$$

whence (5.36) follows. ■

The following theorem seems to be new.

Theorem 5.21 *On any weighted manifold (M,μ) and for any $D>2$, there exists a positive continuous function $\Phi(t,x)$ on $\mathbb{R}_+\times M$, which is decreasing in t and such that the following inequality holds*

$$p_t(x,y) \leq \Phi(t,x)\Phi(t,y)\exp\left(-\lambda_{\min}(M)t - \frac{d^2(x,y)}{2Dt}\right), \tag{5.37}$$

for all $x,y\in M$ and $t>0$.

Proof. Let us first set

$$\Phi(t,x)=\sqrt{E_D(\frac{1}{2}t,x)}. \tag{5.38}$$

By Corollary 5.20, this function is finite. By Lemma 5.16, the function $\Phi(t,x)$ is decreasing in t. By Lemma 5.15, we obtain

$$p_t(x,y)\le\Phi(t,x)\Phi(t,y)\exp\left(-\frac{d^2(x,y)}{2Dt}\right). \tag{5.39}$$

This estimates still does not match (5.37) because of the lack of the term λt in the exponential, where $\lambda=\lambda_{\min}(M)$. To handle this term, let us find a positive smooth function h satisfying on M the equation

$$\Delta_\mu h+\lambda h=0$$

(by [197], [216], such a function h exists for any $\lambda\le\lambda_{\min}(M)$). Consider the measure $\widetilde{\mu}$ defined by $d\widetilde{\mu}=h^2d\mu$ and the heat kernel $\widetilde{p}_t$ on the weighted manifold $(M,\widetilde{\mu})$. Applying (5.39) on $(M,\widetilde{\mu})$, we obtain that there exists a function $\widetilde{\Phi}(t,x)$ decreasing in t such that

$$\widetilde{p}_t(x,y)\le\widetilde{\Phi}(t,x)\widetilde{\Phi}(t,y)\exp\left(-\frac{d^2(x,y)}{2Dt}\right). \tag{5.40}$$

By (4.15) we have

$$p_t(x,y)=\widetilde{p}_t(x,y)h(x)h(y)e^{-\lambda t},$$

which together with (5.40) yields (5.37) with $\Phi(t,x)=\widetilde{\Phi}(t,x)h(x)$. ■

Remark 5.22 As it follows from the construction of the function $\Phi(t,x)$ and from the proof of Corollary 5.20, for any compact set $K\subset M$ there exist positive constants C and T such that

$$\Phi(t,x)\le Ct^{-n/4}\quad\text{for all } x\in K \text{ and } 0<t<T. \tag{5.41}$$

Example 5.23 For the heat kernel p_t on $(\mathbb{R},e^{x^2}dx)$ given by (4.18), one obtains the following estimate of the form (5.37):

$$\begin{aligned} p_t(x,y) &= \frac{1}{(2\pi\sinh 2t)^{1/2}}\exp\left(-\frac{(x^2+y^2)\,e^t}{2\cosh t}-\frac{(x-y)^2}{2\sinh 2t}-t\right) \\ &\le \Phi(t,x)\,\Phi(t,y)\exp\left(-\frac{(x-y)^2}{4t}-t\right), \end{aligned} \tag{5.42}$$

with the function

$$\Phi(t,x):=\frac{1}{(2\pi\sinh 2t)^{1/4}}\exp\left(-x^2\left(\frac{e^t}{2\sinh t}-\frac{1}{2t}\right)\right),$$

which is decreasing in t.

5.3 On-diagonal lower bounds

In this section, we collect the heat kernel lower estimates that use only the volume function $V(x,r)$.

Theorem 5.24 ([44]) *Let (M,μ) be a geodesically complete weighted manifold, and assume that for some point $x\in M$ and all $r\geq r_0$,*

$$V(x,r)\leq Cr^{\alpha}, \tag{5.43}$$

where C,α,r_0 are some positive constants. Then, for all $t\geq t_0$,

$$p_t(x,x)\geq \frac{1/2}{V(x,\sqrt{At\log t})}, \tag{5.44}$$

where $A>0$ depend on x,r_0,α,C and $t_0=t_0\left(r_0\right)$.

In particular, we obtain from (5.43) and (5.44)

$$p_t\left(x,x\right)\geq \frac{c}{\left(t\log t\right)^{\alpha/2}}.$$

Clearly, in the case when $\alpha=n$ and $M=\mathbb{R}^n$ this estimate is not sharp because of the factor $(\log t)^{n/2}$. However, in general assuming only the volume growth condition (5.43) one cannot get rid of this factor as it was shown on example in [44] (see also [16] and [103]).

For a more general volume function, we have the following generalization of Theorem 5.24.

Theorem 5.25 ([44]) *Let (M,μ) be a geodesically complete weighted manifold, and assume that, for some point $x\in M$ and all $r\geq r_0$,*

$$V(x,r)\leq \mathcal{V}(r)\,, \tag{5.45}$$

where $\mathcal{V}(r)>1$ is a continuous increasing function on (r_0,∞) such that the function

$$r\mapsto \frac{r^2}{\log \mathcal{V}(r)}\quad \text{is an increasing bijection from } (r_0,+\infty) \text{ to } (t_0,+\infty)\,, \tag{5.46}$$

and r_0,t_0 are positive reals. Let $\mathcal{R}(t)$ be the inverse function to (5.46), that is, $\mathcal{R}\left(t\right)$ is defined for $t>t_0$ by the identity

$$t=\frac{\mathcal{R}^2(t)}{\log \mathcal{V}(\mathcal{R}(t))} \tag{5.47}$$

Then, for all $t>t_0$,

$$p_t(x,x)\geq \frac{1/2}{V(x,\mathcal{R}(At))}, \tag{5.48}$$

where the constant $A>0$ depends on x and r_0.

Of course, (5.48) implies also

$$p_t\left(x,x\right)\geq \frac{1/2}{\mathcal{V}(\mathcal{R}(At))}.$$

Example 5.26 If $\mathcal{V}(r)=\exp\left(r^{\alpha}\right)$ for $0<\alpha<1$, then we obtain from (5.47) $\mathcal{R}\left(t\right)\simeq t^{\frac{1}{2-\alpha}}$ and hence

$$p_t(x,x)\geq c\exp\left(-Ct^{\frac{\alpha}{2-\alpha}}\right). \tag{5.49}$$

If $\mathcal{V}\left(r\right)=r^{\alpha}$ then (5.47) gives $\mathcal{R}\left(t\right)\simeq\sqrt{t\log t}$, and we obtain the statement of Theorem 5.24.

Sketch of proof of Theorems 5.24 and 5.25. Fix some $R > 0$ and set $\Omega = B(x, R)$. Using (3.12), the Cauchy-Schwarz inequality, and the stochastic completeness of (M, μ) (which follows from Theorem 3.13), we obtain

$$\begin{aligned} p_{2t}(x,x) &= \int_M p_t^2(x,y)d\mu(y) \geq \int_\Omega p_t^2(x,y)d\mu(y) \\ &\geq \frac{1}{\mu(\Omega)}\left(\int_\Omega p_t(x,y)d\mu(y)\right)^2 = \frac{1}{\mu(\Omega)}\left(1 - \int_{\Omega^c} p_t(x,y)d\mu(y)\right)^2. \end{aligned} \tag{5.50}$$

Assume that $R = R(t)$ is so that

$$\int_{B(x,R)^c} p_t(x,y)d\mu(y) \leq \varepsilon < 1. \tag{5.51}$$

Then (5.50) yields

$$p_t(x,x) \geq \frac{(1-\varepsilon)^2}{V(x, R(t))},$$

which will be the desired estimate.

Applying again the Cauchy-Schwarz inequality and using the notation $E_D(t,x)$ defined in (5.27), we obtain

$$\left(\int\limits_{B(x,R)^c} p_t(x,y)d\mu(y)\right)^2 \leq E_D(t,x) \int\limits_{B(x,R)^c} \exp\left(-\frac{d^2(x,y)}{Dt}\right) d\mu(y). \tag{5.52}$$

If $D > 2$ then, by Corollary 5.20, $E_D(t,x)$ is finite and decreasing in t so that for large t it can be replaced by a constant. Estimating the integral in (5.52) by the volume growth hypothesis, one obtains (5.51). ∎

5.4 Relative Faber-Krahn inequality

In this section we always assume that (M, μ) is a geodesically complete weighted manifold. We say that (M, μ) satisfies the *relative Faber-Krahn inequality* if there exist positive constants δ, c such that, for any geodesic ball $B(x,r)$ on M and for any non-empty relatively compact open set $\Omega \subset B(x,r)$,

$$\lambda_1(\Omega) \geq \frac{c}{r^2}\left(\frac{V(x,r)}{\mu(\Omega)}\right)^\delta. \tag{5.53}$$

For example, the relative Faber-Krahn inequality holds in $\mathbb{R}^n$ with $\delta = 2/n$ since $V(x,r) = cr^n$ and hence (5.53) amounts to the uniform Faber-Krahn inequality (5.1) with $\Lambda(v) = cv^{-2/n}$.

Theorem 5.27 ([88]) *If M has non-negative Ricci curvature and μ is the Riemannian volume then (M, μ) satisfies the relative Faber-Krahn inequality.*

Approach to the proof. The following key property of manifolds of non-negative Ricci curvature is used in the proof of this theorem and Theorem 6.4 below. For any $x \in M$ and $0 < s < 1$, define a homothety $\Gamma_s^x : M \to M$ by

$$\Gamma_s^x(y) = \gamma_{x,y}(sl),$$

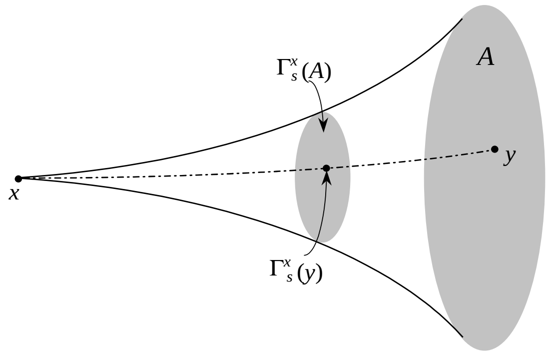


Figure 3: Homothety Γ_s^x

where $\gamma_{x,y} : [0,l] \to M$ is a shortest geodesics between x and y such that $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(l) = y$. Then, for any Borel set $A \subset M$,

$$\mu\left(\Gamma_s^x(A)\right) \geq c\mu(A), \tag{5.54}$$

where $c = c(s) > 0$ (see Fig. 3). A similar but somewhat different measure contraction property was considered in [196]. ■

The class of weighted manifolds with the relative Faber-Krahn inequality is much wider than those with non-negative Ricci curvature. In particular, this class is stable under quasi-isometry. Another example of stability: a connected sum of k copies of the same manifold satisfying the relative Faber-Krahn inequality also satisfies the relative Faber-Krahn inequality (see [112]).

We say that a weighted manifold (M, μ) satisfies the *volume doubling* property if, for all $x \in M$ and $r > 0$,

$$V(x, 2r) \leq CV(x, r).$$

Theorem 5.28 ([89]) *The following conditions are equivalent:*

(a) (M, μ) *satisfies the relative Faber-Krahn inequality.*

(b) (M, μ) *satisfies the volume doubling property and the heat kernel on* (M, μ) *admits the estimate*

$$p_t(x, x) \leq \frac{C}{V\left(x, \sqrt{t}\right)}, \tag{5.55}$$

for all $x \in M$ *and* $t > 0$.

(c) (M, μ) *satisfies the volume doubling property and the heat kernel on* (M, μ) *admits the estimate*

$$p_t(x, y) \leq \frac{C}{V\left(x, \sqrt{t}\right)} \exp\left(-c\frac{d^2(x, y)}{t}\right), \tag{5.56}$$

for all $x, y \in M$ *and* $t > 0$.

Approach to the proof. The proof of $(a) \Rightarrow (b)$ is based on a certain mean value inequality for solutions to the heat equation (see [88], [95], [148]). The proof of $(b) \Rightarrow (a)$ is similar to that of Theorem 5.2. The implication $(c) \Rightarrow (b)$ is trivial whereas $(b) \Rightarrow (c)$ follows by Corollary 5.18 with additional application of the volume doubling property. ■

As it is clear from Corollary 5.18, the constant c in the exponential in (5.56) can be made arbitrarily close to $\frac{1}{4}$. In fact, it can be taken exactly $\frac{1}{4}$ at the expense of additional factors as in the following statement.

Theorem 5.29 ([95]) *Let (M,μ) satisfy the relative Faber-Krahn inequality. Assume in addition that for some $\alpha>0$ and for all $0<r<R$ and $x\in M$,*

$$\frac{V(x,R)}{V(x,r)}\leq C\left(\frac{R}{r}\right)^{\alpha}. \tag{5.57}$$

Then, for all $x,y\in M$ and $t>0$,

$$p_t(x,y)\leq\frac{C}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}}\left(1+\frac{d\left(x,y\right)}{\sqrt{t}}\right)^{\alpha-1}\exp\left(-\frac{d\left(x,y\right)^2}{4t}\right). \tag{5.58}$$

Note that the condition (5.57) with *some* α follows from the volume doubling property. Hence, (5.58) holds on any complete manifold satisfying the relative Faber-Krahn inequality. The exponent $\alpha-1$ in the middle factor in (5.58) is sharp as one can see from the example $M=\mathbb{S}^n$ with $\alpha=n$ (see [165]).

Theorem 5.30 ([44]) *Assume that for some point $x\in M$,*

$$V\left(x,2r\right)\leq CV\left(x,r\right)\quad \textit{for all } r>0$$

and

$$p_t\left(x,x\right)\leq\frac{C}{V\left(x,\sqrt{t}\right)},\quad \textit{for all } t>0.$$

Then, for all $t>0$,

$$p_t\left(x,x\right)\geq\frac{c}{V\left(x,\sqrt{t}\right)}. \tag{5.59}$$

Approach to the proof. The proof is similar to that of Theorem 5.24, with the enhancement that the term $E_D(t,x)$ in (5.52) is estimated not just by a constant but using Theorem 5.17, which in the present setting yields

$$E_D(t,x)\leq\frac{C}{V\left(x,\sqrt{t}\right)}.$$

This allows to replace the term $\sqrt{t\log t}$ in (5.44) by $\sqrt{t}$ as in (5.59). ■

Corollary 5.31 *The following conditions are equivalent:*

(a) *(M,μ) satisfies the relative Faber-Krahn inequality.*

(b) *The heat kernel on (M,μ) satisfies the estimates*

$$p_t\left(x,y\right)\leq\frac{C}{V\left(x,\sqrt{t}\right)}\exp\left(-c\frac{d^2\left(x,y\right)}{t}\right). \tag{5.60}$$

for all $x,y\in M$ and $t>0$, and

$$p_t\left(x,x\right)\geq\frac{c}{V\left(x,\sqrt{t}/2\right)}, \tag{5.61}$$

for all $x\in M$ and $t>0$.

Proof. The implication $(a) \Longrightarrow (b)$ follows from Theorems 5.28 and 5.30. To prove $(b) \Longrightarrow (a)$ observe that (5.60) and (5.61) imply for $t = (2r)^2$

$$\frac{c}{V(x,r)} \leq p_t(x,x) \leq \frac{C}{V(x,2r)},$$

whence the volume doubling property follows. By Theorem 5.28, the upper bound (5.60) and the volume doubling property imply (a). ■

Corollary 5.32 ([150]) *If M has non-negative Ricci curvature and μ is the Riemannian volume then the heat kernel on (M,μ) satisfies the upper bounds* (5.56), (5.58) *and the lower bound* (5.59).

In fact, Li and Yau [150] proved also a Gaussian lower bound for $p_t(x,y)$, which will be discussed below in Section 6.1.

5.5 On-diagonal estimates on model manifolds

In this section, (M,μ) is always a weighted model introduced in Section 2.4. Denote by o the origin of M and set $B_r = B(o,r)$. Recall that $V(r)$ is the volume of B_r and $S(r)$ is the boundary area of B_r.

The next theorem is a version of Theorem 5.1 when all conditions are spherically symmetric.

Theorem 5.33 ([44]) *Assume that, for any $r > 0$,*

$$\lambda_1(B_r) \geq \Lambda(V(r)), \tag{5.62}$$

where the function Λ satisfies the same hypotheses as in Theorem 5.1. *Then, for all $t > 0$,*

$$p_t(o,o) \leq \frac{4}{\gamma(t/2)} \tag{5.63}$$

where the function γ is determined via Λ as in (5.6).

Corollary 5.34 ([94]) *We have, for all $t > 0$,*

$$p_t(o,o) \leq \frac{4}{V(\mathcal{R}(t/8))}, \tag{5.64}$$

where the function $\mathcal{R}(t)$ is defined by

$$t = \int_0^{\mathcal{R}(t)} F(r)\frac{S(r)}{V(r)}dr, \tag{5.65}$$

and F is defined by

$$F(r) := \sup_{0<\xi<r}\left[V(\xi)\int_\xi^r \frac{dt}{S(t)}\right]. \tag{5.66}$$

Proof. By Theorem 2.10, the condition (5.62) holds with the function $\Lambda(v)$ defined implicitly by the identity

$$\Lambda(V(r)) = \frac{1}{4F(r)}.$$

Changing $v = V(r)$ in the integral (5.6), we obtain

$$t = 4 \int^{V^{-1}(\gamma(t))} F(r) \frac{S(r)}{V(r)} dr,$$

whence $\gamma(t) = V(\mathcal{R}(t/4))$. Substituting into (5.63) we obtain (5.64). ■

Of course, obtaining explicit estimates for the function $F(r)$ may be a difficult task. Under additional hypotheses, one can do it as in the next statement.

Given a constant $Q \geq 1$ and a positive function $f(r)$ on an interval (a, b), we say that f is *Q-quasi-increasing* on (a, b) if

$$f(r_1) \leq Q f(r_2) \quad \text{whenever } a < r_1 \leq r_2 < b. \tag{5.67}$$

Of course, if $Q = 1$ then f is an increasing function. If $f \in C^1(a, b)$ then a sufficient condition for f to be Q-quasi-increasing is

$$\int_a^b \left[\frac{f'}{f}\right]_- dr \leq \log Q. \tag{5.68}$$

We say that f is quasi-increasing if it is Q-quasi-increasing for some $Q \geq 1$.

Corollary 5.35 ([44]) *Assume that the function*

$$r \mapsto \frac{V(r)}{S(r)} \quad \text{is } Q\text{-quasi-increasing on } \mathbb{R}_+. \tag{5.69}$$

Then, for all $t > 0$,

$$p_t(o, o) \leq \frac{4}{V(\mathcal{R}(ct))}, \tag{5.70}$$

where $c = \frac{1}{8Q^2}$ *and the function* $\mathcal{R}(t)$ *defined by*

$$t = \int_0^{\mathcal{R}(t)} \frac{V(r)}{S(r)} dr. \tag{5.71}$$

Proof. It is easy to see that the function

$$\xi \mapsto V(\xi) \int_\xi^r \frac{dt}{S(t)}, \tag{5.72}$$

used in the definition (5.66) of $F(r)$, vanishes at both ends $\xi = 0$ and $\xi = r$. Hence, there is a point $\xi \in (0, r)$ where function (5.72) takes the maximum value so that

$$F(r) = V(\xi) \int_\xi^r \frac{dt}{S(t)}.$$

At this point ξ, we have

$$0 = \left(V(\xi) \int_\xi^r \frac{dt}{S(t)}\right)' = S(\xi) \int_\xi^r \frac{dt}{S(t)} - \frac{V(\xi)}{S(\xi)}$$

and hence

$$F(r) = \frac{V(\xi)}{S(\xi)} S(\xi) \int_\xi^r \frac{dt}{S(t)} = \left(\frac{V(\xi)}{S(\xi)}\right)^2.$$

By the condition (5.69),

$$\frac{V(\xi)}{S(\xi)} \leq Q \frac{V(r)}{S(r)}$$

whence

$$F(r) \leq Q^2 \left(\frac{V(r)}{S(r)}\right)^2.$$

Substituting this inequality in (5.65) we obtain

$$t \leq Q^2 \int_0^{\mathcal{R}(t)} \frac{V(r)}{S(r)} dr,$$

so that the definition of $\mathcal{R}(t)$ can be changed to (5.71) with simultaneous replacing t in (5.64) by t/Q^2. ■

Example 5.36 Let $V(r) = \exp(r^\alpha)$ for large r where $0 < \alpha < 1$. Then $\frac{V(r)}{S(r)} = \alpha^{-1} r^{1-\alpha}$, and (5.71) yields, for large t, $\mathcal{R}(t) \simeq t^{\frac{1}{2-\alpha}}$. By Corollary 5.35, we obtain, for large t,

$$p_t(o,o) \leq C \exp\left(-ct^{\frac{\alpha}{2-\alpha}}\right).$$

The exponent $\frac{\alpha}{2-\alpha}$ is sharp as by Theorem 5.25 we have a matching lower bound (5.49), that is

$$p_t(o,o) \asymp C \exp\left(-ct^{\frac{\alpha}{2-\alpha}}\right).$$

For more general model manifolds, two sided estimates of $p_t(o,o)$ are presented in Theorem 5.42 below.

Corollary 5.37 ([44]) *If the function $r \mapsto \frac{V(r)}{S(r)}$ is monotone increasing on $\mathbb{R}_+$, then, for all $t > 0$,*

$$p_t(o,o) \leq \frac{C}{V\left(\sqrt{t/2}\right)}. \tag{5.73}$$

If in addition the function $V(r)$ satisfies the doubling condition, that is $V(2r) \leq CV(r)$ for all $r > 0$ then

$$p_t(o,o) \simeq \frac{1}{V(\sqrt{t})}. \tag{5.74}$$

Approach to the proof. The proof of (5.73) in [44] consists of showing that $V\left(\sqrt{t/2}\right) \leq eV(\mathcal{R}(t/8))$ where $\mathcal{R}(t)$ is the function from (5.71). The lower bound in (5.74) follows from Theorem 5.30. ■

Remark 5.38 The estimate (5.74) remains true if the hypothesis of monotone increasing of $V(r)/S(r)$ is replaced by $V(r) \leq CrS(r)$, while still assuming the volume doubling (see [94]). Under a stronger hypothesis $V(r) \simeq rS(r)$ one obtains also off-diagonal estimates – see Lemma 6.15 below.

5.6 Estimates with the mean curvature function

In this section, we obtain estimates of $p_t(o,o)$ on a weighted model (M,μ) using the mean curvature function $m(r)=\frac{S'(r)}{S(r)}$. The main result is Theorem 5.42, which we state and prove after some preparation.

Lemma 5.39 *Assume that $m>0$ and $m'\leq 0$ on $(r_0,+\infty)$ for some $r_0\geq 0$. Then, for large enough t,*

$$p_t(o,o)\leq \frac{4}{V(\mathcal{R}(ct))}, \tag{5.75}$$

where function $\mathcal{R}(t)$ is defined by

$$t=\int_{r_0}^{\mathcal{R}(t)}\frac{dr}{m(r)}. \tag{5.76}$$

Proof. In order to use Corollary 5.35, we need to verify that $V(r)/S(r)$ is quasi-increasing on $\mathbb{R}_+$. For $r\leq r_0$ we have $\frac{V(r)}{S(r)}\simeq r$, which is quasi-increasing. Hence, it suffices to prove that $V(r)/S(r)$ is quasi-increasing on $(r_0,+\infty)$. By (5.68), it suffices to show that

$$\int_{r_0}^{\infty}\left[\frac{(V/S)'}{V/S}\right]_- dr<\infty. \tag{5.77}$$

By hypotheses $m>0$, we have $S'(r)>0$ on $(r_0,+\infty)$, and, by hypothesis $m'\leq 0$, the function $S(r)/S'(r)$ is increasing on $(r_0,+\infty)$. Therefore, for any $r>r_0$,

$$V(r)-V(r_0)=\int_{r_0}^{r}S(t)\,dt\leq\frac{S(r)}{S'(r)}\int_{r_0}^{r}S'(t)\,dt=\frac{S(r)}{S'(r)}\left(S(r)-S(r_0)\right),$$

whence

$$\frac{S'(r)}{S(r)}\leq\frac{S(r)-S(r_0)}{V(r)-V(r_0)} \tag{5.78}$$

Using the identity

$$\frac{(V/S)'}{V/S}=\frac{S}{V}-\frac{S'}{S}$$

we obtain

$$\begin{aligned}\frac{(V/S)'}{V/S} &\geq \frac{S(r)}{V(r)}-\frac{S(r)-S(r_0)}{V(r)-V(r_0)}=\frac{S(r_0)V(r)-V(r_0)S(r)}{V(r)(V(r)-V(r_0))}\\ &\geq -\frac{V(r_0)(S(r)-S(r_0))}{V(r)(V(r)-V(r_0))}\end{aligned} \tag{5.79}$$

and hence

$$\left[\frac{(V/S)'}{V/S}\right]_-\leq\frac{V(r_0)(S(r)-S(r_0))}{V(r)(V(r)-V(r_0))}. \tag{5.80}$$

Obviously, we have

$$\int_{r_0}^{\infty}\frac{S(r)\,dr}{V^2(r)}=\frac{1}{V(r_0)}<\infty,$$

because $V(r)\to\infty$ as $r\to\infty$ (this follows from $S'>0$ which implies that $S(r)\geq c$ for large r). Therefore, integrating the right hand side of (5.80) from r_0 to ∞, we obtain (5.77).

To finish the proof we are left to show that the function $\mathcal{R}$ from (5.71) can be replaced by the one from (5.76). For that, it suffices to show that, for large enough R,

$$\int_0^R \frac{V(r)}{S(r)}dr \le C\int_{r_0}^R \frac{dr}{m(r)}. \tag{5.81}$$

Indeed, by (5.78) we have

$$\frac{V(r)-V(r_0)}{S(r)-S(r_0)} \le \frac{S(r)}{S'(r)} = \frac{1}{m(r)} \tag{5.82}$$

whereas

$$\frac{V(r_0)}{S(r_0)} \le \frac{V(r_0)\, m(r_0)}{S(r_0)}\frac{1}{m(r)} = \frac{C}{m(r)}$$

just because $m(r)$ is decreasing. Hence, we obtain

$$\int_{r_0}^R \frac{V(r)}{S(r)}dr \le C\int_{r_0}^R \frac{dr}{m(r)},$$

which together with the trivial estimate

$$\int_0^{r_0} \frac{V(r)}{S(r)}dr \le C$$

implies (5.81). ■

Remark 5.40 If $r_0 = 0$ then the proof is much simplified since by (5.79) the function $\frac{V(r)}{S(r)}$ is increasing on $(0,+\infty)$ and by (5.82) $\frac{V(r)}{S(r)} \le \frac{1}{m(r)}$ for all $r > 0$, which implies that (5.75) holds for all $t > 0$.

Lemma 5.41 *Assume that $m > 0$ and $m' \le 0$ on $(r_0,+\infty)$ for some $r_0 \ge 0$. Then the function*

$$r \mapsto \frac{r^2}{\log V(r)} \tag{5.83}$$

is an increasing bijection from $(r_1,+\infty)$ to $(t_1,+\infty)$ for some (large) positive r_1 and t_1.

Proof. Since the function $S(r)$ is increasing, we have $V(\infty) = \infty$. Since the function $m(r)$ is bounded from above, we have $S(r) \le \exp(Cr)$ and $V(r) \le \exp(Cr)$. In particular, the function (5.83) goes to ∞ as $r \to \infty$. We are left to prove that, for large enough r,

$$\left(\frac{r^2}{\log V(r)}\right)' > 0.$$

Differentiating the left hand side, we reduce this to the inequality

$$\frac{rS}{V\log V} < 2,$$

which will true for large r if we prove that

$$\limsup_{r\to\infty} \frac{rS}{V\log V} \le 1. \tag{5.84}$$

Observe that

$$\frac{(rS)'}{(V\log V)'} = \frac{rS'+S}{S\log V + S} = \frac{rm+1}{\log V + 1}$$

and, using $m' \leq 0$ and (5.78),

$$\frac{(rm+1)'}{(\log V+1)'} = \frac{rm'+m}{S/V} \leq \frac{m}{S/V} \leq \frac{V(r)}{V(r)-V(r_0)} \to 1.$$

Going backwards and using l'Hospital's rule, we obtain (5.84). ■

Theorem 5.42 *Assume that the mean curvature function $m(r)$ of a weighted model (M,μ) satisfies the conditions $m>0$ and $m' \leq 0$ on $(r_0,+\infty)$, for some $r_0>0$. Then, for large t, we have*

$$p_t(o,o) \leq \exp\left(-\frac{\mathcal{R}^2(ct)}{t}\right), \tag{5.85}$$

where function $\mathcal{R}(t)$ is determined from the equation

$$\frac{\mathcal{R}(t)}{m(\mathcal{R}(t))} = t. \tag{5.86}$$

If in addition function m satisfies for large r the inequality

$$\int_{r_0}^{r} m(s)\,ds \leq Crm(r), \tag{5.87}$$

then, for large t,

$$p_t(o,o) \geq \frac{1}{2}\exp\left(-\frac{\mathcal{R}^2(Ct)}{t}\right). \tag{5.88}$$

Example 5.43 If $m(r) \simeq r^{-\beta}$ for large r, where $0<\beta<1$ then $\mathcal{R}(t) \simeq t^{\frac{1}{1+\beta}}$, whence, for large t,

$$p_t(o,o) \asymp c\exp\left(-Ct^{\frac{1-\beta}{1+\beta}}\right).$$

This estimate obviously matches the estimates from Example 5.36, which correspond to $\beta = 1-\alpha$.

Note that the functions $m(r) \simeq \frac{1}{r}$ and $m(r) \simeq \frac{\log^a r}{r}$ do not satisfy (5.87).

Proof. Since

$$\int_{r_0}^{R} \frac{dr}{m(r)} \leq \frac{R}{m(R)},$$

the function $\mathcal{R}(t)$ defined from (5.76) is larger than the function $\mathcal{R}(t)$ defined by (5.86), and hence we obtain from Lemma 5.39 that for the latter function $\mathcal{R}(t)$ and for large enough t,

$$p_t(o,o) \leq \frac{4}{V(\mathcal{R}(ct))}. \tag{5.89}$$

Let us prove that, for large enough R,

$$V(R) \geq 4\exp\left(\frac{1}{2}Rm(R)\right). \tag{5.90}$$

Indeed, using the identity

$$S(r) = S(r_0)\exp\left(\int_{r_0}^{r} m(t)\,dt\right) \tag{5.91}$$

and Jensen's inequality, we obtain

$$\begin{aligned}
V(R) &= \int_0^R S(r)\,dr \\
&= S(r_0)\int_0^R \exp\left(\int_{r_0}^r m(t)\,dt\right)dr \\
&\geq S(r_0)\,R\exp\left(\frac{1}{R}\int_0^R\left(\int_{r_0}^r m(t)\,dt\right)dr\right) \\
&\geq S(r_0)\,R\exp\left(\frac{1}{R}\int_{r_0}^R (r-r_0)\,m(R)\,dr\right) \\
&= S(r_0)\,R\exp\left(\frac{(R-r_0)^2}{2R}m(R)\right)
\end{aligned}$$

so that

$$\log V(R) \geq \log S(r_0) + \log R + \frac{1}{2}Rm(R) - r_0 m(R).$$

Since $m(R) \leq m(r_0)$, for large enough R we obtain (5.90) (the constant factor 4 in (5.90) can be replaced by any other positive number).

For $R = \mathcal{R}(ct)$ we have by (5.86) $m(R) = R/(ct)$ so that (5.90) yields

$$V(\mathcal{R}(ct)) \geq 4\exp\left(\frac{\mathcal{R}^2(ct)}{2ct}\right) \geq 4\exp\left(\frac{\mathcal{R}^2(ct)}{t}\right),$$

(provided $c \leq 1/2$ which can be assume here) which together with (5.89) yields (5.85).

By Lemma 5.41, the function (5.83) satisfies the hypotheses of Theorem 5.25. Therefore, by Theorem 5.25, we obtain, for large enough t,

$$p_t(o,o) \geq \frac{1/2}{V(\mathcal{R}^*(Ct))}, \tag{5.92}$$

where $\mathcal{R}^*(t)$ is the inverse function to (5.83), that is

$$\frac{(\mathcal{R}^*)^2}{\log V(\mathcal{R}^*)} = t. \tag{5.93}$$

In particular, we have

$$V(\mathcal{R}^*) = \exp\frac{(\mathcal{R}^*)^2}{t}$$

so that (5.92) can be rewritten as follows:

$$p_t(o,o) \geq \frac{1}{2}\exp\left(-\frac{\mathcal{R}^*(Ct)^2}{Ct}\right) \geq \frac{1}{2}\exp\left(-\frac{\mathcal{R}^*(Ct)^2}{t}\right). \tag{5.94}$$

The lower bound (5.88) will follow from (5.94) if we prove that, for large t,

$$\mathcal{R}^*(t) \leq \mathcal{R}(Ct).$$

Comparing (5.86) and (5.93), we see that it suffices to verify that for large enough R,

$$\frac{R}{m(R)} \leq C\frac{R^2}{\log V(R)},$$

which in turn will follow from

$$\log\left(V\left(R\right)-V\left(r_0\right)\right)\leq CRm\left(R\right).$$

Since by inequality (5.82) from the proof of Lemma 5.39 and by (5.91)

$$V\left(R\right)-V\left(r_0\right)\leq\frac{S\left(R\right)}{m\left(R\right)}=\exp\left(\int_{r_0}^{R}m\left(t\right)dt\right)\frac{S\left(r_0\right)}{m\left(r\right)},$$

it suffices to prove that, for large enough R,

$$\int_{r_0}^{R}m\left(r\right)dr+\log S\left(r_0\right)+\log\frac{1}{m\left(R\right)}\leq CRm\left(R\right). \tag{5.95}$$

The first term in (5.95) is dominated by the right hand side by hypothesis (5.87). This hypothesis implies also that, for large enough r, say, for $r>R_0$,

$$m\left(r\right)\geq\frac{c}{r}\int_{r_0}^{r}m\left(s\right)ds\geq\frac{c}{r}.$$

Therefore, iterating this estimate, we obtain, for large enough R,

$$m\left(R\right)\geq\frac{c}{R}\int_{R_0}^{R}m\left(r\right)dr\geq\frac{c}{R}\int_{R_0}^{R}\frac{dr}{r}\geq c\frac{\log R}{R}.$$

In particular,

$$Rm\left(R\right)\geq c\log R \tag{5.96}$$

whence it follows that the constant term $\log S\left(r_0\right)$ in (5.95) is dominated by the right hand side, for large enough R. Finally, (5.96) implies

$$\log\frac{1}{m\left(R\right)}\leq\log R+C\leq CRm\left(R\right),$$

so that the term $\log\frac{1}{m(R)}$ in (5.95) is also dominated by the right hand side. ■

5.7 Green function and Green operator

The *Green function* $g\left(x,y\right)$ of a weighted manifold $\left(M,\mu\right)$ is defined by

$$g\left(x,y\right)=\int_0^{\infty}p_t\left(x,y\right)dt, \tag{5.97}$$

and it takes the values in $[0,+\infty]$ including $+\infty$. For example, in $\mathbb{R}^n$ we have $g\left(x,y\right)=c_n\left|x-y\right|^{2-n}$ if $n>2$ and $g\left(x,y\right)\equiv+\infty$ for $n\leq2$ (a fundamental solution $\log\frac{1}{|x-y|}$ in $\mathbb{R}^2$ is not a Green function in our sense because it is signed).

For $x=y$, the integral in (5.97) always diverges at $t=0$ provided $n=\dim M>1$, because $p_t\left(x,x\right)\simeq t^{-n/2}$ as $t\to0$ (see Theorem 3.9). We say that the Green function is *finite* if $g\left(x,y\right)<\infty$ for all *distinct* $x,y\in M$. Various conditions for the finiteness of Green function will be discussed in Section 9.1. Here we state only one simple result, but before that let us notice that if the Green function is finite then it determines the *Green operator* G, which acts on functions on M as follows:

$$Gf\left(x\right)=\int_M g\left(x,y\right)f\left(y\right)d\mu\left(y\right). \tag{5.98}$$

Lemma 5.44 *Assume that* $\lambda_{\min}(M) > 0$. *Then the Green function* g *is finite,* $g(x,\cdot) \in L^1_{loc}(M,\mu)$ *for all* $x \in M$, *and*

$$-\Delta_\mu g(x,\cdot) = \delta_x. \tag{5.99}$$

Moreover, the Green operator G *is the inverse operator to* $-\Delta_\mu|_{W_0^2}$ *in* L^2.

Remark 5.45 The equation (5.99) is understood in the distributional sense so that the Green function is a fundamental solution of the operator $-\Delta_\mu$. It also follows from (5.99) that the function $g(x,\cdot)$ is harmonic outside x and hence is smooth.

Proof. The convergence of the integral in (5.97) for distinct x, y follows from Theorem 5.21 (cf. (5.37) and (5.41)). It also follows from Theorem 5.21 that, for any compact set $K \subset M$ and for any $T > 0$, there is a constant C such that

$$p_t(x,y) \le Ce^{-\lambda t} \quad \text{for all } x, y \in K \text{ and } t \ge T, \tag{5.100}$$

where $\lambda = \lambda_{\min}(M)$ (alternatively, (5.100) can be deduced from (3.19)). Using (3.11), (5.97), and (5.100), we obtain

$$\begin{aligned}
\int_K g(x,y)\, d\mu(y) &= \int_0^T \int_K p_t(x,y)\, d\mu(y)\, dt + \int_K \int_T^\infty p_t(x,y)\, dt\, d\mu(y) \\
&\le T + \frac{C}{\lambda} e^{-\lambda T} \mu(K) < \infty,
\end{aligned}$$

whence it follows that $g(x,\cdot) \in L^1_{loc}$.

The spectrum of operator $H := -\Delta_\mu|_{W_0^2}$ is located in $[\lambda, +\infty)$ and hence, H^{-1} exists, is a bounded operator, and $\|H^{-1}\| \le \lambda^{-1}$. By the functional calculus, we have

$$\int_0^\infty e^{-tH} dt = H^{-1}. \tag{5.101}$$

Comparing (5.101) with (5.97) we see that the Green operator G coincides in L^2 with H^{-1}.

Consequently, for any $f \in L^2$ there is a unique solution $u \in W_0^2$ to the equation

$$-\Delta_\mu u = f, \tag{5.102}$$

and this solution is given by $u = Gf$. To prove (5.99) it suffices to verify that, for any $u \in \mathcal{D}$,

$$-\int_M g(x,y)\, \Delta_\mu u(y)\, d\mu(y) = u(x). \tag{5.103}$$

Indeed, the function u is in W_0^2 and satisfies the equation (5.102) with $f = -\Delta_\mu u$. Hence, by the above remark, $u = Gf$, which is exactly (5.103). ∎

Corollary 5.46 *Let* Ω *be a non-empty relatively compact open subset on* M *such that* $M \setminus \overline{\Omega}$ *is non-empty. Then the Green function* g^Ω *of* (Ω,μ) *is finite and, for any* $f \in L^2(\Omega,\mu)$, *the function*

$$u(x) := \int_\Omega g^\Omega(x,y) f(y)\, d\mu(y)$$

is the unique solution to the equation $-\Delta_\mu u = f$ *in the class* $W_0^2(\Omega,\mu)$, *where* Δ_μ *is understood in the distributional sense.*

Proof. By Theorem 2.3, we have $\lambda_{\min}(\Omega) > 0$. By Lemma 5.44, the Green function g^{Ω} is finite and the Green operator G^{Ω} is the inverse to the Dirichlet Laplace operator in $L^2(\Omega,\mu)$, whence the claim follows. ■

Corollary 5.47 *If the Green function g of a weighted manifold (M,μ) is finite then it is the minimal non-negative fundamental solution of $-\Delta_\mu$ on M.*

Proof. The manifold M is non-compact because otherwise $g \equiv \infty$ by (3.30). Let $\{\Omega_k\}_{k=1}^{\infty}$ be an exhausting sequence on M such that all Ω_k are relatively compact and hence $M \setminus \overline{\Omega_k} \neq \emptyset$. By Theorem 2.3 and Lemma 5.44, the weighted manifold (Ω_k,μ) has a finite Green function g^{Ω_k} satisfying in Ω_k the equation

$$-\Delta_\mu g^{\Omega_k}(x,\cdot) = \delta_x. \tag{5.104}$$

It follows from Theorem 3.5 and (5.97) that the sequence $\left\{g^{\Omega_k}\right\}$ increases and converges to g as $k \to \infty$. On the other hand, (5.104) implies that, for all $m < k$, the function $g^{\Omega_k}(x,\cdot) - g^{\Omega_m}(x,\cdot)$ is harmonic in Ω_m, for any $x \in \Omega_m$. Therefore, the function

$$g(x,\cdot) - g^{\Omega_m}(x,\cdot) = \lim_{k\to\infty}\left(g^{\Omega_k}(x,\cdot) - g^{\Omega_m}(x,\cdot)\right)$$

is also harmonic in Ω_m. Therefore, $g(x,\cdot)$ satisfies (5.99) in Ω_m, and letting $m \to \infty$ we conclude that $g(x,\cdot)$ satisfies (5.99) in M.

If f is another non-negative fundamental solution at x, that is, $-\Delta_\mu f = \delta_x$, then the difference $f - g^{\Omega_k}(x,\cdot)$ is harmonic in Ω_k. The minimum principle implies $f - g^{\Omega_k}(x,\cdot) \geq 0$ whence it follows that $f \geq g(x,\cdot)$. ■

Remark 5.48 As one can see from the proof, if $g(x,y)$ is finite for some couple x,y then it is finite for all distinct x,y.

Example 5.49 Let (M,μ) be a weighted model, o be its origin, and $S(r)$ be its boundary area function. Then the Green function at o has the following explicit value:

$$g(x,o) = \int_r^{\infty} \frac{ds}{S(s)}, \tag{5.105}$$

where $r = |x|$. Furthermore, the Green function of the central ball B_R is given by

$$g^{B_R}(x,o) = \int_r^{R} \frac{ds}{S(s)}. \tag{5.106}$$

The latter is verified by showing that this function is harmonic away from o (cf. (2.12)), satisfies (5.99), and vanishes on the boundary. Then (5.105) follows from (5.106) by passing to the limit as $R \to \infty$ (cf. Corollary 5.47).

Estimates of the Green function can be frequently obtained from estimates of the heat kernel and (5.97). The following lemma is useful is such cases.

Lemma 5.50 *Let $F(t)$ be a positive monotone increasing function on $\mathbb{R}_+$ and set*

$$G(r) = \int_0^{\infty} \frac{1}{F\left(\sqrt{t}\right)} \exp\left(-\frac{r^2}{t}\right) dt. \tag{5.107}$$

If F satisfies the doubling property

$$F(2r) \leq CF(r) \quad \textit{for all } r > 0, \tag{5.108}$$

then
$$G\left(r\right) \simeq \int_{r}^{\infty} \frac{sds}{F\left(s\right)}, \tag{5.109}$$
where the constants bounding the ratio of the both sides in (5.109) depend only on the constant C from (5.108). If in addition
$$\frac{F\left(r\right)}{F\left(s\right)} \geq c\left(\frac{r}{s}\right)^{\alpha}, \quad \text{for all } r > s > 0 \tag{5.110}$$
where $c > 0$ and $\alpha > 2$ then
$$G\left(r\right) \simeq \frac{r^{2}}{F\left(r\right)}, \tag{5.111}$$
where the constants bounding the ratio of the both sides in (5.111) depend only on c, C and α.

Proof. Splitting the integral in (5.107) into the sum of the integral over $\left(0, r^{2}\right)$ and $\left(r^{2}, +\infty\right)$ and noticing that the latter integral is of the order $\int_{r}^{\infty} \frac{sds}{F(s)}$, we obtain the lower bound in (5.109). To prove the upper bound, it suffices to appropriately estimate from above the former integral. Noticing that
$$\int_{r}^{\infty} \frac{sds}{F\left(s\right)} \geq \int_{r}^{2r} \frac{sds}{F\left(s\right)} \geq C^{-1} \frac{r^{2}}{F\left(r\right)},$$
we conclude that it suffices to verify that
$$\int_{0}^{r^{2}} \frac{1}{F\left(\sqrt{t}\right)} \exp\left(-\frac{r^{2}}{t}\right) dt \leq \operatorname{const} \frac{r^{2}}{F\left(r\right)}. \tag{5.112}$$
By (5.108) there are constants $\alpha' > 0$ and $C' > 0$ such that
$$\left(\frac{F\left(r\right)}{F\left(s\right)}\right) \leq C'\left(\frac{r}{s}\right)^{\alpha'}, \quad \text{for all } r > s > 0,$$
so that, for all $r \geq \sqrt{t}$,
$$\frac{F\left(r\right)}{F\left(\sqrt{t}\right)} \leq C'\left(\frac{r}{\sqrt{t}}\right)^{\alpha'}.$$
Substituting into (5.112) and changing in the integral $\tau = t/r^{2}$, we see that (5.112) amounts to
$$\int_{0}^{1} \tau^{\alpha'} \exp\left(-\frac{1}{\tau}\right) d\tau \leq \operatorname{const},$$
which is true.

To prove the second claim, it suffices to show that
$$\int_{r}^{\infty} \frac{sds}{F\left(s\right)} \leq \operatorname{const} \frac{r^{2}}{F\left(r\right)},$$
which by (5.110) amounts to
$$\int_{r}^{\infty} \left(\frac{r}{s}\right)^{\alpha} sds \leq \operatorname{const} r^{2},$$
and which is true by $\alpha > 2$. ■

Example 5.51 Let (M,μ) be a geodesically complete manifold satisfying the relative Faber-Krahn inequality. Using the heat kernel upper bound (5.56) and Lemma 5.50, we obtain that the Green function satisfies the estimate

$$g(x,y) \leq C \int_{d(x,y)}^{\infty} \frac{sds}{V(x,s)}. \tag{5.113}$$

In particular, the Green function is finite provided

$$\int^{\infty} \frac{sds}{V(x,s)} < \infty. \tag{5.114}$$

As we will see later, this condition is also necessary for the finiteness of the Green function (see Theorem 9.7 and Corollary 9.9).

6 Harnack inequality

6.1 The Li-Yau estimate

In this section we assume that (M,μ) is a geodesically complete weighted manifold. We say that the heat kernel on (M,μ) satisfies the *Li-Yau estimate* if, for all $x,y \in M$ and $t>0$,

$$p_t(x,y) \asymp \frac{C}{V\left(x,\sqrt{t}\right)} \exp\left(-c\frac{d^2(x,y)}{t}\right). \tag{6.1}$$

P. Li and S.-T. Yau [150] proved this estimate on geodesically complete Riemannian manifolds with non-negative Ricci curvature using the gradient estimates (see Theorem 6.5 below). Perhaps, for the first time the estimate (6.1) appeared in the work of A.K. Gushchin [117] in the context of parabolic equations in unbounded domains in $\mathbb{R}^n$ (see also [118]). Here we survey the approach to (6.1), which originates from [88] and [187] and which applies to a large class of weighted manifolds.

We say that (M,μ) satisfies the (*uniform parabolic*) *Harnack inequality* if, for any ball $B(z,r)$ on M and for any positive solution $u(t,x)$ of the heat equation in the cylinder $\mathcal{C} = \left(0,r^2\right) \times B(z,r)$, the following holds:

$$\sup_{\mathcal{C}_-} u(t,x) \leq C \inf_{\mathcal{C}_+} u(t,x)$$

where $\mathcal{C}_- = \left(\frac{1}{4}r^2, \frac{1}{2}r^2\right) \times B\left(z,\frac{1}{2}r\right)$ and $\mathcal{C}_+ = \left(\frac{3}{4}r^2, r^2\right) \times B\left(z,\frac{1}{2}r\right)$ (see Fig. 4).

It is well known that the Harnack inequality holds for uniformly parabolic equations in $\mathbb{R}^n$ – see [166]. The relation to heat kernels is given by the following statement, which originated from [146] and [144].

Theorem 6.1 ([73], [122], [193]) *A manifold (M,μ) satisfies the Li-Yau estimate if and only if it satisfies the Harnack inequality.*

To characterize manifolds with the Harnack inequality, we need one more notion. We say that a weighted manifold satisfies the (*weak*) *Poincaré inequality* if there exists $\varepsilon \in (0,1)$ such that for any ball $B(z,r)$ and for any function $u \in C^1(B(z,r))$,

$$\inf_{s\in\mathbb{R}} \int_{B(z,\varepsilon r)} (u-s)^2 \, d\mu \leq Cr^2 \int_{B(z,r)} |\nabla u|^2 \, d\mu. \tag{6.2}$$

(The term "weak" refers here to the factor $\varepsilon < 1$).

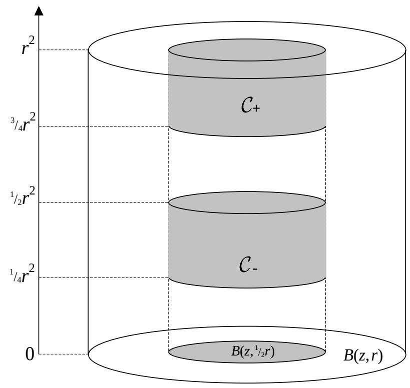


Figure 4: Cylinders $\mathcal{C}_+$ and $\mathcal{C}_-$

Theorem 6.2 ([88], [187]) *A manifold (M, μ) satisfies the Harnack inequality if and only if it satisfies the doubling volume property and the Poincaré inequality.*

Hence, the Li-Yau estimate holds if and only if the doubling volume property and the Poincaré inequality hold.

The proof of Theorem 6.2 uses the following result, which is of its own interest.

Theorem 6.3 ([88], [187]) *If (M, μ) satisfies the Poincaré inequality and the volume doubling property then it satisfies the relative Faber-Krahn inequality.*

Note that the converse to Theorem 6.3 is not true: it is possible to show that a connected sum of two copies of $\mathbb{R}^n$, $n \geq 3$, satisfies the relative Faber-Krahn inequality but not the Poincaré inequality (see Example 6.22 below).

Using Corollary 5.31, we obtain that the Poincaré inequality and the doubling volume property imply the upper bound and the on-diagonal lower bound in (6.1). The off-diagonal lower bound requires additional tools, which we do not touch here and which are similar to Moser's original proof of the Harnack inequality in $\mathbb{R}^n$ (see [166] and [181]).

It is worth mentioning that if (M, μ) satisfies the Li-Yau estimate then the constant c in the exponential in the upper bound in (6.1) can be taken arbitrarily close to $\frac{1}{4}$. Furthermore, by Theorem 5.29, one can take $c = \frac{1}{4}$ at expense of an additional polynomial factor.

Connection to the Ricci curvature comes from the following statement.

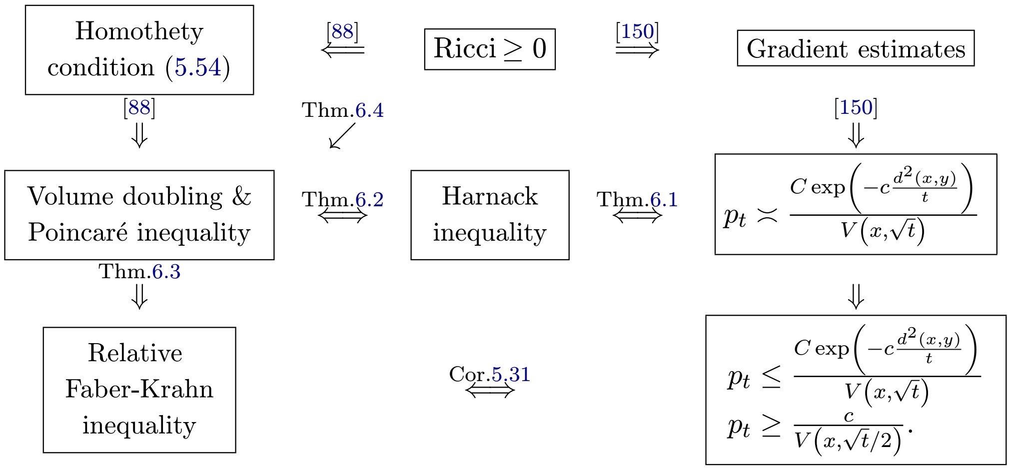
Theorem 6.4 ([27], [88]) *If M has non-negative Ricci curvature and μ is the Riemannian volume then (M, μ) satisfies the Poincaré inequality and the volume doubling property.*

In fact, both Poincaré inequality and volume doubling property come from the property (5.54) of the homothety on such manifolds. Clearly, Theorems 6.4 and 6.3 imply Theorem 5.27.

Successive application of Theorems 6.4, 6.2, and 6.1 yields the following result.

Theorem 6.5 ([150]) *If M has non-negative Ricci curvature and μ is the Riemannian volume then the heat kernel on (M, μ) satisfies the Li-Yau estimate (6.1).*

The above results are schematically presented on the following diagram:



Finally, let us mention that if the heat kernel satisfies the Li-Yau estimate then, by Lemma 5.50, the Green function can be estimated as follows:

$$g(x,y) \simeq \int_{d(x,y)}^{\infty} \frac{sds}{V(x,s)}. \tag{6.3}$$

6.2 Manifolds with relatively connected annuli

For a weighted manifold (M,μ), fix a reference point $o \in M$, which will be called an *origin*, and use the notation

$$|x| := d(x,o) \quad \text{and} \quad V(s) := V(o,s). \tag{6.4}$$

We say that M has *relatively connected annuli* if there exists a positive constant K such that, for all large enough r and all $x,y \in M$ with $|x| = |y| = r$, there exists a continuous path $\gamma : [0,1] \to M$ with $\gamma(0) = x$, $\gamma(1) = y$ whose image is contained in $B(o,Kr) \setminus B(o,K^{-1}r)$ (see Fig. 5).

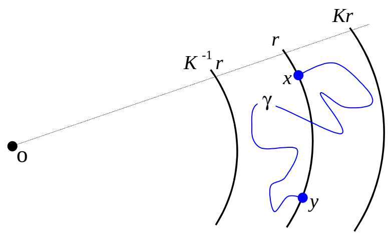


Figure 5: Path γ connects points x and y in $B(o,Kr) \setminus B\left(o,K^{-1}r\right)$

For example, any model manifold of dimension $n \geq 2$ (in particular $\mathbb{R}^n$) has relatively connected annuli.

We say that a weighted manifold (M,μ) satisfies the *volume comparison condition* if, for all $x \in M$ and $r = |x|$,

$$V(r) \leq CV(x, \frac{1}{100}r). \tag{6.5}$$

It is clear that the volume doubling property implies the volume comparison condition.

A ball $B(x,r)$ is called *remote* if $r \leq \frac{1}{2}|x|$, and *central* if $x = o$.

Theorem 6.6 ([110]) *Let (M, μ) be a complete non-compact weighted manifold with relatively connected annuli. Assume that the volume doubling property and the Poincaré inequality hold for remote balls in (M, μ). Then these properties hold for all balls in (M, μ) if and only if (M, μ) satisfies the volume comparison condition.*

Approach to the proof. It is easy to see that it suffices to prove the volume doubling property (VD) and the Poincaré inequality (PI) for central balls. One obtains (VD) for central balls from (VD) for remote balls and from (6.5). Most non-trivial part is to prove (PI) for a central ball. For that, fix a large enough number ρ and split a central ball $B(o, \rho^n)$ into a series of annuli $\{A_k\}_{k=1}^n$ where

$$A_1 = B(o, \rho) \text{ and } A_k = B(o, \rho^k) \setminus B(o, \rho^{k-1}) \text{ for } k > 1.$$

First, one proves a version of (PI) in each annulus A_k covering it by a bounded number of remote balls. Next, consider the family $\Gamma = \{A_k\}_{k=1}^n$ as a graph with vertices A_k and edges connecting A_k and A_{k+1}. Put also weight $m_k = \mu(A_k)$ on each vertex A_k. Then one can prove a certain discrete Poincaré inequality on the weighted graph Γ using the fact that the sequence $\{m_k\}$ of weights grows exponentially in k (which follows from (VD)). Combining it with (PI) in each annulus A_k one obtains (PI) in $B(o, \rho^n)$. ∎

Example 6.7 Let M be a Riemannian model of dimension $n \geq 2$, and let $V(r)$ and $S(r) = V'(r)$ be, respectively, the volume function and the boundary area function of M (see Section 2.4). Using Theorems 6.2 and 6.6, it was shown in [110] that M satisfies the Harnack inequality provided the following conditions hold:

$$V(r) \leq Cr^n \quad \text{and} \quad V(r) \simeq rS(r). \tag{6.6}$$

Under the standing assumption that $S(r_1) \simeq S(r_2)$ if $r_1 \simeq r_2$, these conditions are also necessary for the Harnack inequality.

6.3 Non-uniform change of measure

In this section (M, μ) is a geodesically complete non-compact weighted manifold. It is easy to see that the volume doubling property and the Poincaré inequality are invariant under quasi-isometry. Hence, we obtain from Theorem 6.2 that the Harnack inequality is invariant under quasi-isometry. This highly non-trivial result was first proved in [187]. Let us state for further references the following particular case.

Corollary 6.8 *If the heat kernel on (M, μ) satisfies the Li-Yau estimate and $h \simeq 1$ is a smooth function on M then the heat kernel on $(M, \widetilde{\mu})$ also satisfies the Li-Yau estimate, where $d\widetilde{\mu} = h^2 d\mu$.*

We will state below a more general result about the heat kernel on $(M, \widetilde{\mu})$ when h is not necessarily bounded. We use the notation and terminology from the previous section.

Theorem 6.9 ([110]) *Assume that (M, μ) satisfies the Harnack inequality and has relatively connected annuli. Let h be a smooth positive function on M satisfying the following two conditions, for all $k = 1, 2, \ldots$:*

$$h_k := \sup_{2^{k-1} \leq |x| \leq 2^k} h \leq C \inf_{2^{k-1} \leq |x| \leq 2^k} h \tag{6.7}$$

and

$$\sum_{i=1}^{k} h_i^2 V\left(2^i\right) \leq C h_k^2 V(2^k). \tag{6.8}$$

Define $\widetilde{\mu}$ *by* $d\widetilde{\mu} = h^2 d\mu$*. Then* $(M, \widetilde{\mu})$ *satisfies the Harnack inequality.*

Moreover, if h *satisfies* (6.7) *then* (6.8) *is not only sufficient but also necessary for* $(M, \widetilde{\mu})$ *to satisfy the Harnack inequality.*

Sketch of proof. By Theorem 6.2, we know that (M, μ) satisfies the volume doubling property (VD) and the Poincaré inequality (PI), and we need to verify that (VD) and (PI) hold on $(M, \widetilde{\mu})$. Since function h is nearly constant in any remote ball, both (VD) and (PI) for remote balls on $(M, \widetilde{\mu})$ trivially follow from those properties on (M, μ). Observing that (6.8) is equivalent to the volume comparison condition for $(M, \widetilde{\mu})$, we conclude by Theorem 6.6 that (VD) and (PI) hold for *all* balls on $(M, \widetilde{\mu})$. ■

Remark 6.10 Let us set

$$\overline{h}\left(r\right) = \sup_{|x|=r} h\left(x\right). \tag{6.9}$$

Then conditions (6.7) and (6.8) can be equivalently stated as follows:

$$h\left(x\right) \simeq \overline{h}\left(|x|\right) \quad \text{for all } x \in M \tag{6.10}$$

and

$$\int_1^r \overline{h}^2\left(s\right) V\left(s\right) \frac{ds}{s} \simeq \overline{h}^2\left(r\right) V\left(r\right), \quad \text{for all } r > 2 \tag{6.11}$$

(note that the opposite inequality in (6.8) is trivial).

For example, if $V\left(r\right) \simeq r^{\alpha}$ and $\overline{h}\left(r\right) \simeq r^{\beta}$ for large r then (6.11) holds for $\beta > -\alpha/2$.

Corollary 6.11 ([110]) *Assume that* M *has relatively connected annuli and let the heat kernel on* (M, μ) *satisfy the Li-Yau estimate. Let* h *be a smooth positive function on* M *satisfying the conditions* (6.10) *and* (6.11)*. Then the heat kernel* $\widetilde{p}_t$ *of* $(M, \widetilde{\mu})$*, where* $d\widetilde{\mu} = h^2 d\mu$*, admits the following estimate, for all* $x, y \in M$ *and* $t > 0$*,*

$$\widetilde{p}_t(x, y) \asymp \frac{C}{V(x, \sqrt{t})\overline{h}^2\left(|x| + \sqrt{t}\right)} \exp\left(-c\frac{d^2(x, y)}{t}\right), \tag{6.12}$$

and the Green function $\widetilde{g}$ *admits the estimate*

$$\widetilde{g}\left(x, y\right) \simeq \int_{d(x,y)}^{\infty} \frac{sds}{V(x, s)\overline{h}^2\left(|x| + s\right)}. \tag{6.13}$$

Proof. It follows from Theorems 6.9 and 6.1 that the heat kernel $\widetilde{p}_t$ on $(M, \widetilde{\mu})$ satisfies the Li-Yau estimate, that is

$$\widetilde{p}_t\left(x, y\right) \asymp \frac{C}{\widetilde{V}\left(x, \sqrt{t}\right)} \exp\left(-c\frac{d^2\left(x, y\right)}{t}\right),$$

where

$$\widetilde{V}\left(x, r\right) = \widetilde{\mu}\left(B\left(x, r\right)\right) = \int_{B(x,r)} h^2 d\mu.$$

Using (6.10) and (6.11), it is not difficult to see that

$$\widetilde{V}\left(x, r\right) \simeq V\left(x, r\right) \overline{h}^2(|x| + r), \tag{6.14}$$

whence (6.12) follows. Finally, (6.13) follows from (6.12) and Lemma 5.50. ■

Remark 6.12 Since $\widetilde{p}_t(x,y) = \widetilde{p}_t(y,x)$, the estimate (6.12) can be "symmetrized" as follows:

$$\widetilde{p}_t(x,y) \asymp \frac{C \exp\left(-c\frac{d^2(x,y)}{t}\right)}{\sqrt{V(x,\sqrt{t})V\left(y,\sqrt{t}\right)}\,\overline{h}\left(|x|+\sqrt{t}\right)\overline{h}\left(|y|+\sqrt{t}\right)}. \tag{6.15}$$

Example 6.13 Let $M = \mathbb{R}^n$, $n \geq 2$, and μ be the Lebesgue measure. Set $\langle x\rangle = 2 + |x|$ and let h be a smooth function on $\mathbb{R}^n$ such that $h(x) \simeq \langle x\rangle^\beta$ for all $x \in \mathbb{R}^n$ where $\beta > -\frac{n}{2}$. Then $h(x)$ satisfies the conditions (6.10), (6.11), and we obtain by Corollary 6.11 that the heat kernel $\widetilde{p}_t$ on $(\mathbb{R}^n, \widetilde{\mu})$ satisfies the estimate

$$\widetilde{p}_t(x,y) \asymp C t^{-\frac{n}{2}-\beta}\left(1+\frac{\langle x\rangle}{\sqrt{t}}\right)^{-2\beta}\exp\left(-c\frac{|x-y|^2}{t}\right). \tag{6.16}$$

Example 6.14 In the setting of the previous example, let $h = h(r)$ depend only on $r = |x|$ so that $(\mathbb{R}^n, \widetilde{\mu})$ is a weighted model. Assume that $h(r) = r^{-n/2}$ for $r > 1$ so that the condition (6.8) fails. By Theorem 6.9, the Li-Yau estimate is not true in this case (in fact, the volume doubling property fails). The boundary area function $\widetilde{S}(r)$ on $(\mathbb{R}^n, \widetilde{\mu})$ is given by $\widetilde{S}(r) = cr^{n-1}h^2(r) = cr^{-1}$ whence we obtain the volume function $\widetilde{V}(r) = c\log r + \widetilde{V}(1)$, for $r > 2$. Since the function $\widetilde{V}(r)/\widetilde{S}(r)$ is increasing, Corollary 5.37 applies, and we obtain by (5.74)

$$\widetilde{p}_t(o,o) \simeq \frac{1}{\widetilde{V}\left(\sqrt{t}\right)} \simeq \frac{1}{\log t} \quad \text{for } t > 2.$$

Finally, if $h(x) \simeq \langle x\rangle^\beta$ with $\beta < -\frac{n}{2}$ then $\widetilde{\mu}(\mathbb{R}^n) < \infty$ and by (3.30), $\widetilde{p}_t(x,y) \to \widetilde{\mu}(\mathbb{R}^n)^{-1}$ as $t \to \infty$.

In conclusion of this section, let us prove a certain heat kernel estimate for weighted models. Let (M,μ) be a weighted model with the origin o, $V(r)$ be the volume function of (M,μ), and $S(r) = V'(r)$ be the boundary area function (cf. Section 2.4).

Lemma 6.15 *Let (M,μ) be a weighted model of dimension $n \geq 2$, and assume that, for all $r > 0$,*

$$V(r) \simeq rS(r). \tag{6.17}$$

Then the heat kernel p_t on (M,μ) satisfies the estimate

$$p_t(o,x) \asymp \frac{C}{V\left(\sqrt{t}\right)}\exp\left(-c\frac{|x|^2}{t}\right), \tag{6.18}$$

for all $x \in M$ and $t > 0$.

It is worth mentioning that the condition (6.17) is always satisfied for a bounded range of r, so it suffices to require it for large r. In a particular case $V(r) = r^\alpha$ for large r, where $0 < \alpha < 1$, the upper bound in (6.18) was proved in [91], using a different approach. If (M,μ) is a Riemannian model and $V(r) = r^\alpha$ for large r where $\alpha > n$ then, by Example 6.7, the Harnack inequality on (M,μ) fails, whereas the estimate (6.18) is still true.

Proof. Consider a positive function $h(r)$ defined by the equation

$$S(r) = \omega_n r^{n-1}h^2(r). \tag{6.19}$$

In other words, $S(r)$ coincides with the boundary area function of the weighted model $\left(\mathbb{R}^n, h^2(|x|)\,dx\right)$. Let $\widetilde{p}_t(x,y)$ be the heat kernel of $\left(\mathbb{R}^n, h^2(|x|)\,dx\right)$. Manifold (M,μ) is stochastically complete since it satisfies (3.27). Hence, we conclude by Lemma 4.1 that

$$p_t(o,x) = \widetilde{p}_t(o,x),$$

where we identify M and $\mathbb{R}^n$ by means of the polar coordinates.

The hypothesis (6.17) implies that the function h satisfies the condition (6.11) in $(\mathbb{R}^n, dx)$. Therefore, by Corollary 6.11,

$$\widetilde{p}_t(o,x) \asymp \frac{C}{t^{n/2}h^2\left(\sqrt{t}\right)} \exp\left(-c\frac{d^2(o,x)}{t}\right) \simeq \frac{C}{V\left(\sqrt{t}\right)} \exp\left(-c\frac{|x|^2}{t}\right),$$

whence (6.18) follows. ■

6.4 Conformal change of the metric tensor

For a manifold with relatively connected annuli, we use the notation and terminology from Section 6.2.

Theorem 6.16 ([113]) *Let (M,g,μ) be a geodesically complete weighted manifold that satisfies the Harnack inequality and has relatively connected annuli. Let $a(x)$ be a smooth positive function on M such that*

$$a(x) \simeq \overline{a}(|x|) \text{ for all } x \in M, \tag{6.20}$$

where $\overline{a}(r) := \sup_{|x|=r} a(x)$, and

$$\int_1^r \overline{a}(s)\,ds \approx \overline{a}(r)\,r \text{ for all } r > 2. \tag{6.21}$$

Then the weighted manifold $(M, \widetilde{\mathbf{g}}, \mu)$ with the metric $\widetilde{g} = a^2 g$ is also geodesically complete, satisfies the Harnack inequality, and has relatively connected annuli.

The proof is similar to Theorem 6.9 and also uses Theorem 6.6.

Define a function $\rho(r)$ by the identity

$$r = \int_0^{\rho(r)} \overline{a}(s)\,ds. \tag{6.22}$$

It is possible to show that the volume of the geodesic ball $\widetilde{B}(x,r)$ in the metric $\widetilde{g}$ admits the following estimate

$$\mu(\widetilde{B}(x,r)) \simeq V\left(x, \frac{r}{\overline{a}(|x| + \rho(r))}\right), \tag{6.23}$$

where, as before, $|x|$ is the distance from x to the origin o in the metric g, and $V(x,r)$ is the volume of the geodesic ball $B(x,r)$ in the metric g. Combining Theorems 6.1 and 6.16, we obtain the following result.

Corollary 6.17 ([113]) *Assume that (M,g,μ) is a geodesically complete manifold with relatively connected annuli and let the heat kernel of (M,g,μ) satisfy the Li-Yau estimate. Let $a(x)$ be a* *smooth positive function on M satisfying the conditions (6.20) and (6.21). Then the heat kernel $\widetilde{p}_t$ of $(M, \widetilde{g}, \mu)$, where $\widetilde{g} = a^2 g$, satisfies the following estimate, for all $x, y \in M$ and $t > 0$,*

$$\widetilde{p}_t(x,y) \asymp \frac{C}{V\left(x, \frac{\sqrt{t}}{\overline{a}\left(|x|+\rho\left(\sqrt{t}\right)\right)}\right)} \exp\left(-c\frac{\widetilde{d}(x,y)^2}{t}\right), \tag{6.24}$$

where $\widetilde{d}$ is the geodesic distance in the metric $\widetilde{g}$.

Example 6.18 Consider the manifold $(\mathbb{R}^n, g, \mu)$ where $n \geq 2$, g is the Euclidean metric and μ is the Lebesgue measure. Assume that $a(x) \simeq \overline{a}(|x|)$ and

$$\overline{a}(r) \simeq r^\alpha \text{ for } r > 1.$$

The condition (6.21) is satisfies provided $\alpha > -1$, which will be assumed in the sequel. It follows from (6.22) that $\rho(r) \simeq r^{\frac{1}{1+\alpha}}$, whence (6.24) yields, for $t > 1$,

$$\widetilde{p}_t(x,y) \asymp Ct^{-\frac{n}{2+2\alpha}} \left(1 + \frac{|x|}{t^{\frac{1}{2+2\alpha}}}\right)^{\alpha n} \exp\left(-c\frac{\widetilde{d}(x,y)^2}{t}\right). \tag{6.25}$$

If y is the origin and $|x| > 1$ then $\widetilde{d}(x,y)$ is easily estimated as follows

$$\widetilde{d}(o,x) \simeq \int_0^{|x|} \overline{a}(s)\, ds \simeq |x|^{1+\alpha}, \tag{6.26}$$

and (6.25) yields, for $t, |x| > 1$,

$$\widetilde{p}_t(o,x) \asymp Ct^{-\frac{n}{2+2\alpha}} \exp\left(-c\frac{|x|^{2+2\alpha}}{t}\right).$$

Example 6.19 Let $(\mathbb{R}^n, g, \mu)$ be as above, and consider a simultaneous change of metric and measure given by

$$\widetilde{g} = a^2 g \text{ and } d\widetilde{\mu} = b^2 d\mu,$$

where a and b are smooth positive functions on $\mathbb{R}^n$. Note that Laplace operator $\widetilde{\Delta}_{\widetilde{\mu}}$ of the weighted manifold $(\mathbb{R}^n, \widetilde{g}, \widetilde{\mu})$ is given by

$$\widetilde{\Delta}_{\widetilde{\mu}} u = \frac{1}{b^2} \operatorname{div}\left(\frac{b^2}{a^2}\nabla u\right)$$

(cf. Example 2.1). Let us also mention that if $b \equiv a^{n/2}$ then $\widetilde{\mu}$ is the Riemannian measure of the metric $\widetilde{g}$. However, in general, we do not assume this.

Assume that $a(x) \simeq \overline{a}(|x|)$ and $b(x) \simeq \overline{b}(|x|)$ where

$$\overline{a}(r) \simeq r^\alpha \text{ and } \overline{b}(r) \simeq r^\beta \text{ for } r > 1. \tag{6.27}$$

As in Example 6.13, if $\beta > -\frac{n}{2}$ then the manifold $(\mathbb{R}^n, g, \widetilde{\mu})$ satisfies the Harnack inequality and, hence, $(\mathbb{R}^n, g, \widetilde{\mu})$ can be used as an input manifold in Theorem 6.16. Assuming $\alpha > -1$, we obtain that the function a satisfies the hypotheses of this theorem, whence it follows that $(\mathbb{R}^n, \widetilde{g}, \widetilde{\mu})$ satisfies the Harnack inequality and, hence, its heat kernel $\widetilde{\widetilde{p}}_t(x,y)$ admits the Li-Yau estimate.

Applying successively (6.14) and (6.23), we obtain, for $r > 1$,

$$\widetilde{\mu}(\widetilde{B}(x,r)) \simeq r^n \left(|x| + r^{\frac{1}{1+\alpha}}\right)^{2\beta-\alpha n},$$

and the Li-Yau estimate for $\widetilde{\widetilde{p}}_t(x,y)$ yields, for $t > 1$,

$$\widetilde{\widetilde{p}}_t(x,y) \asymp Ct^{-\frac{n+2\beta}{2+2\alpha}} \left(1 + \frac{|x|}{t^{\frac{1}{2+2\alpha}}}\right)^{\alpha n-2\beta} \exp\left(-c\frac{\widetilde{d}(x,y)^2}{t}\right)$$

(cf. (6.16) and (6.25)). In particular, (6.26) implies, for $t, |x| > 1$,

$$\widetilde{\widetilde{p}}_t(o,x) \asymp Ct^{-\frac{n+2\beta}{2+2\alpha}} \exp\left(-c\frac{|x|^{2+2\alpha}}{t}\right).$$

Example 6.20 Recall that in the above example we assumed $\alpha > -1$ and $\beta > -n/2$, which allowed us to apply Theorems 6.9 and 6.16. Here we consider the borderline case

$$\alpha = -1 \quad \text{and} \quad \beta = -\frac{n}{2}, \tag{6.28}$$

and prove that the weighted manifold $(\mathbb{R}^n, \widetilde{g}, \widetilde{\mu})$ still satisfies the Harnack inequality. Since the Harnack inequality is stable under quasi-isometries, we can assume without loss of generality that the functions a and b are radial. The conditions (6.27) and (6.28) imply $b(r) \simeq a^{n/2}(r)$, so that we can assume $b(r) \equiv a^{n/2}(r)$. The latter means that measure $\widetilde{\mu}$ is actually the Riemannian measure of metric $\widetilde{g}$, that is, $(\mathbb{R}^n, \widetilde{g}, \widetilde{\mu})$ is a *Riemannian* model (cf. Section 2.4).

Let $\widetilde{d}(x,y)$ be the geodesic distance of metric $\widetilde{g}$. Setting $\widetilde{r}(x) := \widetilde{d}(o,x)$ and $r(x) := |x|$, we obtain

$$\widetilde{r} = \int_0^r a(s)\, ds.$$

The metric $\widetilde{g}$ is represented in the polar coordinates as follows:

$$\widetilde{g} = a^2(r)\left(dr^2 + r^2 d\theta^2\right) = d\widetilde{r}^2 + r^2 a^2(r)\, d\theta^2,$$

and the boundary area function $S(\widetilde{r})$ of $(\mathbb{R}^n, \widetilde{g}, \widetilde{\mu})$ is given by

$$S(\widetilde{r}) = \omega_n r^{n-1} a(r)^{n-1}.$$

Since $a(r) \simeq r^{-1}$ for large r, we obtain that $S(\widetilde{r}) \simeq 1$ for large $\widetilde{r}$, that is, the manifold $(\mathbb{R}^n, \widetilde{g}, \widetilde{\mu})$ is cylinder-like. The function $S(\widetilde{r})$ obviously satisfies the conditions of Example 6.7, which yields the Harnack inequality.

It is easy to see that, for all $x \in \mathbb{R}^n$ and $s > 0$,

$$\widetilde{\mu}(\widetilde{B}(x,s)) \simeq \min(s, s^n),$$

which gives the following estimate of the heat kernel of $(\mathbb{R}^n, \widetilde{g}, \widetilde{\mu})$:

$$\widetilde{\widetilde{p}}_t(x,y) \asymp \frac{C}{\min\left(t^{1/2}, t^{n/2}\right)} \exp\left(-c\frac{\widetilde{d}(x,y)^2}{t}\right).$$

Since $\widetilde{d}(o,x) = \int_0^{|x|} a(r)\, dr \simeq \log|x|$ for large $|x|$, we obtain that

$$\widetilde{\widetilde{p}}_t(o,x) \asymp \frac{C}{t^{1/2}} \exp\left(-c\frac{\log^2|x|}{t}\right),$$

for large t and $|x|$.

6.5 Manifolds with ends

Let (M, μ) be a complete non-compact weighted manifold. Let $K \subset M$ be a non-empty compact set with smooth boundary such that $M \setminus K$ has k connected components $E_1, \ldots, E_k$ and each *end* E_i is non-compact. Assume further that each E_i is isometric (as a weighted manifold) to the exterior of a compact set on another complete non-compact weighted manifold (M_i, μ_i). In this case, we say that M is a *connected sum* of the manifolds $M_1, ..., M_k$ and write $M = M_1\#...\#M_k$ (see Fig. 6).

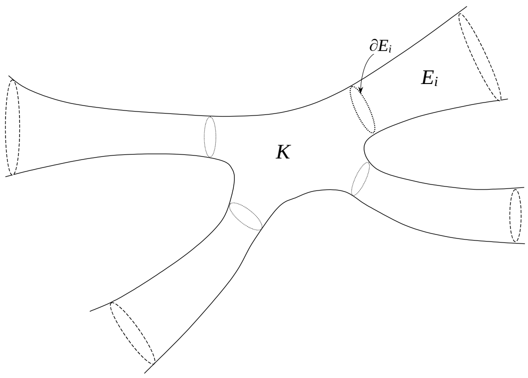


Figure 6: Manifold with ends

We are interested in estimating the heat kernel p_t on the connected sum (M, μ) given enough information about the heat kernels on (M_i, μ_i). Fix a reference point $o_i \in M_i$ and let $V_i(r)$ be the μ_i-measure of the geodesic ball on M_i centered at o_i. Also, for any point $x \in M$, set $|x| = \max_{z\in K} d(x, z)$ where d is the geodesic distance on M.

Theorem 6.21 ([107], [111]) *Under the above hypotheses, assume that each of the manifolds (M_i, μ_i) satisfies the Li-Yau estimate. Assume further that, for any $i = 1, ..., k$, there exists $\alpha_i > 2$ such that $V(r) \simeq r^{\alpha_i}$ for $r > 1$.*

(a) (Non-parabolic case) *Set*

$$\alpha = \min_{1\le i\le k} \alpha_i$$

and suppose that $\alpha > 2$. Then, for all $x \in E_i$ and $y \in E_j$ with $i \neq j$, and for all $t > 1$,

$$p_t(x, y) \asymp C\left(\frac{1}{t^{\alpha/2}|x|^{\alpha_i-2}|y|^{\alpha_j-2}} + \frac{1}{t^{\alpha_j/2}|x|^{\alpha_i-2}} + \frac{1}{t^{\alpha_i/2}|y|^{\alpha_j-2}}\right)\exp\left(-c\frac{d^2(x, y)}{t}\right).$$

(b) (Mixed case) *Let all manifolds M_i have relatively connected annuli* (see Section 6.2). *Assume that $\alpha_i \neq 2$ for all i, and there are ends with $\alpha_i < 2$ and with $\alpha_i > 2$. Set*

$$\alpha_i^* = \begin{cases} \alpha_i, & \text{if } \alpha_i > 2, \\ 4 - \alpha_i, & \text{if } \alpha_i < 2, \end{cases}$$

and

$$\alpha^* = \min_{1\le i\le k} \alpha_i^*.$$

Then, for all $x \in E_i$ and $y \in E_j$ with $i \neq j$, and for all $t > 1$,

$$\begin{aligned} p_t(x, y) \asymp\ & C\left(\frac{1}{t^{\alpha^*/2}|x|^{\alpha_i^*-2}|y|^{\alpha_j-2}} + \frac{1}{t^{\alpha_j^*/2}|x|^{\alpha_i^*-2}} + \frac{1}{t^{\alpha_i^*/2}|y|^{\alpha_j^*-2}}\right) \\ & \times |x|^{(2-\alpha_i)_+}|y|^{(2-\alpha_j)_+}\exp\left(-c\frac{d^2(x, y)}{t}\right). \end{aligned}$$

Note that the long time behavior of the heat kernel is determined by the factor $t^{-\alpha^*/2}$. Therefore, the end M_i which is dominant in the long term has the smallest exponent α_i^*, which means, that the dominant volume growth exponent α_i is the nearest to 2!

The assumption $V_i(r) \simeq r^{\alpha_i}$ is used here to simplify the statement. See [111] for general functions $V_i(r)$ as well as for the estimate of $p_t(x,y)$ for all $x, y \in M$, $t > 0$.

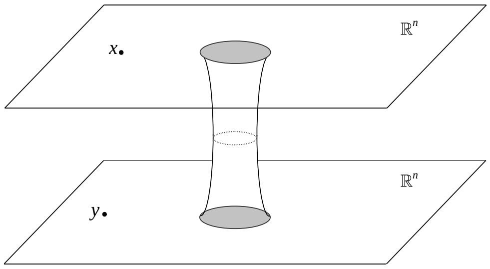


Figure 7: Manifold $\mathbb{R}^n \# \mathbb{R}^n$

Example 6.22 Let $M = \mathbb{R}^n \# \mathbb{R}^n$ with $n > 2$ (see Fig. 7). Then Theorem 6.21 yields, for x and y on different sheets,

$$p_t(x,y) \asymp \frac{C}{t^{n/2}} \left(\frac{1}{|x|^{n-2}} + \frac{1}{|y|^{n-2}} \right) \exp\left(-c \frac{d^2(x,y)}{t} \right).$$

Example 6.23 Let $M = \mathcal{R} \# \mathbb{R}^3$ where $\mathcal{R}$ is a 3-manifold such that the exterior of a compact in $\mathcal{R}$ is isometric to $\mathbb{R}_+ \times \mathbb{S}^2$ (see Fig. 8). Then we have $\alpha_1 = 1$ and $\alpha_2 = 3$ whence $\alpha_1^* = \alpha_2^* = 3$. Theorem 6.21 yields, for $x \in \mathcal{R}$ and $y \in \mathbb{R}^3$,

$$p_t(x,y) \asymp \frac{C}{t^{3/2}} \left(1 + \frac{|x|}{|y|} \right) \exp\left(-c \frac{d^2(x,y)}{t} \right).$$

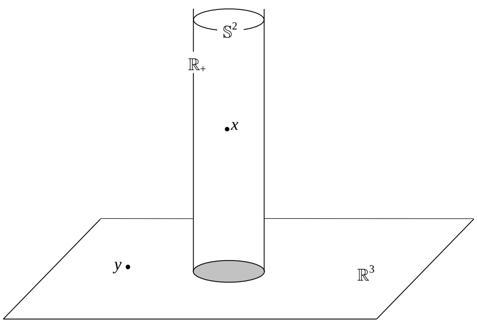


Figure 8: Manifold $\mathcal{R} \# \mathbb{R}^3$

Approach to the proof. The proof of case (a) of Theorem 6.21 is quite involved and uses a certain probabilistic technique based on the strong Markov property (cf. Theorem 8.4 below), estimates of hitting probabilities [109], estimates of the heat kernel in E_i with the Dirichlet condition on ∂E_i [108], and other tools.

The case (b) can be reduced to the case (a) by a certain Doob transform. Firstly, one constructs a positive harmonic function h on M such that, on each end E_i, $h(x) \simeq |x|^{(2-\alpha_i)_+}$. Consider a new measure $\widetilde{\mu}$ defined by $d\widetilde{\mu} = h^2 d\mu$. By (4.13), we have

$$p_t(x,y) = \widetilde{p}_t(x,y)\, h(x)\, h(y),$$

so it suffices to estimate $\widetilde{p}_t$. By Theorem 6.9, each manifold $(M_i, \widetilde{\mu})$ satisfies the Li-Yau estimate. On the other hand, the construction of h implies that the volume growth exponent of $(M_i, \widetilde{\mu})$ is $\alpha_i^* > 2$ so that the case (a) applies. See [110] and [111] for further details. ■

Some preliminary results on heat kernels on manifolds with ends were obtained in [21], [32], [57].

7 Eigenvalues of Schrödinger operators

7.1 Negative eigenvalues

The following theorem generalizes Corollary 5.3.

Theorem 7.1 ([115]) *Let (M,μ) be a weighted manifold, and let σ be a Radon measure on M such that the following inequality holds, for any non-empty relatively compact open set $\Omega \subset M$,*

$$\lambda_1(\Omega) \geq c\sigma(\Omega)^{-\alpha}, \tag{7.1}$$

where $\alpha, c > 0$. Then, for any such Ω and for all $k = 1, 2, ...,$

$$\lambda_k(\Omega) \geq c' \left(\frac{k}{\sigma(\Omega)}\right)^{\alpha}, \tag{7.2}$$

where $c' = c'(\alpha, c) > 0$.

Approach to the proof. Similarly to Corollary 5.3, the proof of Theorem 7.1 goes via heat kernel estimates although in the case $\sigma \neq \mu$ one has to use some integral estimates as opposed to the pointwise estimates of Theorems 5.1 and 5.2. See [115] for the details. ■

Theorem 7.2 ([115], [147], [149], [188]) *Let (M,μ) be a geodesically complete weighted manifold and assume that it satisfies the Faber-Krahn inequality*

$$\lambda_1(\Omega) \geq c\mu(\Omega)^{-1/p}, \tag{7.3}$$

where $p > 1$. Consider the operator

$$H = -\Delta_\mu + \Phi,$$

where $\Phi \in L^2_{loc}(M,\mu)$ and $\Phi_- \in L^p(M,\mu)$. Then $H|_{\mathcal{D}}$ is essentially self-adjoint in $L^2(M,\mu)$, the negative part of the spectrum of H is discrete, and the number $\operatorname{Neg}(H)$ *of the negative eigenvalues of H satisfies the estimate*

$$\operatorname{Neg}(H) \leq C \int_M \Phi_-^p \, d\mu, \tag{7.4}$$

where $C = C(p,c)$.

For example, (7.3) holds in $\mathbb{R}^n$ with $p = 2/n$ and hence the conclusion of Theorem 7.2 is valid for $\mathbb{R}^n$ provided $n > 2$. This case of Theorem 7.2 constitutes a celebrated theorem of Cwikel - Lieb - Rozenblum proved in [49], [151], [186] (see also [183]). If $\Phi(x) \to 0$ as $|x| \to \infty$ then it is known that the spectrum of H contains $[0, +\infty)$. Hence, in this case Theorem 7.2 says that the additional negative spectrum consists of a finite number of eigenvalues, estimated by (7.4).

Sketch of proof. Assume for simplicity that the function Φ is smooth and negative, and consider on M the new measure $\widetilde{\mu}$ defined by

$$d\widetilde{\mu} = |\Phi|\, d\mu.$$

The Faber-Krahn inequality (7.3) with $p > 1$ implies the Sobolev inequality

$$\left(\int_M u^{\frac{2p}{p-1}} d\mu\right)^{1-\frac{1}{p}} \leq C \int_M |\nabla u|^2 \, d\mu, \tag{7.5}$$

for any non-negative function $u \in \mathcal{D}$ (cf. Theorem 5.6). Using the Hölder inequality, we obtain, for any relatively compact open set $\Omega \subset M$ and for any non-negative function $u \in \mathcal{D}(\Omega)$,

$$\int_\Omega u^2 d\widetilde{\mu} = \int_\Omega u^2 \, |\Phi| \, d\mu \leq \left(\int_\Omega u^{\frac{2p}{p-1}} d\mu\right)^{1-\frac{1}{p}} \left(\int_\Omega |\Phi|^p \, d\mu\right)^{\frac{1}{p}}. \tag{7.6}$$

Define yet another measure σ on M by

$$d\sigma = |\Phi|^p \, d\mu,$$

so that (7.5) and (7.6) yield

$$\frac{\int_\Omega |\nabla u|^2 \, d\mu}{\int_\Omega u^2 d\widetilde{\mu}} \geq c\sigma(\Omega)^{-1/p}. \tag{7.7}$$

Let g be the Riemannian metric of M and consider a conformal change of metric

$$\widetilde{g} = |\Phi| \, g.$$

Denoting by $\widetilde{\nabla}$ the gradient of the metric $\widetilde{g}$ and noticing that

$$|\widetilde{\nabla} u|^2 = \frac{1}{|\Phi|} \, |\nabla u|^2,$$

we obtain

$$\int_\Omega |\widetilde{\nabla} u|^2 d\widetilde{\mu} = \int_\Omega |\nabla u|^2 d\mu.$$

Hence, (7.7) can be rewritten in the form

$$\frac{\int_\Omega |\widetilde{\nabla} u|^2 d\widetilde{\mu}}{\int_\Omega u^2 d\widetilde{\mu}} \geq c\sigma(\Omega)^{-1/p}, \tag{7.8}$$

whence by (2.9)

$$\widetilde{\lambda}_1(\Omega) \geq c\sigma(\Omega)^{-1/p},$$

where $\widetilde{\lambda}_k(\Omega)$ stands for the k-th eigenvalue of the Dirichlet Laplacian in $(\Omega, \widetilde{g}, \widetilde{\mu})$. We conclude that the manifold $(M, \widetilde{g}, \widetilde{\mu})$ satisfies the hypothesis of Theorem 7.1, which yields, for all $k = 1, 2, ...$,

$$\widetilde{\lambda}_k(\Omega) \geq c\left(\frac{k}{\sigma(\Omega)}\right)^{1/p}. \tag{7.9}$$

As in Example 2.1, the Laplace operator $\widetilde{\Delta}_{\widetilde{\mu}}$ of the manifold $(M, \widetilde{g}, \widetilde{\mu})$ is equal to $\frac{1}{|\Phi|}\Delta_\mu$ where Δ_μ is the Laplacian of (M, g, μ). Hence, the total multiplicity of the negative eigenvalues of $-\Delta_\mu - |\Phi|$ in Ω is equal to the total multiplicity of those eigenvalues of the operator $-\widetilde{\Delta}_{\widetilde{\mu}}$ in Ω, which are smaller than 1. The latter is bounded above by $C\sigma(\Omega)$ as it follows from (7.9), whence

$$\operatorname{Neg}(H, \Omega) \leq C\sigma(\Omega) = C \int_\Omega |\Phi|^p \, d\mu.$$

Exhausting M by a sequence of subsets Ω, we finish the proof. ■

7.2 Stability index of minimal surfaces

If M is a two-dimensional minimal surface in $\mathbb{R}^3$ then its *stability index* $\operatorname{ind}(M)$ is the maximum number of linearly independent local deformations of M, which decrease the area. More precisely, if Ω is a non-empty relatively compact open subset of M then $\operatorname{ind}(\Omega)$ is the number of the negative eigenvalues of the Dirichlet problem in Ω

$$\begin{cases} \Delta u - 2Ku + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7.10}$$

where Δ is the Laplace operator of the induced Riemannian metric on M and K is the Gauss curvature of M. The index of M is then define by $\operatorname{ind}(M) = \sup_\Omega \operatorname{ind}(\Omega)$. If M is an area minimizer then $\operatorname{ind}(M) = 0$. However, for most interesting classes of minimal surfaces one has $\operatorname{ind}(M) > 0$ (see for example [125]).

Theorem 7.3 ([115]) *For any two-dimensional immersed oriented minimal surface M in $\mathbb{R}^3$, we have*

$$\operatorname{ind}(M) \leq C \int_M |K| \, d\mu, \tag{7.11}$$

where μ is the Riemannian measure on M and C is an absolute constant.

For *geodesically complete* minimal surfaces, the estimate (7.11) was first proved in [204]. The proof of Theorem 7.3 uses the following localized version of Corollary 5.3 and Theorem 7.1.

Theorem 7.4 ([40], [115]) *If (M, μ) is a weighted manifold and, for any non-empty relatively compact open set $\Omega \subset M$ such that $\mu(\Omega) < v_0$,*

$$\lambda_1(\Omega) \geq c\mu(\Omega)^{-\alpha},$$

where $\alpha, c, \nu_0 > 0$, then, for any non-empty relatively compact open set $\Omega \subset M$,

$$\lambda_k(\Omega) \geq c' \left(\frac{k}{\mu(\Omega)} \right)^\alpha, \quad \text{for all } k > 24\frac{\mu(\Omega)}{v_0}, \tag{7.12}$$

where $c' = c'(\alpha, c) > 0$.

Denote by $\mathcal{N}_\lambda(\Omega)$ the counting function of the sequence $\{\lambda_k(\Omega)\}$, that is

$$\mathcal{N}_\lambda(\Omega) = \max\{k \geq 1 : \lambda_k(\Omega) < \lambda\}.$$

It easily follows from (7.12) that

$$\mathcal{N}_\lambda(\Omega) \leq C\left(\lambda^{1/\alpha} + v_0^{-1}\right)\mu(\Omega). \tag{7.13}$$

Sketch of proof of Theorem 7.3. Let g be the Riemannian metric on M. Assume for simplicity that $K(x) \neq 0$ on M (and hence $K < 0$), and define a new Riemannian metric $\widetilde{g}$ by

$$\widetilde{g} = |K| \, g.$$

Since $\dim M = 2$, we obtain that the Riemannian measure $\widetilde{\mu}$ of the metric $\widetilde{g}$ satisfies the identity

$$d\widetilde{\mu} = |K| \, d\mu.$$

As in the proof of Theorem 7.2, the Laplace operator $\widetilde{\Delta}$ of the Riemannian manifold $(M, \widetilde{g})$ is given by

$$\widetilde{\Delta} = \frac{1}{|K|} \Delta,$$

where Δ is the Laplace operator of (M, g). It follows that

$$-\Delta + 2K = |K| \left(-\widetilde{\Delta} - 2 \right),$$

which implies that the number of negative eigenvalues of the operator $-\Delta + 2K$ in Ω is equal to the number of the eigenvalues of $-\widetilde{\Delta}$ in Ω, which are smaller than 2. If we can apply the estimate (7.13) to the weighted manifold $(M, \widetilde{\mu})$ then it will give (with a fixed v_0 and $\lambda = 2$)

$$\operatorname{ind}(\Omega) \leq C \widetilde{\mu}(\Omega) = C \int_\Omega |K| \, d\mu,$$

which will finish the proof. By Theorem 7.4, it suffices to show that the Riemannian manifold $(M, \widetilde{g})$ satisfies a restricted Faber-Krahn inequality. The latter was proved in [115] using the fact that the metric $\widetilde{g}$ is the pull-back of the standard metric on $\mathbb{S}^2$ for the Gauss map from M to $\mathbb{S}^2$. $\blacksquare$

The following lower estimate of the index partly complements (7.11).

Theorem 7.5 ([105]) *For any two-dimensional connected complete oriented surface M minimally embedded in $\mathbb{R}^3$, we have*

$$\operatorname{ind}(M) \geq c \left(\int_M |K| \, d\mu \right)^{1/2}, \tag{7.14}$$

where $c > 0$ is an absolute constant.

8 The Brownian motion

8.1 Construction of the Brownian motion

If a manifold M is compact then let $\overline{M} = M$, and if not then let $\overline{M}$ be an one point compactification of M so that $\overline{M} \setminus M$ consists of a single point ∞ whose open neighborhoods are the complements in $\overline{M}$ of compact sets in M. By a *path* on M we will mean any continuous mapping $\omega : [0, +\infty) \to \overline{M}$ such that if $\omega(t_0) = \infty$ then also $\omega(t) = \infty$ for any $t > t_0$. The point ∞ is also called the *cemetery*, and the *lifetime* $\zeta(\omega)$ of ω is hence defined by

$$\zeta(\omega) := \inf \{ t \geq 0 : \omega(t) = \infty \} = \sup \{ t \geq 0 : \omega(t) \in M \}.$$

Denote by $\mathbf{\Omega}$ the set of all paths on M, and by $\mathbf{\Omega}_x$ the set of all paths starting from the point $x \in \overline{M}$, that is

$$\mathbf{\Omega}_x = \{ \omega \in \mathbf{\Omega} : \omega(0) = x \}.$$

By a *stochastic process with continuous paths* we will mean a family $\{\mathbb{P}_x\}_{x\in M}$ of probability measures on spaces $\mathbf{\Omega}_x$ such that the σ-algebra of $\mathbb{P}_x$-measurable events is generated by the following events

$$\{\omega \in \mathbf{\Omega}_x : \omega(t_1) \in A_1, \omega(t_2) \in A_2, ..., \omega(t_k) \in A_k\}, \tag{8.1}$$

for any positive integer k, for all $0 < t_1 < t_2 < ... < t_k$, and for all Borel sets $A_1, ..., A_k \subset \overline{M}$. For any $t \geq 0$ denote by X_t the random variable (on each of the spaces $\mathbf{\Omega}_x$) defined by $X_t(\omega) = \omega(t)$ so that $\{X_t\}_{t\geq 0}$ is a random path on $\overline{M}$. Hence, the following *transition probabilities* are well-defined:

$$\mathbb{P}_x(X_{t_1} \in A_1, ..., X_{t_k} \in A_k),$$

where $t_1, ..., t_k$ and $A_1, ..., A_k$ are as above.

Let now (M, μ) be a weighted manifold. The heat kernel p_t of (M, μ) can be used to define the transition probabilities as follows. First, consider the *transition function* $\mathcal{P}_t(x, \cdot)$, which, for any $t > 0$ and $x \in \overline{M}$, is a Borel probability measure on $\overline{M}$ defined by

$$\begin{aligned} &\mathcal{P}_t(x, A) = \int_A p_t(x, y)\, d\mu(y) \quad \text{if } x \in M, A \subset M, \\ &\mathcal{P}_t(\infty, A) = 0, \qquad\qquad\quad \text{if } A \subset M, \\ &\mathcal{P}_t(x, \{\infty\}) = 1 - \mathcal{P}_t(x, M). \end{aligned} \tag{8.2}$$

Comparing (8.2) with (3.7), we see that the heat semigroup $P_t = e^{t\Delta_\mu}$ relates to the transition function as follows

$$P_t \mathbf{1}_A(x) = \mathcal{P}_t(x, A). \tag{8.3}$$

It also follows from (8.2) that, for $x \in M$,

$$\mathcal{P}_t(x, M) = \int_M p_t(x, y)\, d\mu(y).$$

If (M, μ) is stochastically incomplete then $\mathcal{P}_t(x, M) < 1$ for positive t and therefore $\mathcal{P}_t(x, \infty) > 0$. This clarifies the purpose of introducing the cemetery ∞, which is to ensure that $\mathcal{P}_t(x, \cdot)$ is a probability measure. If (M, μ) is stochastically complete then $\mathcal{P}_t(x, M) = 1$ for all $x \in M$ so that $\mathcal{P}_t(x, \cdot)$ is a probability measure on M and hence the cemetery can be neglected. In particular, if M is compact then the cemetery ∞ is not defined at all, which agrees with the fact that, by Corollary 3.12, (M, μ) is stochastically complete.

Now, define the (minimal) *Brownian motion* on a weighted manifold (M, μ) as the stochastic process $\left(\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in M}\right)$ with continuous paths such that

$$\mathbb{P}_x(X_{t_1} \in A_1, ..., X_{t_k} \in A_k) = \int_{A_k} ... \int_{A_1} \mathcal{P}_{t_1}(x, dx_1)\, \mathcal{P}_{t_2 - t_1}(x_1, dx_2) ... \mathcal{P}_{t_k - t_{k-1}}(x_{k-1}, dx_k), \tag{8.4}$$

where $t_1, ..., t_k$, $A_1, ..., A_k$ are as above. In particular, we have the identity

$$\mathbb{P}_x(X_t \in A) = \mathcal{P}_t(x, A) = P_t \mathbf{1}_A(x). \tag{8.5}$$

The term "minimal" relates to the fact that $p_t(x, y)$ is the *minimal* regular fundamental solution to the heat equation. In order to define the transition probabilities, one can use another regular fundamental solution possessing the semigroup property, hence obtaining a different Brownian motion. Here we consider only the minimal Brownian motion.

The definition (8.4) satisfies the Kolmogorov consistency condition: for any $1 \leq i \leq k$,

$$\mathbb{P}_x\left(X_{t_1} \in A_1, ..., X_{t_i} \in \overline{M}, ..., X_{t_k} \in A_{t_k}\right) = \mathbb{P}_x\left(X_{t_1} \in A_1, ..., \overset{i}{\checkmark}, ..., X_{t_k} \in A_k\right),$$

where in the right hand side the i-th condition is omitted. Indeed, consider, for example, the case $k = 2$, $x \in M$, $A_j \subset M$ for $j \neq i$. If $i = 1$ then using the semigroup identity (3.9) and $\mathcal{P}_t(\infty, A_2) = 0$, we obtain

$$\begin{aligned}
\mathbb{P}_x\left(X_{t_1} \in \overline{M}, X_{t_2} \in A_2\right) &= \int\limits_{A_2}\int\limits_{M} \mathcal{P}_{t_1}(x, dx_1)\, \mathcal{P}_{t_2 - t_1}(x_1, dx_2) \\
&\quad + \int\limits_{A_2} \mathcal{P}_{t_1}(x, \infty)\, \mathcal{P}_{t_2 - t_1}(\infty, dx_2) \\
&= \int\limits_{A_2} \mathcal{P}_{t_2}(x, dx_2) = \mathbb{P}_x\left(X_{t_2} \in A_2\right).
\end{aligned}$$

If $i = 2$ then using $\mathcal{P}_t\left(x, \overline{M}\right) \equiv 1$ we obtain

$$\begin{aligned}
\mathbb{P}_x\left(X_{t_1} \in A_1, X_{t_2} \in \overline{M}\right) &= \int\limits_{\overline{M}}\int\limits_{A_1} \mathcal{P}_{t_1}(x, dx_1)\, \mathcal{P}_{t_2 - t_1}(x_1, dx_2) \\
&= \int\limits_{A_1} \mathcal{P}_{t_1}(x, dx_1)\, \mathcal{P}_{t_2 - t_1}\left(x_1, \overline{M}\right) = \mathbb{P}_x\left(X_{t_1} \in A_1\right).
\end{aligned}$$

The consistency condition together with additional argument related to the continuity of paths, allows to prove the following result.

Theorem 8.1 *The minimal Brownian motion exists on any weighted manifold* (M, μ).

A complete proof can be extracted from the theory of Markov processes[2] – see [25], [69], [70], [74]. The properties of the Brownian motion considered below in this section can also be found in these references in a more general context.

Theorem 8.2 *For any bounded or non-negative Borel function f on M and for all $x \in M$, $t \geq 0$, we have*

$$P_t f(x) = \mathbb{E}_x f(X_t). \tag{8.6}$$

Furthermore, for all $0 \leq t_1 \leq t_2 \leq ... \leq t_k$ and bounded or non-negative Borel functions $f_1, f_2, ..., f_k$,

$$P_{t_1}\left(f_1 P_{t_2 - t_1}\left(f_2 .... P_{t_k - t_{k-1}} f_k\right)\right)(x) = \mathbb{E}_x\left(f_1(X_{t_1})\, f_2(X_{t_2}) ... f_k(X_{t_k})\right). \tag{8.7}$$

Sketch of proof. Approximating f by indicator functions, we obtain from (8.3)

$$P_t f(x) = \int_M f(y)\, \mathcal{P}_t(x, dy). \tag{8.8}$$

[2]For alternative approaches for construction of diffusion processes on Riemannian manifolds and more general underlying spaces, see [8], [71], [131], [156], [160], [194].

Since X_t is $\mathbb{P}_x$-measurable and f is Borel, $f(X_t)$ is also $\mathbb{P}_x$-measurable. By (8.5), we obtain

$$\mathbb{E}_x f(X_t) = \int_{\mathbf{\Omega}_x} f(\omega(t))\, d\mathbb{P}_x(\omega) = \int_M f(y)\, \mathcal{P}_t(x, dy),$$

whence (8.6) follows.

The identity (8.7) in the case $k = 1$ coincides with (8.6). The proof for $k > 1$ is done by induction. For simplicity, we consider only the case $k = 2$ and the indicator functions $f_i = \mathbf{1}_{A_i}$. Also, assume $0 < t_1 < t_2$ since the case $t_1 = 0$ or $t_2 = t_1$ reduces to $k = 1$. Then we have by (8.4)

$$\mathbb{E}_x(f_1(X_{t_1}) f_2(X_{t_2})) = \mathbb{P}_x(X_{t_1} \in A_1, X_{t_2} \in A_2) = \int\limits_{A_2}\int\limits_{A_1} \mathcal{P}_{t_1}(x, dx_1)\, \mathcal{P}_{t_2 - t_1}(x_1, dx_2)$$

whereas by (8.8)

$$P_{t_1}(f_1 P_{t_2-t_1} f_2)(x) = \int\limits_{A_1} P_{t_2-t_1} f_2(x_1)\, \mathcal{P}_{t_1}(x, dx_1) = \int\limits_{A_1}\int\limits_{A_2} \mathcal{P}_{t_2-t_1}(x_1, dx_2)\, \mathcal{P}_{t_1}(x, dx_1),$$

so that we obtain (8.7) by Fubini's theorem. Passage to arbitrary functions $f_1, ..., f_k$ is done by a standard approximation argument. ■

By definition, the σ-algebra of all $\mathbb{P}_x$-measurable events is generated by the elementary events (8.1). If we restrict in (8.1) all t_i by the condition $t_i \le t$ where $t > 0$ is fixed, then the elementary events generate the σ-algebra $\mathcal{F}_t$ of all events happening by the time t.

Define the *time shift operator* θ_t on $\mathbf{\Omega}$ by $\theta_t(\omega) = \omega(\cdot + t)$.

Theorem 8.3 (The Markov property) *Let $t > 0$. If η is a bounded random variable on $\mathbf{\Omega}$ and ϕ is a bounded $\mathcal{F}_t$-measurable random variable on $\mathbf{\Omega}$ then, for all $x \in M$,*

$$\mathbb{E}_x(\phi \mathbb{E}_{X_t} \eta) = \mathbb{E}_x(\phi\, \eta \circ \theta_t). \tag{8.9}$$

Sketch of proof. Let first $\phi \equiv 1$ and $\eta = f(X_s)$ where f is a bounded Borel function. Then we have $\mathbb{E}_y \eta = P_s f(y)$ and $\eta \circ \theta_t = f(X_s \circ \theta_t) = f(X_{t+s})$. Using again (8.6) and the semigroup property $P_t P_s = P_{t+s}$, we obtain

$$\mathbb{E}_x(\mathbb{E}_{X_t}\eta) = \mathbb{E}_x((P_s f)(X_t)) = P_t(P_s f)(x) = P_{t+s} f(x) = \mathbb{E}_x f(X_{t+s}) = \mathbb{E}_x(\eta \circ \theta_t).$$

If $\phi = g(X_r)$ where $r \le t$ then we argue similarly using (8.7):

$$\begin{aligned}
\mathbb{E}_x(\phi \mathbb{E}_{X_t}\eta) &= \mathbb{E}_x(g(X_r)(P_s f)(X_t)) = P_r(g P_{t-r}(P_s f))(x) \\
&= P_r(g P_{t-r+s} f)(x) = \mathbb{E}_x(g(X_r) f(X_{t+s})) = \mathbb{E}_x(\phi\, \eta \circ \theta_t).
\end{aligned}$$

The case of random variables ϕ and η of the form $f_1(X_{t_1}) ... f_k(X_{t_k})$ is treated similarly, and the passage to arbitrary ϕ and η is done by a suitable approximation argument. ■

In fact, a stronger statement is true. Let us say that a random variable $\tau : \mathbf{\Omega} \to [0, +\infty]$ is a *stopping time* if the event $\{\tau \le t\}$ is $\mathcal{F}_t$-measurable for any $t > 0$. For a stopping time τ, define $\mathcal{F}_\tau$ as the σ-algebra of all events $\mathcal{A}$ with the property that $\mathcal{A} \cap \{\tau \le t\} \in \mathcal{F}_t$, for any $t > 0$. The random time shift operator θ_τ is defined by $\theta_\tau(\omega) = \omega(\cdot + \tau(\omega))$ on the paths $\{\omega : \tau(\omega) < \infty\}$.

Theorem 8.4 (The strong Markov property) *Let τ be a stopping time. If η is a bounded random variable on $\mathbf{\Omega}$ and ϕ is a bounded $\mathcal{F}_\tau$-measurable random variable on $\mathbf{\Omega}$ such that $\phi|_{\{\tau = \infty\}} = 0$ then, for all $x \in M$,*

$$\mathbb{E}_x(\phi \mathbb{E}_{X_\tau}\eta) = \mathbb{E}_x(\phi\, \eta \circ \theta_\tau). \tag{8.10}$$

Approach to the proof. If τ takes discrete values then the strong Markov property easily amounts to (8.9). Indeed, suppose that τ takes only values $t_1, ..., t_k$. Then we have, using (8.9) with $t = t_i$,

$$\begin{aligned}
\mathbb{E}_x \left(\phi \mathbb{E}_{X_\tau} \eta \right) &= \sum\nolimits_{i=1}^k \mathbb{E}_x \left(\phi \mathbf{1}_{\{\tau = t_i\}} \mathbb{E}_{X_\tau} \eta \right) \\
&= \sum\nolimits_{i=1}^k \mathbb{E}_x \left(\phi \mathbf{1}_{\{\tau = t_i\}} \mathbb{E}_{X_{t_i}} \eta \right) \\
&= \sum\nolimits_{i=1}^k \mathbb{E}_x \left(\phi \mathbf{1}_{\{\tau = t_i\}} \eta \circ \theta_{t_i} \right) \\
&= \mathbb{E}_x \left(\phi \, \eta \circ \theta_\tau \right).
\end{aligned}$$

Observe that by hypothesis $\phi \mathbf{1}_{\{\tau = t_i\}}$ is $\mathcal{F}_{t_i}$-measurable, which justifies application of (8.9). For a general τ, one uses an approximation argument. ∎

We conclude this section with one more characterization of the stochastic completeness of (M, μ).

Lemma 8.5 *A weighted manifold (M, μ) is stochastically complete if and only if the lifetime ζ of the Brownian motion $\{X_t\}$ is equal to ∞ with $\mathbb{P}_x$-probability 1 for all $x \in M$.*

Proof. We have, for any $x \in M$,

$$\begin{aligned}
\mathbb{P}_x (\zeta = \infty) &= \mathbb{P}_x \left(X_t \in M \quad \text{for all } t > 0 \right) = \lim_{T \to \infty} \mathbb{P}_x \left(X_t \in M \quad \text{for all } 0 < t \leq T \right) \\
&= \lim_{T \to \infty} \mathbb{P}_x (X_T \in M) = \lim_{T \to \infty} \int_M p_T (x, y) \, d\mu (y).
\end{aligned}$$

In particular, it follows from this computation that the function

$$f(T) = \int_M p_T (x, y) \, d\mu (y)$$

is monotone decreasing in T. Since $f(T) \leq 1$, the limit $\lim_{T \to \infty} f(T)$ is equal to 1 if and only if $f(T) \equiv 1$. Hence, $\mathbb{P}_x (\zeta = \infty) = 1$ if and only if (M, μ) is stochastically complete, which was to be proved. ∎

8.2 The first exit time

Let (M, μ) be a weighted manifold, and $\Omega \subset M$ be a relatively compact open set such that $M \setminus \overline{\Omega}$ is non-empty. Let τ be the *first exit time* of the Brownian motion from Ω, that is

$$\tau = \inf \{ t > 0 : X_t \notin \Omega \}.$$

It is possible to prove that τ is a stopping time. It is obvious that $X_t \in \Omega$ for $t < \tau$ and $X_\tau \in \partial \Omega$ provided $\tau < \infty$.

Theorem 8.6 ([74]) *For any bounded or non-negative Borel function f on Ω,*

$$P_t^\Omega f(x) = \mathbb{E}_x \left(\mathbf{1}_{\{t < \tau\}} f(X_t) \right), \tag{8.11}$$

for all $x \in \Omega$, $t > 0$.

Corollary 8.7 *The exit time τ is finite $\mathbb{P}_x$-almost surely.*

Proof. Applying (8.11) for $f = 1$, observing that $\lambda_{\min}(\Omega) > 0$ (cf. Theorem 2.3), and using (3.14), we obtain, for any $x \in \Omega$,

$$\mathbb{P}_x(t < \tau) = P_t^\Omega 1(x) \simeq \exp(-\lambda_{\min}(\Omega)\, t) \to 0 \quad \text{as } t \to \infty$$

whence $\mathbb{P}_x(\tau = \infty) = 0$. ■

Corollary 8.8 *For any $x \in \Omega$,*

$$\mathbb{P}_x(\tau \le t) = o(t) \quad \textit{as } t \to 0.$$

Proof. From (8.6) and (8.11), we obtain

$$\mathbb{P}_x(\tau \le t) = P_t 1(x) - P_t^\Omega 1(x).$$

Lemma 3.2 implies that the function $v(t,x) := \mathbb{P}_x(\tau \le t)$ extended by 0 for $t \le 0$, satisfies in $\mathbb{R} \times \Omega$ the heat equation in the distributional sense. Hence, by Weyl's lemma, it is C^∞-smooth in $\mathbb{R} \times \Omega$ and satisfies in this domain the heat equation in the classical sense. Observing that by Theorem 3.1 $v(0,\cdot) \equiv 0$, we obtain

$$\left.\frac{\partial}{\partial t} v(t,\cdot)\right|_{t=0} = \Delta_\mu v(0,\cdot) = 0,$$

whence $v(t,x) = o(t)$ as $t \to 0$, which was to be proved. ■

Many applications of the first exist time come from the *Feynman-Kac formula.*

Theorem 8.9 ([26], [41], [74], [77]) (A Feynman-Kac formula) *Let $\Omega \subset M$ be a relatively compact open set with a smooth non-empty boundary, and let τ be the first exit time from Ω. Then, for any $0 \le \Phi \in C\left(\overline{\Omega}\right)$ and $f \in C(\partial\Omega)$, the function*

$$u(x) = \mathbb{E}_x\left(\exp\left(-\int_0^\tau \Phi(X_t)dt\right) f(X_\tau)\right) \tag{8.12}$$

solves in Ω the Dirichlet problem

$$\begin{cases} \Delta_\mu u - \Phi u = 0 & \textit{in } \Omega, \\ u = f & \textit{on } \partial\Omega. \end{cases} \tag{8.13}$$

Remark 8.10 The condition $\Phi \ge 0$ implies that the Dirichlet problem (8.13) has a *unique* solution, which hence coincides with (8.12). The statement of Theorem 8.9 remains true for a class of signed perturbations Φ (see [36], [200], [201], [226]).

Sketch of proof. It is possible to prove that if $x \to y \in \partial\Omega$ then $\tau \to 0$ and $X_\tau \to y$ which implies the boundary condition in (8.13). Let us verify that u satisfies the equation in (8.13). Consider the function $t, x \mapsto P_t^\Omega u(x)$ which solves the heat equation in Ω with the initial function u. Therefore, it suffices to show that

$$\frac{P_t^\Omega u - u}{t} \to \Phi u \quad \text{in } \Omega \text{ as } t \to 0.$$

By (8.11) and (8.12), we have

$$P_t^\Omega u(x) = \mathbb{E}_x\left(\mathbf{1}_{\{t<\tau\}} u(X_t)\right) = \mathbb{E}_x\left(\mathbf{1}_{\{t<\tau\}} \mathbb{E}_{X_t}\left(e^{-\int_0^\tau \Phi(X_s)ds} f(X_\tau)\right)\right).$$

Applying the Markov property (8.9) with

$$\eta = e^{-\int_0^\tau \Phi(X_s)ds} f(X_\tau)$$

and noticing that on the path set $\{\omega : t < \tau(\omega)\}$

$$\eta \circ \theta_t = e^{-\int_t^\tau \Phi(X_s)ds} f(X_\tau),$$

because $X_\tau \circ \theta_t = X_\tau$ (see Fig. 9), we obtain

$$P_t^\Omega u(x) = \mathbb{E}_x \left(\mathbf{1}_{\{t<\tau\}} e^{-\int_t^\tau \Phi(X_s)ds} f(X_\tau) \right). \tag{8.14}$$

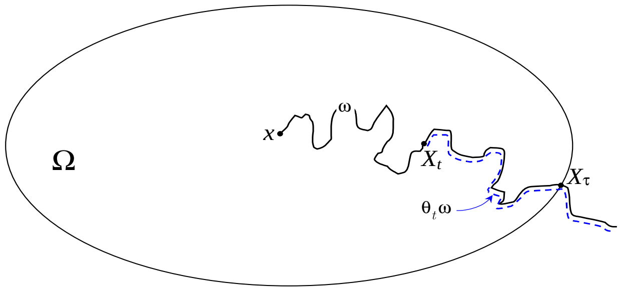


Figure 9: The paths ω and $\theta_t\omega$ have the same exit point provided $t < \tau$.

It follows from (8.12) and (8.14) that

$$P_t^\Omega u(x) - u(x) = \mathbb{E}_x \left(\mathbf{1}_{\{t<\tau\}} \left(e^{\int_0^t \Phi(X_s)ds} - 1 \right) e^{-\int_0^\tau \Phi(X_s)ds} f(X_\tau) \right) \tag{8.15}$$

$$-\mathbb{E}_x \left(\mathbf{1}_{\{t\geq\tau\}} e^{-\int_0^\tau \Phi(X_s)ds} f(X_\tau) \right). \tag{8.16}$$

By Corollary 8.8, the term (8.16) is $o(t)$ as $t \to 0$. Hence, dividing (8.15) by t and taking t to 0, we obtain

$$\begin{aligned} \lim_{t\to 0} \frac{P_t^\Omega u(x) - u(x)}{t} &= \lim_{t\to 0} \mathbb{E}_x \left(\mathbf{1}_{\{t<\tau\}} \frac{e^{\int_0^t \Phi(X_s)ds} - 1}{t} e^{-\int_0^\tau \Phi(X_s)ds} f(X_\tau) \right) \\ &= \Phi(x) u(x), \end{aligned}$$

which was to be proved. ■

8.3 The Dirichlet problem for a Schrödinger operator

The main result of this section is the following estimate of the solution to the Dirichlet problem (8.13).

Theorem 8.11 ([99], [121]) *Let (M, μ) be a weighted manifold and let $\Omega \subset M$ be a relatively compact open set with a smooth non-empty boundary. Let $h \in C(\overline{\Omega})$ be a positive function, which is harmonic in Ω, $0 \leq \Phi \in C(\overline{\Omega})$, and assume that u is the solution of the Dirichlet problem*

$$\begin{cases} \Delta_\mu u - \Phi u = 0 & \text{in } \Omega, \\ u = h & \text{on } \partial\Omega. \end{cases} \tag{8.17}$$

Then, for any $x \in \Omega$,

$$\frac{u(x)}{h(x)} \geq \exp\left(-\frac{\int_\Omega g^\Omega(x,y)h(y)\Phi\left(y\right)d\mu(y)}{h(x)}\right), \tag{8.18}$$

where g^Ω is the Green function of (Ω, μ).

Remark 8.12 Note that by the maximum principle, $u\left(x\right) \leq h\left(x\right)$, and that the Green function g^Ω is finite by Corollary 5.46. The proof below is taken from [99] (see also [123] for a similar argument). The hypothesis $\Phi \geq 0$ is used only to allow application of Theorem 8.9. In fact, the statement of Theorem 8.11 holds also for those signed Φ for which the Dirichlet problem (8.13) has a unique solution given by the Feynman-Kac formula (8.12).

Proof. Let τ be the first exit time from Ω. Then function u is represented by (8.12) with $f = h$ and, since $\Delta_\mu h = 0$ in Ω, we obtain from (8.12) that

$$h(x) = \mathbb{E}_x h(X_\tau) \quad \text{for all } x \in \Omega. \tag{8.19}$$

Consider random variables

$$\eta = h(X_\tau) \quad \text{and} \quad \xi = \int_0^\tau \Phi(X_t)dt,$$

so that by (8.19) and (8.12)

$$h(x) = \mathbb{E}_x \eta \quad \text{and} \quad u(x) = \mathbb{E}_x\left(e^{-\xi}\eta\right).$$

Using Jensen's inequality with the probability measure

$$\mathbb{Q} = \frac{\eta}{\mathbb{E}_x \eta}\mathbb{P}_x$$

in the path space, we obtain that

$$\begin{aligned}\frac{u\left(x\right)}{h\left(x\right)} &= \frac{\mathbb{E}_x\left(e^{-\xi}\eta\right)}{\mathbb{E}_x\eta} = \int e^{-\xi}d\mathbb{Q} \\ &\geq \exp\left(-\int \xi\, d\mathbb{Q}\right) = \exp\left(-\frac{\int \xi\eta\, d\mathbb{P}_x}{\mathbb{E}_x\eta}\right) = \exp\left(-\frac{\mathbb{E}_x\left(\eta\xi\right)}{h\left(x\right)}\right). \end{aligned} \tag{8.20}$$

Observe that, by (8.22) and (8.26)

$$\mathbb{E}_x\left(\eta\xi\right) = \mathbb{E}_x\left(h(X_\tau)\int_0^\tau \Phi(X_t)dt\right) = \int_0^\infty \mathbb{E}_x\left(1_{\{t<\tau\}}\Phi(X_t)h(X_\tau)\right)dt. \tag{8.21}$$

Let us show that, for any bounded $\mathcal{F}_t$-measurable random variable ϕ,

$$\mathbb{E}_x\left(\mathbf{1}_{\{t<\tau\}}\phi\, h\left(X_\tau\right)\right) = \mathbb{E}_x\left(\mathbf{1}_{\{t<\tau\}}\phi\, h\left(X_t\right)\right). \tag{8.22}$$

Indeed, on the path family $\{\omega : t < \tau\left(\omega\right)\}$ we have $X_t \in \Omega$ and hence, by (8.19),

$$h\left(X_t\right) = \mathbb{E}_{X_t} h\left(X_\tau\right)$$

which implies

$$\mathbb{E}_x\left(\mathbf{1}_{\{t<\tau\}}\phi h\left(X_t\right)\right) = \mathbb{E}_x\left(\mathbf{1}_{\{t<\tau\}}\phi\, \mathbb{E}_{X_t} h\left(X_\tau\right)\right). \tag{8.23}$$

On the other hand, by the Markov property (8.9),

$$\mathbb{E}_x\left(\mathbf{1}_{\{t<\tau\}}\phi\,\mathbb{E}_{X_t}h\left(X_\tau\right)\right)=\mathbb{E}_x\left(\mathbf{1}_{\{t<\tau\}}\phi\,h\left(X_\tau\circ\theta_t\right)\right)=\mathbb{E}_x\left(\mathbf{1}_{\{t<\tau\}}\phi\,h\left(X_\tau\right)\right), \tag{8.24}$$

because $X_\tau\circ\theta_t=X_\tau$ on $\{t<\tau\}$ (see Fig. 9). Clearly, (8.22) follows from (8.23) and (8.24).

Hence, we obtain from (8.21) and (8.22)

$$\mathbb{E}_x\left(\eta\xi\right)=\int_0^\infty\mathbb{E}_x\left(1_{\{t<\tau\}}\Phi(X_t)h(X_t)\right)dt=\mathbb{E}_x\left(\int_0^\tau\Phi(X_t)h(X_t)dt\right). \tag{8.25}$$

We are left to show that

$$\mathbb{E}_x\left(\int_0^\tau\Phi(X_t)h(X_t)dt\right)=\int_\Omega g^\Omega\left(x,y\right)\Phi\left(y\right)h\left(y\right)d\mu\left(y\right), \tag{8.26}$$

which together with (8.20), (8.25) will finish the proof. Indeed, using (8.11) we obtain, for $f=\Phi h$,

$$\mathbb{E}_x\left(\int_0^\tau f(X_t)dt\right)=\int_0^\infty\mathbb{E}_x\left(\mathbf{1}_{\{t<\tau\}}f(X_t)dt\right)=\int_0^\infty P_t^\Omega f\left(x\right)dt=G^\Omega f\left(x\right),$$

which was to be proved. ■

Corollary 8.13 ([100]) *Let a weighted manifold (M,μ) have a finite Green function $g\left(x,y\right)$ and let Φ be a non-negative continuous function on M. Then the Green function $g^\Phi\left(x,y\right)$ of the operator $\Delta_\mu-\Phi$ satisfies the estimate*

$$1\geq\frac{g^\Phi\left(x,y\right)}{g\left(x,y\right)}\geq\exp\left(-\frac{\int_M g\left(x,z\right)g\left(z,y\right)\Phi\left(z\right)d\mu\left(z\right)}{g\left(x,y\right)}\right), \tag{8.27}$$

for all distinct $x,y\in M$.

Sketch of proof. Let Ω be a non-empty open subset of M with smooth boundary such that $M\setminus\overline{\Omega}\neq\emptyset$. Fix a point $y\in\Omega$ and choose a shrinking sequence $\{U_k\}_{k=1}^\infty$ of open sets with smooth boundaries, such that $\overline{U_k}\subset\Omega$ and $\bigcap_{k=1}^\infty U_k=\{y\}$. Set $\Omega_k=\Omega\setminus\overline{U_k}$ and consider the function $h_k:=g^{\Omega_k}\left(\cdot,y\right)$ which is clearly harmonic in Ω_k. Let u_k solve the Dirichlet problem

$$\begin{cases}\Delta_\mu u_k-\Phi u_k=0 & \text{in }\Omega_k,\\ u_k=h_k & \text{on }\partial\Omega_k.\end{cases}$$

By Theorem 8.11 we obtain, for any $x\in\Omega_k$,

$$\frac{u_k(x)}{h_k(x)}\geq\exp\left(-\frac{\int_{\Omega_k}g^{\Omega_k}(x,\cdot)h_k\Phi d\mu}{h_k\left(x\right)}\right).$$

Passing to the limit as $k\to\infty$ and noticing that $h_k\left(x\right)\to g^\Omega\left(\cdot,y\right)$ and $u_k\to g^{\Phi,\Omega}\left(\cdot,y\right)$, where $g^{\Phi,\Omega}$ is the Green function of the operator $\Delta_\mu-\Phi$ in (Ω,μ), we obtain

$$\frac{g^{\Phi,\Omega}\left(x,y\right)}{g^\Omega\left(x,y\right)}\geq\exp\left(-\frac{\int_\Omega g^\Omega\left(x,z\right)g^\Omega\left(z,y\right)\Phi\left(z\right)d\mu\left(z\right)}{g^\Omega\left(x,y\right)}\right).$$

Finally, passing to the limit as $\Omega\to M$ we obtain (8.27). ■

9 Path properties of stochastic processes

As in the previous section, let $\left(\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in M}\right)$ be the minimal Brownian motion on a weighted manifold (M,μ).

9.1 Recurrence and transience

We say that the Brownian motion $\{X_t\}$ is *recurrent* if, for any non-empty open set $\Omega \subset M$ and for any point $x \in M$,

$$\mathbb{P}_x \left(\text{there is a sequence } t_k \to \infty \text{ such that } X_{t_k} \in \Omega\right) = 1.$$

Otherwise, the process $\{X_t\}$ is called *transient*.

Theorem 9.1 ([1], [96], [128], [140], [189]) *The following conditions are equivalent:*

- *The Brownian motion on (M,μ) is recurrent.*
- *The manifold (M,μ) is parabolic (that is, any bounded subharmonic function on M is constant).*
- *The Green function of (M,μ) is identical $+\infty$.*
- *For some/all $x \in M$,*

$$\int_1^\infty p_t(x,x)\,dt = \infty. \tag{9.1}$$

In this section, we are concerned with conditions for the parabolicity of (M,μ), which, hence, is equivalent to the recurrence of the Brownian motion.

Example 9.2 $\mathbb{R}^n$ is parabolic if and only $n \leq 2$ because $p_t(x,x) \simeq t^{-n/2}$.

Example 9.3 Let $\lambda_{\min}(M) > 0$. Then by Corollary 3.7 the heat kernel $p_t(x,x)$ decays exponentially in t as $t \to \infty$ so that the integral in (9.1) converges and hence (M,μ) is non-parabolic (cf. Lemma 5.44). For example, the hyperbolic space $\mathbb{H}^n$ is non-parabolic for any $n \geq 2$.

Example 9.4 If M is geodesically complete and $\mu(M) < \infty$ then (M,μ) is parabolic by (3.30). In particular, a compact manifold is parabolic.

Example 9.5 Let (M,μ) be a weighted model (see Section 2.4). Then it is parabolic if and only if

$$\int^\infty \frac{dr}{S(r)} = \infty \tag{9.2}$$

(see [82], [96]). For example, if $V(r) = cr^{2+\varepsilon}$ for large r (where $\varepsilon > 0$) and hence $S(r) = cr^{1+\varepsilon}$ then M is non-parabolic,

Example 9.6 If manifold M is geodesically complete and, if for some $x \in M$ and all large r,

$$V(x,r) \leq Cr^2 \tag{9.3}$$

then Theorem 5.24 yields, for large t,

$$p_t(x,x) \geq \frac{C}{t\log t}.$$

In particular, we see that (9.1) holds and, hence, the manifold (M,μ) is parabolic.

The next result allows to relax the condition (9.3).

Theorem 9.7 ([81], [82], [137], [210]) *Let (M,μ) be a geodesically complete weighted manifold. If for some point $x \in M$,*

$$\int^{\infty} \frac{dt}{V\left(x,\sqrt{t}\right)} = \infty \tag{9.4}$$

then (M,μ) is parabolic.

Corollary 9.8 ([38]) *Let (M,μ) be a geodesically complete weighted manifold. If, for some point $x \in M$ and for a sequence $r_k \to \infty$,*

$$V(x,r_k) \le C\,r_k^2 \tag{9.5}$$

then (M,μ) is parabolic.

The converse to Theorem 9.7 is not true: it is easy to construct an example of a parabolic manifold with arbitrarily fast growing function $r \mapsto V(x,r)$, in particular, with convergent integral (9.4). Indeed, take a model manifold from Example 9.5 with a prescribed volume function $V(r)$ and then slightly change $V(r)$ along a rare sequence $r_k \to \infty$ so that the function $V(r)$ still remains big but its derivative around r_k is nearly 0. Since $S(r) = V'(r)$, one can easily satisfy (9.2) so that the manifold is parabolic.

However, for some important classes of manifolds, the condition (9.4) is still equivalent to the parabolicity.

Corollary 9.9 *Let (M,μ) be a geodesically complete weighted manifold. The parabolicity of (M,μ) is equivalent to (9.4) provided one of the following conditions hold:*

(a) (M,μ) satisfies the relative Faber-Krahn inequality.

(b) M has non-negative Ricci curvature and μ is the Riemannian volume.

(c) M is a regular cover of a compact Riemannian manifold and μ is the Riemannian volume.

Proof. We only need to prove that if

$$\int^{\infty} \frac{dt}{V\left(x,\sqrt{t}\right)} < \infty \tag{9.6}$$

then (M,μ) is non-parabolic.

(a) Condition (9.6) and the upper bound (5.55) of the heat kernel imply that

$$\int^{\infty} p_t(x,x)\,dt < \infty,$$

so that (M,μ) is non-parabolic by Theorem 9.1.

(b) This follows from part (a) because the relative Faber-Krahn inequality holds for manifolds with non-negative Ricci curvature (see Theorem 6.4). Alternatively, this follows also from the estimate (6.3) of the Green function.

(c) Using the heat kernel upper bound of Corollary 5.11, it suffices to prove that

$$\int^{\infty} \frac{dt}{\gamma(t)} < \infty \tag{9.7}$$

where $\gamma(t)$ is defined by (5.25) via the volume function $V(r) = V(x_0, r)$. Changing in the integral in (9.7) $r = V^{-1}(\gamma(t))$ and noticing that by (5.25)

$$\frac{dt}{dr} = \frac{r^2 V'(r)}{V(r)},$$

we obtain

$$\int^{\infty} \frac{dt}{\gamma(t)} = \int^{\infty} r^2 \frac{V'(r)}{V^2(r)} dr \leq \text{const} + 2\int^{\infty} \frac{rdr}{V(r)},$$

where in last part we have used integration by parts. Finally, changing $r = \sqrt{t}$, we obtain the integral (9.4). ∎

The parts (b), (c) of Corollary 9.9 were first proved in [207], [208], [209].

We conclude this section with a sufficient condition for non-parabolicity.

Corollary 9.10 ([96]) *Assume that (M, μ) satisfies the Faber-Krahn inequality with a positive decreasing function Λ such that*

$$\int^{\infty} \frac{dv}{v^2 \Lambda(v)} < \infty. \tag{9.8}$$

Then (M, μ) is non-parabolic.

Sketch of proof. If in addition Λ satisfies the condition (5.4), that is,

$$\int_0 \frac{dv}{v\Lambda(v)} < \infty \tag{9.9}$$

then by Theorem 5.2 we have the heat kernel upper bound (5.5). Hence, it suffices to verify the condition (9.7) where $\gamma(t)$ is given by (5.6). Changing $v = \gamma(t)$ and noticing that by (5.6)

$$\frac{dt}{dv} = \frac{1}{v\Lambda(v)}$$

we obtain

$$\int^{\infty} \frac{dt}{\gamma(t)} = \int^{\infty} \frac{dv}{v^2 \Lambda(v)}$$

whence the claim follows.

In general, in absence of condition (9.9), one needs to repeat the argument of the proof of Theorem 5.2 with some modification (see [96]). ∎

9.2 Escape rate

In this section we assume that (M, μ) is a geodesically complete weighted manifold. An increasing function $R(t)$ is called an *upper radius* for the Brownian motion $\{X_t\}$ on M if $X_t \in B(x, R(t))$ for all t large enough almost surely, that is,

$$\mathbb{P}_x \{\exists T : \forall t > T \quad X_t \in B(x, R(t))\} = 1, \quad \text{for all } x \in M.$$

Similarly, a positive increasing function $r(t)$ is called a *lower radius* if X_t eventually leaves the ball $B(x, r(t))$ almost surely, that is

$$\mathbb{P}_x \{\exists T : \forall t > T \quad X_t \notin B(x, r(t))\} = 1, \quad \text{for all } x \in M$$

(see Fig. 10). For example, the function $r(t) = \text{const}$ is a lower radius if and only if the Brownian motion is transient, that is, (M, μ) is non-parabolic.

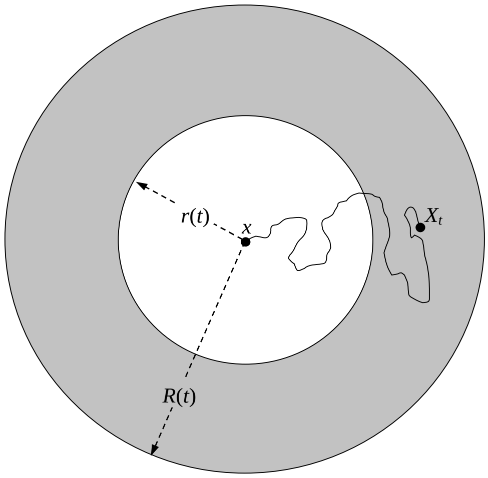


Figure 10: Upper radius $R\,(t)$ and lower radius $r\,(t)$

Theorem 9.11 ([93],[102]) *Assume that, for some $x_0 \in M$ and all r large enough,*

$$V(x_0, r) \le C\, r^{\alpha}.$$

Then, for any $\varepsilon > 0$, the function

$$R(t) = \sqrt{(\alpha+\varepsilon)\, t \log t} \tag{9.10}$$

is an upper radius for the Brownian motion $\{X_t\}$ on M.

Theorem 9.12 ([93],[102]) *Assume that the relative Faber-Krahn inequality holds on (M, μ). Then, for any $\varepsilon > 0$, the function*

$$R(t) = \sqrt{(4+\varepsilon) t \log\log t} \tag{9.11}$$

is an upper radius for Brownian motion $\{X_t\}$ on M.

By Khinchin's law of the iterated logarithm, the function (9.11) is an upper radius in $\mathbb{R}^n$ if $\varepsilon > 0$ and is not if $\varepsilon \le 0$ (see [24], [132]). Theorem 9.12 matches the law of the iterated logarithm, whereas Theorem 9.11 gives a rougher upper radius. However, assuming only the volume growth condition, one cannot in general improve the function (9.10) even to $\sqrt{ct \log t}$ with small enough c (see [16], [103]). Note that the function $\sqrt{t \log t}$ appears in Theorem 9.11 for the same reason as in the lower bound of the heat kernel in Theorem 5.24.

Theorem 9.13 ([93]) *Assume that the relative Faber-Krahn inequality holds on (M, μ), and let (M, μ) be non-parabolic. Fix a point $x \in M$ and set*

$$\varphi(r) := \left(\int_r^{\infty} \frac{s ds}{V(x, s)} \right)^{-1}. \tag{9.12}$$

If $r(t)$ is an increasing positive function on $(0, \infty)$ such that

$$\int^{\infty} \frac{\varphi(r(t))}{V(x, \sqrt{t})} dt < \infty, \tag{9.13}$$

then $r(t)$ is a lower radius for the Brownian motion $\{X_t\}$ on M.

Note that by Corollary 9.9, the non-parabolicity of (M,μ) is equivalent to the convergence of the integral in (9.12).

Example 9.14 Let $V(x,r)\simeq r^{\alpha}$ for all r large enough, for some $\alpha>2$. We obtain from (9.12) $\varphi(r)\simeq r^{\alpha-2}$, and (9.13) amounts to

$$\int^{\infty}\left(\frac{r(t)}{\sqrt{t}}\right)^{\alpha-2}\frac{dt}{t}<\infty. \tag{9.14}$$

By a theorem of Dvoretzky-Erdös [68], in $\mathbb{R}^n$, where $\alpha=n$, the condition (9.14) is sufficient and necessary for $r(t)$ to be a lower radius provided $r(t)/\sqrt{t}$ is decreasing. For example, the function

$$r(t)=\frac{C\sqrt{t}}{\log^{\frac{1+\varepsilon}{n-2}}t} \tag{9.15}$$

is a lower radius in $\mathbb{R}^n$ if $\varepsilon>0$ and is not if $\varepsilon\leq 0$.

9.3 Recurrence and transience of α-process

It is known that, for any $\alpha\in(0,2]$, the operator $(-\Delta_{\mu})^{\alpha/2}$ on a weighted manifold (M,μ) is a generator of a Hunt process $\left(\{X_t^{(\alpha)}\}_{t\geq 0},\{\mathbb{P}_x^{(\alpha)}\}_{x\in M}\right)$ on M, which is called *the α-process*. In $\mathbb{R}^n$ the α-process coincides with the symmetric stable Levy process of index α.

The following result follows from a general semigroup theory of subordinated processes (see [64], [159], [220]).

Theorem 9.15 *The α-process on a weighted manifold is recurrent if and only if*

$$\int_1^{\infty}t^{\alpha/2-1}p_t(x,x)dt=\infty. \tag{9.16}$$

Corollary 9.16 ([96]) *Let (M,μ) be a geodesically complete weighted manifold. If, for some $x\in M$ and all large r,*

$$V(x,r)\leq Cr^{\alpha} \tag{9.17}$$

then the α-process is recurrent.

Proof. By Theorem 5.24, the condition (9.17) implies for large t,

$$p_t(x,x)\geq\frac{\text{const}}{t^{\alpha/2}\log^{\alpha/2}t},$$

and hence (9.16) is satisfied. ■

Corollary 9.17 *Let M be a geodesically complete manifold and assume that the relative Faber-Krahn inequality holds on (M,μ). Then the recurrence of the α-process is equivalent to*

$$\int^{\infty}\frac{dt}{V(x,t^{1/\alpha})}=\infty. \tag{9.18}$$

Proof. Indeed, by Corollary 5.31, we have $p_t(x,x)\simeq\frac{1}{V(x,\sqrt{t})}$. Substituting into (9.16) and making change in the integral, we obtain (9.18). ■

It is not known yet if (9.18) alone, without the relative Faber-Krahn inequality, implies that the α-process is recurrent (except for the case $\alpha=2$, which is covered by Theorem 9.7).

Similarly to Corollary 9.10 one obtains the following transience test for the α-process.

Corollary 9.18 *Let (M,μ) satisfy the uniform Faber-Krahn inequality with a positive decreasing function Λ. If, for some $\alpha \in (0,2]$,*

$$\int^{\infty}\left[\int_1^v \frac{d\xi}{\xi\Lambda(\xi)}\right]^{\alpha/2-1}\frac{dv}{v^2\Lambda(v)} < \infty \tag{9.19}$$

then the α-process is transient.

Example 9.19 If $\Lambda(v) \simeq v^{-2/\beta}$ for large v, then (9.19) holds if and only if $\alpha < \beta$. For example, let (M,μ) be a manifold of bounded geometry. As was mentioned in Section 5.1, (M,μ) satisfies the uniform Faber-Krahn inequality with the function (5.3), that is $\Lambda(v) \simeq v^{-2}$, for large v. Therefore, if $\alpha < 1$ then the α-process is transient on any manifold of bounded geometry.

9.4 Asymptotic separation of trajectories of α-process

Consider a stochastic process $(\{Y_t\}_{t\geq 0}, \{\mathbb{Q}_x\}_{x\in M})$ on a weighted manifold (M,μ) and assume that it has infinite lifetime. Denote by $Y_t(x)$ the process started at x, with the law $\mathbb{Q}_x$. Given a sequence $\overline{x} := (x_1,...,x_k)$ of k points of M, consider k independent processes $Y_t(x_1),...,Y_t(x_k)$ with the joint law $\mathbb{Q}_{\overline{x}} := \mathbb{Q}_{x_1}\times ... \times \mathbb{Q}_{x_k}$. The processes $Y_t(x_1),...,Y_t(x_k)$ are said to be *asymptotically separated* if, for some $\varepsilon > 0$, the following condition holds with $\mathbb{Q}_{\overline{x}}$-probability 1:

$$\text{there exists } T>0 \text{ such that, for all } t_1,...,t_k > T, \quad \max_{1\leq i,j\leq k} d(Y_{t_i}(x_i), Y_{t_j}(x_j)) \geq \varepsilon.$$

Otherwise, we say that the processes are asymptotically close (see Fig. 11).

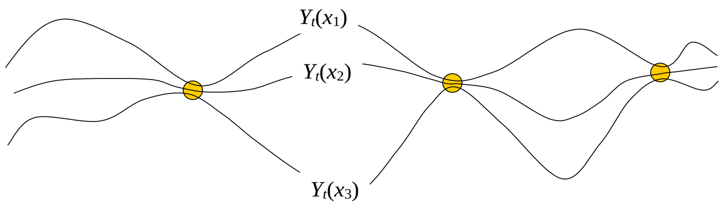


Figure 11: The processes $Y_t(x_1), Y_t(x_2), Y_t(x_3)$ are asymptotically close if they approach each other arbitrarily closely with a positive probability.

Theorem 9.20 ([104]) *Let (M,μ) be a weighted manifold with bounded geometry. Assume that, for some $\beta > 0$ and all t large enough,*

$$\sup_{x\in M} p_t(x,x) \leq \frac{C}{t^{\beta/2}}, \tag{9.20}$$

and, for some integer $k \geq 2$ and for $\alpha \in (0,2]$,

$$\frac{\alpha}{\beta} + \frac{1}{k} < 1. \tag{9.21}$$

Then k independent α-processes $X_t^{(\alpha)}(x_1),...,X_t^{(\alpha)}(x_k)$ on M are asymptotically separated.

Example 9.21 Let (M,μ) be $\mathbb{R}^n$ so that (9.20) holds with $\beta = n$. It was proved in [203] that if $\frac{\alpha}{n} + \frac{1}{k} > 1$ then k independent α-processes intersect with probability 1, which of course implies that they are asymptotically close. In the borderline case $\frac{\alpha}{n} + \frac{1}{k} = 1$, k independent α-processes do not intersect but they are still asymptotically close. Finally, if $\frac{\alpha}{n} + \frac{1}{k} < 1$ then k independent α-processes are asymptotically separated by Theorem 9.20.

Example 9.22 Since the bounded geometry condition already implies (9.20) with $\beta = 1$ (cf. Corollary 5.5), we see that, for $\alpha < 1$, k independent α-processes are asymptotically separated whenever $k > \frac{1}{1-\alpha}$.

10 Heat kernels of Schrödinger operators

10.1 Heat kernel and a ground state

Let Φ be a smooth function on a weighted manifold (M, μ). Recall that by Lemma 4.7, if the equation

$$-\Delta_\mu h + \Phi h = 0 \tag{10.1}$$

admits a positive solution h on M then the Schrödinger operator $\Delta_\mu - \Phi$ possesses the heat kernel p_t^Φ, and the latter can be expressed via the heat kernel $\widetilde{p}_t$ of the Laplace operator $\Delta_{\widetilde{\mu}}$ on $(M, \widetilde{\mu})$, where $d\widetilde{\mu} = h^2 d\mu$, by the following formula:

$$p_t^\Phi (x, y) = \widetilde{p}_t (x, y) \, h (x) \, h (y) \, . \tag{10.2}$$

Using this approach, we obtain in this section some explicit estimates for p_t^Φ, most of them being new.

We start with a simple observation.

Lemma 10.1 *For any smooth function $\Phi \geq 0$ on M, there exists a smooth positive solution of (10.1) on M.*

Proof. Fix a reference point $o \in M$ and let $\{\Omega_k\}_{k=1}^\infty$ be an exhausting sequence in M of relatively compact open sets with smooth boundaries such that o belongs to all Ω_k. Since $\Phi \geq 0$, the Dirichlet problem in Ω_k

$$\begin{cases} \Delta_\mu u - \Phi u = 0 & \text{in } \Omega_k \\ u = 1 & \text{on } \partial\Omega_k \end{cases}$$

has a unique positive solution $u = u_k$. Set $h_k = u/u(o)$ so that h_k satisfies the equation $\Delta_\mu h_k - \Phi h_k = 0$ in Ω_k and the condition $h_k(o) = 1$. Since the sequence $\{h_k\}$ converges at o, by the local properties of elliptic equations there is a subsequence of $\{h_k\}$ that converges on M to a function h, which is positive and solves (10.1). ■

Corollary 10.2 *Let (M, μ) be a geodesically complete weighted manifold, and let Φ be a smooth function on M, which is bounded below by a (negative) constant. Then the operator $-\Delta_\mu + \Phi|_{\mathcal{D}}$ is essentially self-adjoint, and the associated heat semigroup P_t^Φ has a smooth positive heat kernel p_t^Φ.*

Proof. Let $\Phi \geq -K$ where $K \geq 0$ is a constant. Then $\Psi := \Phi + K \geq 0$ and hence by Lemma 10.1 there is a positive solution h to the equation $-\Delta_\mu h + \Psi h = 0$. By Corollary 4.5, the operator $-\Delta_\mu + \Psi|_{\mathcal{D}}$ is essentially self-adjoint and, by Lemma 4.7, its heat semigroup P_t^Ψ has a smooth positive heat kernel p_t^Ψ. Since $\Delta_\mu - \Phi = (\Delta_\mu - \Psi) + K \,\mathrm{id}$, we conclude that $-\Delta_\mu + \Psi|_{\mathcal{D}}$ is also essentially self-adjoint, and $P_t^\Phi = e^{Kt} P_t^\Psi$. Hence, P_t^Φ has a smooth heat kernel $p_t^\Phi (x, y) = e^{Kt} p_t^\Psi (x, y)$, which finishes the proof. ■

10.2 Green bounded potentials

Recall that the Green function $g(x,y)$ of a weighted manifold (M,μ) is defined by (5.97), that is,

$$g(x,y) := \int_0^\infty p_t(x,y)\,dt,$$

where $p_t(x,y)$ is the heat kernel of (M,μ). We say that a continuous function $\Phi \geq 0$ on M is *Green bounded* if either $\Phi \equiv 0$ or the Green function $g(x,y)$ is finite (that is, (M,μ) is non-parabolic - cf. Theorem 9.1), and

$$\sup_{x\in M} \int_M g(x,y)\,\Phi(y)\,d\mu(y) < \infty. \tag{10.3}$$

Lemma 10.3 *Let $\Phi \geq 0$ be a continuous function on a weighted manifold (M,μ).*

(a) If (M,μ) is non-parabolic and $\operatorname{supp}\Phi$ *is compact then Φ is Green bounded.*

(b) If Φ is Green bounded then the equation $\Delta_\mu h - \Phi h = 0$ has a solution $h \simeq 1$ on M.

Remark 10.4 By [87] or [98], the condition $G\Phi(x) < \infty$ implies that there is a non-zero solution h to the equation (10.1) such that $0 \leq h \leq 1$. The Green boundedness of Φ, that is, $\sup G\Phi < \infty$, is needed to ensure that h has a positive lower bound.

Proof. (a) The function

$$v(x) := \int_M g(x,y)\,\Phi(y)\,d\mu(y)$$

is finite for all $x \in M$ because $g(x,\cdot) \in L^1_{loc}$. Let $\{\Omega_k\}_{k=1}^\infty$ be an exhausting sequence on M of relatively compact open sets with smooth boundaries, such that $S := \operatorname{supp}\Phi \subset \Omega_k$ for all k. Let v_k be the solution in Ω_k of the Dirichlet problem

$$\begin{cases} \Delta_\mu v_k = -\Phi & \text{in } \Omega_k, \\ v_k = 0 & \text{on } \partial\Omega_k, \end{cases}$$

so that

$$v_k(x) = \int_{\Omega_k} g^{\Omega_k}(x,y)\,\Phi(y)\,d\mu(y),$$

where g^{Ω_k} is the Green function of (Ω_k,μ). Since the function v_k is harmonic outside S and v_k vanishes on $\partial\Omega_k$, we have

$$\sup_{\Omega_k} v_k = \sup_S v_k.$$

Since the sequence $\{v_k(x)\}_{k=1}^\infty$ increases and converges pointwise to $v(x)$, passing to the limit we obtain

$$\sup_M v = \sup_S v < \infty,$$

that is, Φ is Green bounded.

(b) If $\Phi \equiv 0$ then set $h \equiv 1$. Assume in the sequel that $\Phi \not\equiv 0$ and g is finite. Let Ω_k be as above, and solve in each set Ω_k the Dirichlet problem

$$\begin{cases} \Delta_\mu u_k - \Phi u_k = 0 & \text{in } \Omega_k, \\ u_k = 1 & \text{in } \partial\Omega_k. \end{cases}$$

Since $\Phi \geq 0$, it follows from the maximum principle that $0 \leq u_k \leq 1$. By Theorem 8.11, for any $x \in \Omega$,

$$u_k(x) \geq \exp\left(-\int_\Omega g^{\Omega_k}(x,y)\Phi(y)\,d\mu(y)\right) \geq \exp\left(-\int_M g(x,y)\Phi(y)\,d\mu(y)\right) \geq c > 0.$$

By the comparison principle, the sequence $\{u_k\}$ decreases and hence it converges to a function h that solve on M the equation $\Delta_\mu h - \Phi h = 0$ and satisfies the estimates $c \leq h \leq 1$, which was to be proved. ■

Theorem 10.5 *Let (M,μ) be a geodesically complete noncompact weighted manifold. Assume that the heat kernel p_t of (M,μ) satisfies the Li-Yau estimate (6.1). Then, for any non-negative smooth function Φ on M, which is Green bounded, the heat kernel p_t^Φ satisfies the Li-Yau estimate, that is*

$$p_t^\Phi(x,y) \asymp \frac{C}{V(x,\sqrt{t})}\exp\left(-c\frac{d^2(x,y)}{t}\right). \tag{10.4}$$

In other words, the perturbation potential Φ satisfying (10.3) is so small that it does not affect the heat kernel in essential way[3].

Proof. By Lemma 10.3, there exists a positive solution $h \simeq 1$ to the equation $\Delta_\mu h - \Phi h = 0$ on M. Let $\widetilde{\mu}$ be the measure defined by $d\widetilde{\mu} = h^2 d\mu$. By Lemma 4.7, the heat kernel p_t^Φ of the operator $\Delta_\mu - \Phi$ and the heat kernel $\widetilde{p}_t$ of the weighted manifold $(M,\widetilde{\mu})$ are related by (10.2), whence it follows $p_t^\Phi \simeq \widetilde{p}_t$. Using the fact that (M,μ) satisfies the Li-Yau estimate and Corollary 6.8, we conclude that $\widetilde{p}_t$ satisfies the Li-Yau estimate and, hence, so does p_t^Φ. ■

Example 10.6 Under the hypotheses of Theorem 10.5, assume that, for some $\alpha > 2$,

$$V(x,r) \simeq r^\alpha, \quad \text{for all } x \in M \text{ and } r > 1.$$

Then (6.3) gives for all x,y such that $d(x,y) > 1$,

$$g(x,y) \simeq d(x,y)^{2-\alpha}.$$

Fix a reference point $o \in M$, set $\langle x \rangle = 2 + d(x,o)$ and consider a non-negative function Φ such that

$$\Phi(x) \leq C\langle x\rangle^{-\gamma},$$

for some constant γ. A straightforward computation shows (see [100]) that if $\gamma > 2$ then Φ is Green bounded. Hence, in this case the heat kernel p_t^Φ satisfies for large t the estimate

$$p_t^\Phi(x,y) \asymp \frac{C}{t^{\alpha/2}}\exp\left(-c\frac{d^2(x,y)}{t}\right).$$

The assumption $\gamma > 2$ cannot be relaxed because for the potentials $\Phi(x) \simeq \langle x\rangle^{-\gamma}$ with $\gamma \leq 2$ one has entirely different estimates of the heat kernel – see Section 10.4 and Example 10.21.

Remark 10.7 By Lemma 10.3, if (M,μ) is non-parabolic and $\Phi \geq 0$ has a compact support then Φ is Green bounded and hence Theorem 10.5 applies. The non-parabolicity is essential

[3]After the first version of this survey was circulated, Takeda [202] obtained necessary and sufficient condition for the heat kernel p_t^Φ to satisfy (10.4).

here. Indeed, as it was shown in [167], on the parabolic manifold $\mathbb{R}^2$, for any non-zero function $\Phi \geq 0$ with compact support, the heat kernel $p_t^\Phi(x,y)$ satisfies the following estimate

$$p_t^\Phi(x,y) \simeq \frac{\log\langle x\rangle \log\langle y\rangle}{t\log^2 t}, \tag{10.5}$$

for fixed x, y and for large t, where $\langle x\rangle = 2 + |x|$. Obviously, in this case p_t^Φ does not satisfy the Li-Yau estimate. See also Example 10.14 below where the estimate (10.5) will be proved.

Remark 10.8 Recall that if the heat kernel p_t satisfies the Li-Yau estimate then the Green function $g(x,y)$ satisfies the estimate (6.3). By integrating (10.4) in time, we obtain that the Green function $g^\Phi(x,y)$ of the operator $\Delta_\mu - \Phi$ also satisfies (6.3), which implies that

$$g^\Phi(x,y) \simeq g(x,y). \tag{10.6}$$

In $\mathbb{R}^n$ with $n > 2$ we have $g(x,y) = c_n|x-y|^{2-n}$. In this case, the fact that the Green boundedness of Φ implies (10.6) is well known and goes back to [2].

Assume that the Green function of a weighted manifold (M,μ) satisfies the following so-called $3G$-condition:

$$\min(g(x,z), g(z,y)) \leq Cg(x,y).$$

Then the estimate (8.27) of Corollary 8.13 implies

$$1 \geq \frac{g^\Phi(x,y)}{g(x,y)} \geq \exp\left(-\int_M (g(x,z) + g(z,y))\,\Phi(z)\,d\mu(z)\right).$$

Therefore, if Φ is Green bounded then (10.6) holds (see [26], [41], [48], [120], [169], [175] for this result in various settings, and [123], [200], [225], [226] for related results).

In general, the Li-Yau estimate does not seem to imply the $3G$-condition but, as we have seen above, the same conclusion holds.

Remark 10.9 The first result about comparison of heat kernels p_t and p_t^Φ goes back to [7], where it was shown that if $\Phi \in L^p(\mathbb{R}^n)$ with $p > n/2$ then the estimate

$$p_t^\Phi(x,y) \asymp \frac{C}{t^{n/2}} \exp\left(-c\frac{|x-y|^2}{t}\right) \tag{10.7}$$

holds for a *bounded* range of t (see also [153]). On the other hand, in $\mathbb{R}^n$ for a potential $\Phi(x) \simeq \langle x\rangle^{-\gamma}$ with $\gamma > 2$, it has been known for long time that $p_t^\Phi \simeq t^{-n/2} \simeq p_t$ as $t \to \infty$ (see, for example, [168], [176], [191], [214]). The estimate (10.7) for *all* $t > 0$ under various conditions of smallness of Φ was obtained in a number of works. It was proved in [224] that if $\Phi \geq 0$ and

$$\Phi \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \quad \text{with } p > n/2 \text{ and } 1 < q < n/2, \tag{10.8}$$

then (10.7) holds for all x, y and $t > 0$ with the sharp constant $c = \frac{1}{4}$. Note that any $\Phi \geq 0$ satisfying (10.8) is Green bounded, which easily follows from the Hölder inequality and from

$$g(o,\cdot) \in L^{\frac{p}{p-1}}(B(o,R)) \quad \text{and} \quad g(o,\cdot) \in L^{\frac{q}{q-1}}(B(o,R)^c).$$

Hence, our Theorem 10.5 applies also to the potentials (10.8) but it does not recover the sharp value of constant c in the exponential. In a harder case $\Phi \leq 0$, the estimate (10.7) with $c = \frac{1}{4}$ was proved in [152] assuming that the norm $\|\Phi\|_{L^{n/2}}$ and the Green bound constant of $|\Phi|$ are small enough.

10.3 Potentials with a polynomial ground state

In the next statement, we use terminology and notation from Section 6.2. In particular, $|x| = d(x,o)$ where o is the origin of a manifold with relatively connected annuli, $V(r) = V(o,r)$, and $\overline{h}(r) = \sup_{|x|=r} h(x)$.

Theorem 10.10 *Let (M,μ) be a geodesically complete noncompact weighted manifold with relatively connected annuli. Assume that the heat kernel on (M,μ) satisfies the Li-Yau estimate. Let h be a smooth positive function on M satisfying (6.10), (6.11). Let Ψ be a smooth function on M with compact support, and set*

$$\Phi = \frac{\Delta_\mu h}{h} + \Psi.$$

Let $\widetilde{\mu}$ be a measure on M defined by $d\widetilde{\mu} = h^2 d\mu$, and assume that one of the following conditions is satisfied:

(i) either $\Psi \equiv 0$,

(ii) or $\Psi \geq 0$ and $(M,\widetilde{\mu})$ is non-parabolic,

(iii) or $\Phi \geq 0$ and $(M,\widetilde{\mu})$ is non-parabolic.

Then the heat kernel p_t^Φ of the operator $\Delta_\mu - \Phi$ in $L^2(M,\mu)$ admits the following estimate, for all $x,y \in M$ and $t > 0$,

$$p_t^\Phi(x,y) \asymp \frac{Ch(x)\,h(y)}{V(x,\sqrt{t})\overline{h}^2\left(|x| + \sqrt{t}\right)} \exp\left(-c\frac{d^2(x,y)}{t}\right). \tag{10.9}$$

Also, the Green function g^Φ of the operator $\Delta_\mu - \Phi$ satisfies the estimate

$$g^\Phi(x,y) \simeq h(x)\,h(y)\int_{d(x,y)}^\infty \frac{sds}{V(x,s)\,\overline{h}^2(|x|+s)}. \tag{10.10}$$

Remark 10.11 Note that if $h \equiv 1$ then in the case (ii) the function $\Phi = \Psi$ is Green bounded on (M,μ), and the claim of Theorem 10.10 follows from Theorem 10.5.

Remark 10.12 Recall that the conditions (6.10), (6.11) are as follows: $h(x) \simeq \overline{h}(|x|)$ and

$$\int_1^r V(s)\,\overline{h}^2(s)\,\frac{ds}{s} \simeq V(r)\,\overline{h}^2(r), \quad \text{for all } r > 2. \tag{10.11}$$

By Corollary 6.11, $(M,\widetilde{\mu})$ is non-parabolic if and only if

$$\int_1^\infty \frac{sds}{V(s)\,\overline{h}^2(s)} < \infty. \tag{10.12}$$

Remark 10.13 By symmetrizing (10.9), we obtain

$$p_t^\Phi(x,y) \asymp \frac{Ch(x)\,h(y)}{\sqrt{V(x,\sqrt{t})V\left(y,\sqrt{t}\right)\overline{h}\left(|x|+\sqrt{t}\right)\overline{h}\left(|y|+\sqrt{t}\right)}} \exp\left(-c\frac{d^2(x,y)}{t}\right). \tag{10.13}$$

Similarly, one can symmetrize (10.10) but technically it is more convenient to estimate the right hand side of (10.10) as follows: assuming $|x| \geq |y|$ and hence $2|x| \geq d(x,y)$, and splitting the integration in (10.10), we obtain

$$g^\Phi(x,y) \simeq \frac{h(y)}{h(x)}\int_{d(x,y)}^{2|x|} \frac{sds}{V(x,s)} + h(x)\,h(y)\int_{2|x|}^\infty \frac{sds}{V(s)\,\overline{h}^2(s)}. \tag{10.14}$$

Proof of Theorem 10.10. Recall that, by Corollary 6.11, the heat kernel $\widetilde{p}_t$ on $(M, \widetilde{\mu})$ satisfies the estimate (6.12), and the Green function $\widetilde{g}$ satisfies the estimate (6.13).

(i) If $\Psi \equiv 0$ then h satisfies the equation $\Delta_\mu h - \Phi h = 0$, and hence by Lemma 4.7

$$p_t^\Phi(x,y) = h(x)\, h(y)\, \widetilde{p}_t(x,y) \quad \text{and} \quad g^\Phi(x,y) = h(x)\, h(y)\, \widetilde{g}(x,y)\,.$$

Therefore, (6.12) implies (10.9) and (6.13) implies (10.10).

(ii) Let us show that there exists a smooth positive function h' on M such $h' \simeq h$ and

$$\Delta_\mu h' - \Phi h' = 0 \quad \text{on } M. \tag{10.15}$$

Since h satisfies the equation $\Delta_\mu h - (\Phi - \Psi)\, h = 0$, we obtain from (4.10)

$$\Delta_{\widetilde{\mu}} - \Psi = \frac{1}{h} \circ (\Delta_\mu - \Phi) \circ h \tag{10.16}$$

so that by the change $h' = hu$ (10.15) becomes

$$\Delta_{\widetilde{\mu}} u - \Psi u = 0. \tag{10.17}$$

By Lemma 10.3, Ψ is Green bounded on $(M, \widetilde{\mu})$ and hence the equation (10.17) has a solution $u \simeq 1$. Consequently, we obtain a solution $h' = hu$ of (10.15) such that $h' \simeq h$. Clearly, the function h' satisfies the conditions (6.10), (6.11). By the case (i), p_t^Φ and g^Φ satisfy respectively the estimates (10.9) and (10.10) where h is replaced everywhere by h'. However, since $h \simeq h'$, we can replace h' back by h thus finishing proof.

(iii) If $\Phi \equiv 0$ then there is nothing to prove, so we assume that $\Phi \not\equiv 0$. The strategy is the same as in the case (ii) - it suffices to prove that there exists a solution $u \simeq 1$ to (10.17). However, in this case it is easier to start with the equation (10.15) and observe that by Lemma 10.1 this equation has a positive solution h'. Hence, the function $u = h'/h$ is positive and solves (10.17).

Let us show that u is bounded. By (10.17) function u is $\Delta_{\widetilde{\mu}}$-harmonic outside $K := \operatorname{supp} \Psi$. The fact that $(M, \widetilde{\mu})$ has relatively connected annuli and satisfies the Harnack inequality implies that any positive harmonic function in $M \setminus K$ has a limit at infinity, and the limit is always finite if the manifold is non-parabolic, which is the case now (see Lemma 10.22 in Appendix). Therefore, u has a finite limit at ∞ and hence is bounded.

Let us show that the limit of u is positive, which will finish the proof. Assume from the contrary that

$$\lim_{|x| \to \infty} u(x) = 0. \tag{10.18}$$

Set $v = hu$ and prove that

$$\int_M \left(|\nabla v|^2 + \Phi v^2 \right) d\mu = 0, \tag{10.19}$$

which together with $v > 0$ and $\Phi \geq 0$ implies $\Phi \equiv 0$, hence contradicting the hypothesis.

To prove (10.19), observe that, for any $\varepsilon > 0$, the set $\Omega_\varepsilon = \{x \in M : u(x) > \varepsilon\}$ is relatively compact, and function $u_\varepsilon := u - \varepsilon$ vanishes on $\partial \Omega_\varepsilon$. Multiplying the equation $-\Delta_{\widetilde{\mu}} u + \Psi u = 0$ by u_ε and integrating over Ω_ε, we obtain

$$\int_{\Omega_\varepsilon} \left(|\nabla u_\varepsilon|^2 \, d\widetilde{\mu} + \Psi u_\varepsilon u \right) d\widetilde{\mu} = 0.$$

Therefore,

$$\int_{\Omega_\varepsilon} \left(|\nabla u_\varepsilon|^2 + \Psi u_\varepsilon^2 \right) d\widetilde{\mu} = \int_{\Omega_\varepsilon} \Psi u_\varepsilon (u_\varepsilon - u)\, d\widetilde{\mu} = -\varepsilon \int_{\Omega_\varepsilon} \Psi u_\varepsilon d\widetilde{\mu}$$

whence it follows that

$$\left|\int_{\Omega_\varepsilon}\left(|\nabla u_\varepsilon|^2+\Psi u_\varepsilon^2\right)d\widetilde{\mu}\right|\le\varepsilon\left(\sup u\right)\int_M|\Psi|\,d\widetilde{\mu}. \tag{10.20}$$

On the other hand, setting $v_\varepsilon=hu_\varepsilon$ and using the Green formula and (10.16), we obtain

$$\begin{aligned}\int_{\Omega_\varepsilon}\left(|\nabla u_\varepsilon|^2+\Psi u_\varepsilon^2\right)d\widetilde{\mu} &= \int_{\Omega_\varepsilon}u_\varepsilon\left(-\Delta_{\widetilde{\mu}}+\Psi\right)u_\varepsilon\,d\widetilde{\mu}\\ &= \int_{\Omega_\varepsilon}v_\varepsilon\left(-\Delta_\mu+\Phi\right)v_\varepsilon\,d\mu\\ &= \int_{\Omega_\varepsilon}\left(|\nabla v_\varepsilon|^2+\Phi v_\varepsilon^2\right)d\mu. \end{aligned}\tag{10.21}$$

Combining (10.21) with (10.20), we obtain

$$\lim_{\varepsilon\to0}\int_{\Omega_\varepsilon}\left(|\nabla v_\varepsilon|^2+\Phi v_\varepsilon^2\right)d\mu=0.$$

As $\varepsilon\to0$, we have $v_\varepsilon=v-\varepsilon h\to v$ and $\nabla v_\varepsilon=\nabla v-\varepsilon\nabla h\to\nabla v$ pointwise. Therefore, (10.19) follows by Fatou's lemma. ■

Example 10.14 Let $\Phi\ge0$ be a smooth non-zero function with compact support in $\mathbb{R}^2$. By Lemma 10.1, there exists a function $h>0$ in $\mathbb{R}^2$ satisfying the equation $\Delta h-\Phi h=0$. The function h is unbounded because h is subharmonic and $h\not\equiv$ const. Since h is harmonic outside $\operatorname{supp}\Phi$, comparing it with $\log|x|$, which is also harmonic, it is not difficult to show that $h(x)\simeq\log|x|$ for large $|x|$ (cf. [84], [108], [169], [198]). Since $V(r)=\pi r^2$, we see that the conditions (6.10), (6.11) are satisfied. By Theorem 10.10(i), we obtain

$$p_t^\Phi(x,y)\asymp\frac{C\log\langle x\rangle\log\langle y\rangle}{t\log\left(\langle x\rangle+\sqrt{t}\right)\log\left(\langle y\rangle+\sqrt{t}\right)}\exp\left(-c\frac{|x-y|^2}{t}\right), \tag{10.22}$$

for all $x,y\in\mathbb{R}^2$ and $t>0$, where

$$\langle x\rangle:=2+|x|\,.$$

In particular, in the range $t\ge\langle x\rangle^2+\langle y\rangle^2$, we obtain

$$p_t^\Phi(x,y)\simeq\frac{\log\langle x\rangle\log\langle y\rangle}{t\log^2t}$$

(cf. Remark 10.7). Also, (10.14) yields, for $\langle x\rangle\ge\langle y\rangle$,

$$g^\Phi(x,y)\simeq\log\langle y\rangle+\frac{\log\langle y\rangle}{\log\langle x\rangle}\log_+\frac{1}{|x-y|}. \tag{10.23}$$

Example 10.15 Let h be a smooth positive function in $\mathbb{R}^3$ and $h(x)=\frac{1}{|x|}$ for $|x|>1$. Then the function $\Phi=\frac{\Delta h}{h}$ vanishes for $|x|>1$ and hence has a compact support. If Φ were non-negative then, by Theorem 10.5, p_t^Φ would have satisfied the Li-Yau estimate in $\mathbb{R}^3$. However, Φ must take negative values[4] and hence Theorem 10.5 does not apply. Let us show that in fact the

[4]Otherwise, h would be a positive subharmonic function in $\mathbb{R}^3$, which tends to 0 at ∞ and hence must be identical 0 by the maximum principle.

Li-Yau estimate does not hold for p_t^Φ. The function h satisfies the conditions (6.10) and (6.11) because $V(r) = cr^3$ and, for large r,

$$\int_1^r \overline{h}^2(s) V(s) \frac{ds}{s} \simeq r \simeq \overline{h}^2(r) V(r).$$

Hence, by Theorem 10.10(i), we conclude that

$$p_t^\Phi(x,y) \asymp \frac{1}{t^{3/2}} \left(1 + \frac{\sqrt{t}}{\langle x \rangle}\right) \left(1 + \frac{\sqrt{t}}{\langle y \rangle}\right) \exp\left(-c \frac{|x-y|^2}{t}\right). \tag{10.24}$$

As we see, $p_t^\Phi(x,y)$ is *larger* than the unperturbed heat kernel in $\mathbb{R}^3$ as one should expect for a negative perturbation Φ. In particular, for $t \geq \langle x \rangle^2 + \langle y \rangle^2$, we obtain

$$p_t^\Phi(x,y) \simeq \frac{1}{\sqrt{t} \langle x \rangle \langle y \rangle}$$

and hence $g^\Phi \equiv \infty$.

The next section contains important examples of application of Theorem 10.10(ii) and (iii).

10.4 Potentials of quadratic decay in $\mathbb{R}^n$

Here we will estimate the heat kernel p_t^Φ of the operator $\Delta - \Phi$ in $\mathbb{R}^n$, $n \geq 2$, where

$$\Phi(x) = \frac{b}{|x|^2} \quad \text{for } |x| > 1, \tag{10.25}$$

and $b \neq 0$ is a real constant.

Let $h(x)$ be a smooth positive function on $\mathbb{R}^n$ such that

$$h(x) = |x|^\beta \quad \text{for } |x| > 1.$$

As it was noticed in Example 6.13, if $\beta > -n/2$ then h satisfies the conditions (6.10), (6.11). The condition (10.12) of the non-parabolicity of $(\mathbb{R}^n, h^2 dx)$ holds provided $\beta > 1 - n/2$. By a direct computation (cf. Example 4.6), we have, for $|x| > 1$,

$$\frac{\Delta h}{h} = \frac{\beta^2 + (n-2)\beta}{|x|^2}.$$

The equation $\beta^2 + (n-2)\beta = b$ has a root $\beta \geq 1 - n/2$ given by

$$\beta = -(\frac{n}{2} - 1) + \sqrt{(\frac{n}{2} - 1)^2 + b}, \tag{10.26}$$

provided

$$b \geq b_0 := -(\frac{n}{2} - 1)^2,$$

which will be assumed henceforth[5]. With β as in (10.26), the function h satisfies the equation $\Delta h - \Phi h = 0$ for $|x| > 1$. Let us make the following assumptions about $\Phi(x)$ for $|x| \leq 1$:

[5] If $b < b_0$ then the operator $\Delta_\mu - \Phi$ is *supercritical* (see [169]). In particular, the bottom of the spectrum is negative so that the heat kernel cannot satisfy estimates like (10.9), and the equation $\Delta_\mu h - \Phi h = 0$ has no positive solution. For heat kernel estimates in this case see [223].

1. If $b = b_0$ and hence $\beta = 1 - n/2$ then set $\Phi = \frac{\Delta h}{h}$ also for $|x| \leq 1$. Note that in this case manifold $\left(\mathbb{R}^n, h^2 dx\right)$ is parabolic, and we will use part (i) of Theorem 10.10, where the equation $\Delta h - \Phi h = 0$ must be satisfied on the entire manifold.

2. If $b_0 < b < 0$ then set $\Phi = \frac{\Delta h}{h} + \Psi$ where Ψ is a smooth non-negative function compactly supported in $|x| \leq 1$, so that the hypotheses of part (ii) of Theorem 10.10 hold.

3. If $b > 0$ then extend Φ to $|x| \leq 1$ arbitrarily, only preserving the smoothness and non-negativity of Φ. In this case, the hypotheses of part (iii) of Theorem 10.10 hold.

Applying in each case the estimate (10.13) of Theorem 10.10 and using the notation

$$\langle x \rangle := 2 + |x|$$

so that $h(x) \simeq \langle x \rangle^{\beta}$ for all x, we obtain

$$p_t^{\Phi}(x, y) \asymp \frac{C}{t^{n/2}} \left(1 + \frac{\sqrt{t}}{\langle x \rangle}\right)^{-\beta} \left(1 + \frac{\sqrt{t}}{\langle y \rangle}\right)^{-\beta} \exp\left(-c\frac{|x-y|^2}{t}\right), \tag{10.27}$$

for all $t > 0$ and $x, y \in \mathbb{R}^n$. In particular, in the most interesting range $t \geq \langle x \rangle^2 + \langle y \rangle^2$ (10.27) becomes

$$p_t^{\Phi}(x, y) \simeq \frac{\langle x \rangle^{\beta} \langle y \rangle^{\beta}}{t^{\nu/2}}$$

where

$$\nu = n + 2\beta = 2 + \sqrt{(n-2)^2 + 4b}.$$

The estimate (10.27) is new in its entirety. The following results were known before, all in the case $n \geq 3$.

1. It was proved in [61] that if $b_0 < b < 0$ then, for any $\varepsilon > 0$ and all $t > 1$,

$$p_t^{\Phi}(x, y) \leq \frac{C}{t^{\nu/2 - \varepsilon}} \exp\left(-c\frac{|x-y|^2}{t}\right) \tag{10.28}$$

   and

$$\sup_{x \in \mathbb{R}^n} p_t^{\Phi}(x, x) \geq \frac{c}{t^{\nu/2 + \varepsilon}}.$$

   In this case, we have $\beta < 0$ and therefore (10.28) follows from (10.27) by $1 + \frac{\sqrt{t}}{\langle x \rangle} \leq 2\sqrt{t}$.

2. In [221], [222], [223] upper and lower bounds for p_t^{Φ} were obtained similar to (10.27) but with non-sharp values of the exponent β different for upper and lower bound.

3. For a *singular* potential

$$\Phi(x) = \frac{b}{|x|^2} \quad \text{for all } x \in \mathbb{R}^n \setminus \{o\}$$

   with $b \geq b_0$, the following estimate was proved in [162]:

$$p_t^{\Phi}(x, y) \asymp \frac{C}{t^{n/2}} \left(1 + \frac{\sqrt{t}}{|x|}\right)^{-\beta} \left(1 + \frac{\sqrt{t}}{|y|}\right)^{-\beta} \exp\left(-c\frac{|x-y|^2}{t}\right). \tag{10.29}$$

This estimate obviously matches (10.27) except for small values of $|x|$ and $|y|$ where the singularity of the potential at o becomes dominant. The upper bound in (10.29) was also proved in [154], where a similar lower bounds was obtained as well, but with a different (non-sharp) value of β. Sharp upper bounds for the heat kernel of $\Delta - \Phi$ in *bounded* domains were obtained in [11].

If $b = b_0$ then the Green function g^Φ is infinite because the manifold $\left(\mathbb{R}^n, h^2 dx\right)$ is parabolic. If $b > b_0$ then (10.14) yields the following estimates of g^Φ, assuming $\langle x\rangle \geq \langle y\rangle$:

$$\begin{aligned} g^\Phi(x,y) &\simeq \frac{1}{|x-y|^{n-2}} \frac{\langle y\rangle^\beta}{\langle x\rangle^\beta}, & n>2 \\ g^\Phi(x,y) &\simeq \left(\log_+ \frac{\langle x\rangle}{|x-y|} + 1\right) \frac{\langle y\rangle^\beta}{\langle x\rangle^\beta}, & n=2 \end{aligned}$$

The upper bound in these estimates was also proved in [169].

An interesting phenomenon can be observed if we consider the function

$$h(x) = |x|^{\beta_0} \log|x| \quad \text{for } |x| > 2,$$

with the critical exponent $\beta_0 = 1 - n/2$. A direct computation shows that it solves the same equation $\Delta h - \Phi h = 0$ with the same potential $\Phi(x) = b_0 |x|^{-2}$, for $|x| > 2$ (cf. (4.12)). In this case, the corresponding manifold $(M, \widetilde{\mu})$ has the volume growth function $\widetilde{V}(r) \simeq r^2 \log^2 r$ and, hence, is *non-parabolic*. Extending $\Phi(x)$ for $|x| < 2$ so that $\Phi(x) \geq \frac{\Delta h}{h}$ in this domain and using part *(ii)* of Theorem 10.10, we obtain

$$p_t^\Phi(x,y) \asymp \frac{C}{t^{n/2}} \left(1 + \frac{\sqrt{t}}{\langle x\rangle}\right)^{\frac{n}{2}-1} \left(1 + \frac{\sqrt{t}}{\langle y\rangle}\right)^{\frac{n}{2}-1} \frac{\log\langle x\rangle \log\langle y\rangle}{\log(\langle x\rangle + \sqrt{t}) \log(\langle y\rangle + \sqrt{t})} e^{-c\frac{|x-y|^2}{t}}. \tag{10.30}$$

The corresponding Green function is estimated as follows, assuming $\langle x\rangle \geq \langle y\rangle$:

$$g^\Phi(x,y) \simeq \left(\frac{1}{|x-y|^{n-2}} + \frac{\log\langle x\rangle}{\langle x\rangle^{n-2}}\right) \frac{\langle x\rangle^{\frac{n}{2}-1}}{\langle y\rangle^{\frac{n}{2}-1}} \frac{\log\langle y\rangle}{\log\langle x\rangle},$$

if $n > 2$ and as in (10.23) if $n = 2$.

This example shows that the estimate (10.27) for the critical value $\beta = \beta_0$ is *unstable* in the following sense: if $\Phi(x) = b_0 |x|^{-2}$ for large $|x|$ then (10.27) holds only if Φ is extended to small $|x|$ in a specific way, whereas (10.30) holds whenever $\Phi(x)$ is large enough in a neighborhood of the origin (see Fig. 12). It is natural to conjecture that (10.27) holds whenever $\Phi(x)$ is *small* enough in a neighborhood of the origin. It would be interesting to find out the exact borderline between these two cases.

For example, in the case $n = 2$ we have $b_0 = \beta = 0$, and (10.27) amounts to the standard Gaussian estimate, which holds for $\Phi \equiv 0$. The estimate (10.30) amounts to (10.22), which holds for any non-zero potential $\Phi \geq 0$ (see Example 10.14). Hence, in $\mathbb{R}^2$ the two cases are distinguished according to whether $\Phi \equiv 0$ or $\Phi \not\equiv 0$.

10.5 Spherically symmetric potentials

In this section, (M, μ) is a weighted model and as before, $V(r)$ is the volume function of (M, μ), $S(r)$ is the boundary area function, and $m(r)$ is the mean curvature functions (see Sections 2.4, 5.5, 6.3). Let us start with a modification of example of Section 10.4.

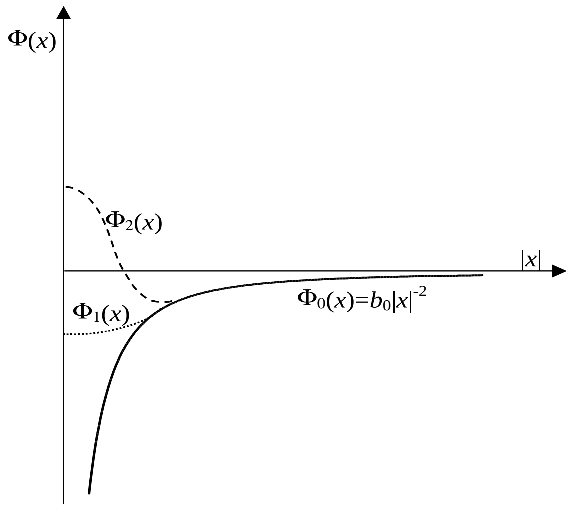


Figure 12: The heat kernel p_t^Φ satisfies (10.29) for $\Phi = \Phi_0$, (10.27) for $\Phi = \Phi_1$, and (10.30) for $\Phi \geq \Phi_2$ (case $n > 2$).

Example 10.16 Let $V(r) = \operatorname{const} r^\alpha$ for large r, where $\alpha > 0$. Let $h = h(r)$ be a positive smooth function on M such that $h(r) = r^\beta$ for $r > 1$. Set $\Phi = \frac{\Delta_\mu h}{h}$ and observe that, by (2.12),

$$\Phi(r) = \frac{\beta(\alpha + \beta - 2)}{r^2} \quad \text{for } r > 1.$$

Let $\widetilde{\mu}$ be the measure on M defined by $d\widetilde{\mu} = h^2 d\mu$ and $\widetilde{V}, \widetilde{S}$ be respectively the volume function and the boundary area function of the model manifold $(M, \widetilde{\mu})$. Then we have

$$\widetilde{S}(r) = h^2 S(r) = \operatorname{const} r^{\alpha+2\beta-1}$$

and, assuming $\alpha + 2\beta > 0$,

$$\widetilde{V} \simeq r^{a+2\beta} \simeq r\widetilde{S}(r).$$

Therefore, $(M, \widetilde{\mu})$ satisfies the hypotheses of Lemma 6.15, and we conclude that the heat kernel $\widetilde{p}_t$ of $(M, \widetilde{\mu})$ admits the estimate

$$\widetilde{p}_t(o, x) \asymp \frac{C}{\widetilde{V}(\sqrt{t})} \exp\left(-c\frac{|x|^2}{t}\right).$$

Applying Lemma 4.7, we obtain, for $t > 1$,

$$p_t^\Phi(o, x) \asymp \frac{C(1 + |x|)^\beta}{t^{\alpha/2+\beta}} \exp\left(-c\frac{|x|^2}{t}\right).$$

In $\mathbb{R}^n$ where $\alpha = n$, this estimate matches (10.27) with $y = o$. However, if $\alpha > n = \dim M$ then (M, μ) does not satisfy the Li-Yau estimate and hence the approach of Section 10.4 does not work. Furthermore, if we try to treat the potential $\Phi(r) = r^{-\gamma}$ with $\gamma < 2$ then we have to consider a function h of a superpolynomial growth so that Lemma 6.15 is no longer applicable either. Nevertheless, we still can apply Theorem 5.42 to estimate $\widetilde{p}_t(o, o)$. The techniques for that will be developed in the rest of this section (see also Example 10.21 below).

Lemma 10.17 *Let $\Phi = \Phi(r)$ be a smooth function on M, which depends only on the polar radius r. Assume that there exists a smooth positive function $h = h(r)$ on M satisfying on M the equation*

$$\Delta_\mu h - \Phi h = 0.$$

Assume that the function

$$\widetilde{m} := m + 2\frac{h'}{h} \tag{10.31}$$

satisfies the conditions $\widetilde{m} > 0$ and $\widetilde{m}' \leq 0$ on $(r_0, +\infty)$ for some $r_0 > 0$. Then the heat kernel p_t^Φ of the Schrödinger operator $\Delta_\mu - \Phi$ on (M, μ) satisfies the following estimate, for large t,

$$p_t^\Phi(o,o) \leq \exp\left(-\frac{\mathcal{R}^2(ct)}{t}\right),$$

where function $\mathcal{R}(t)$ is determined from the equation

$$\frac{\mathcal{R}(t)}{\widetilde{m}(\mathcal{R}(t))} = t. \tag{10.32}$$

If in addition function $\widetilde{m}$ satisfies the condition

$$\int_{r_0}^{r} \widetilde{m}(s)\, ds \leq Cr\, \widetilde{m}(r), \tag{10.33}$$

for large enough r, then we have, for large t,

$$p_t^\Phi(o,o) \geq \frac{1}{2}\exp\left(-\frac{\mathcal{R}^2(Ct)}{t}\right).$$

Proof. Multiplying h by a constant, we can assume $h(o) = 1$. Consider the weighted model $(M, \widetilde{\mu})$ where $d\widetilde{\mu} = h^2 d\mu$. If $S(r)$ and $\widetilde{S}(r)$ are the boundary area functions of (M, μ) and $(M, \widetilde{\mu})$, respectively, then $\widetilde{S} = Sh^2$, whence it follows that the function $\widetilde{m}$ defined by (10.31) is the mean curvature function of $(M, \widetilde{\mu})$.

By Corollary 4.5, the operator $\Delta_\mu - \Phi|_{\mathcal{D}}$ is essentially self-adjoint and, by Lemma 4.7, the corresponding heat kernel p_t^Φ is determined by

$$p_t^\Phi(x,y) = \widetilde{p}_t(x,y)\, h(x)\, h(y), \tag{10.34}$$

where $\widetilde{p}_t$ is the heat kernel of $(M, \widetilde{\mu})$. By Theorem 5.42, we obtain the estimates for $\widetilde{p}_t(o,o)$, and from (10.34) and $h(o) = 1$ we obtained the required bounds for $p_t^\Phi(o,o)$. ■

Lemma 10.18 *Let $\Phi = \Phi(r) \geq 0$ be a smooth function on M. Then there exists a smooth positive function $h = h(r)$ on M satisfying the equation $\Delta_\mu h - \Phi h = 0$ and the condition $h'(r) \geq 0$ for all $r \geq 0$.*

Proof. The proof is similar to that of Lemma 10.1. Since $\Phi \geq 0$, the Dirichlet problem in the ball $B_k = B(o,k)$, $k = 1, 2, \ldots$,

$$\begin{cases} \Delta_\mu u - \Phi u = 0 & \text{in } B_k \\ u = 1 & \text{on } \partial B_k \end{cases}$$

has a unique positive solution u, which depends only on r. By the maximum principle, for any $r \in (0, k)$,

$$\sup_{B_r} u = \sup_{\partial B_r} u = u(r),$$

which implies that $u(r)$ is an increasing function of r and thus $u' \geq 0$. Set $h_k(r) = u(r)/u(0)$ so that h_k satisfies the equation $\Delta_\mu h_k - \Phi h_k = 0$ in B_k and the condition $h_k(0) = 1$. Since the sequence $\{h_k\}$ on M converges at o, by the local properties of elliptic equations there is a subsequence of $\{h_k\}$ that converges on M to a function $h = h(r)$, which hence satisfies all the required properties. ■

Lemma 10.19 *Let Ψ be a smooth function on $(r_0, +\infty)$ such that*

$$\Psi > 0, \;\; \Psi' \leq 0, \quad \text{and} \quad \lim_{r\to\infty} \Psi(r) = \lim_{r\to\infty} \frac{\Psi'}{\Psi^{3/2}} = 0. \tag{10.35}$$

Let u be a function on $(r_0, +\infty)$ satisfying the equation

$$u' + \frac{1}{2}u^2 = \Psi. \tag{10.36}$$

If $\sup_{(r_0,+\infty)} u > 0$ then $u(r) > 0$ and $u'(r) \leq 0$ for all large enough r, and $u(r) \sim \sqrt{2\Psi(r)}$ as $r \to \infty$.

Proof. By hypotheses $u(r_1) > 0$ for some $r_1 > r_0$. Let show that $u(r) > 0$ for all $r > r_1$. Indeed, if $u(r_2) \leq 0$ for some $r_2 > r_1$ then there is an intermediate point $\xi \in [r_1, r_2]$ such that $u(\xi) = 0$ and $u'(\xi) \leq 0$. However, the equation (10.36) fails at this point because $\Psi(\xi) > 0$.

If $u'(r) \geq 0$ for all $r > r_0$ then $u(r) \geq c > 0$ as $r \to \infty$ whereas it follows from (10.35) and (10.36) that $u(r) \leq \sqrt{2\Psi(r)} \to 0$. Therefore, $u'(r_1) < 0$ for some $r_1 > r_0$. We claim that $u'(r) \leq 0$ for all $r > r_1$. Indeed, if $u'(r_2) > 0$ for some $r_2 > r_1$ then the function u in $[r_1, r_2]$ takes the minimal value at a point $\xi \in (r_1, r_2)$. We have then $u'(\xi) = 0$ and hence

$$\Psi(\xi) = \frac{1}{2}u^2(\xi) < u'(r_2) + \frac{1}{2}u^2(r_2) = \Psi(r_2),$$

which contradicts to the hypothesis that Ψ is decreasing. This contradiction shows that $u'(r) \leq 0$ for $r > r_1$.

Consequently, we see that $u \geq \sqrt{2\Psi}$ for large r. Setting $F = 1/\sqrt{2\Psi}$ and $v = \frac{1}{Fu}$, rewrite the equation (10.36) in the form

$$2Fv' + 2F'v + v^2 = 1.$$

The last condition in (10.35) is equivalent to $F'(r) \to 0$ as $r \to \infty$. Since $0 < v(r) \leq 1$ for large r, we have $2vF' \to 0$ and hence

$$2Fv' + v^2 \to 1 \quad \text{as } r \to \infty. \tag{10.37}$$

Let us show that $v(r) \to 1$ as $r \to \infty$, which will finish the proof. Fix $0 < a < 1$ and verify that $v(r) \geq a$ for large enough r. By (10.37) there exists R such that

$$2Fv' + v^2 \geq b^2 \text{ for } r > R, \tag{10.38}$$

where $b \in (a, 1)$. If $v \leq a$ in $(R, +\infty)$ then

$$v' \geq \frac{b^2 - a^2}{2F}.$$

The condition $F' \to 0$ implies $F(r) \leq Cr$ whence $v'(r) \geq \frac{c}{r}$, which yields by integration $v(r) \to \infty$, contradicting $v \leq 1$. Hence, for some $r_1 > R$, we have $v(r_1) > a$. We claim that $v(r) \geq a$ for all $r > r_1$. Indeed, if $v(r_2) < a$ for some $r_2 > r_1$ then there exists $\xi \in (r_1, r_2)$ such that $v(\xi) = a$ and $v'(\xi) \leq 0$, which is however impossible by (10.38). ■

Theorem 10.20 *Let (M,μ) be a weighted model, and let $\Phi=\Phi(r)\geq 0$ be a smooth function on M. Set*

$$\Psi := m' + \frac{1}{2}m^2 + 2\Phi, \tag{10.39}$$

where $m(r)$ is the mean curvature function of (M,μ). If the function Ψ satisfies the hypotheses (10.35) on some interval $(r_0,+\infty)$ and $\sup_{(r_0,+\infty)} m(r) > 0$ then the heat kernel p_t^Φ of the operator $\Delta_\mu - \Phi$ on (M,μ) admits for large t the estimate

$$p_t^\Phi(o,o) \leq \exp\left(-\frac{\mathcal{R}^2(ct)}{t}\right),$$

where function $\mathcal{R}(t)$ is determined by

$$\frac{\mathcal{R}(t)}{\sqrt{\Psi(\mathcal{R}(t))}} = t. \tag{10.40}$$

If in addition function Ψ satisfies for large enough r the condition

$$\int_{r_0}^{r} \sqrt{\Psi(s)}ds \leq Cr\sqrt{\Psi(r)}, \tag{10.41}$$

then p_t^Φ admits for large t the lower bound

$$p_t^\Phi(o,o) \geq \frac{1}{2}\exp\left(-\frac{\mathcal{R}^2(Ct)}{t}\right).$$

Proof. By Lemma 10.18, there exists a smooth positive function $h=h(r)$ on M satisfying the equation $\Delta_\mu h - \Phi h = 0$ and the condition $h'(r)\geq 0$ for all $r\geq 0$. Let $\widetilde{m}(r)$ be the mean curvature function of $(M,\widetilde{\mu})$ where $d\widetilde{\mu}=h^2 d\mu$ so that

$$\widetilde{m} = m + 2\frac{h'}{h}.$$

Set $f=\log h$ and observe that f satisfies the equation

$$\Delta_\mu f + |\nabla f|^2 = \Phi,$$

that is

$$f'' + mf' + (f')^2 = \Phi.$$

Substituting $f'=\frac{1}{2}(\widetilde{m}-m)$, we conclude that $\widetilde{m}$ satisfies the equation

$$\widetilde{m}' + \frac{1}{2}\widetilde{m}^2 = m' + \frac{1}{2}m^2 + 2\Phi = \Psi.$$

Since $\sup_{(r_0,+\infty)} m > 0$ and $h'\geq 0$, we obtain $\sup_{(r_0,+\infty)}\widetilde{m} > 0$. By Lemma 10.19, we conclude that $\widetilde{m}(r) > 0$ and $\widetilde{m}'(r)\leq 0$ for large r so that Lemma 10.17 can be applied. Lemma 10.19 also yields $\widetilde{m}(r)\sim\sqrt{2\Psi(r)}$ as $r\to\infty$ and hence

$$\frac{R}{\widetilde{m}(R)} \simeq \frac{R}{\sqrt{\Psi(R)}},$$

which implies that the definition (10.32) of the function $\mathcal{R}$ in Lemma 10.17 can be replaced by (10.40) at the expense of an additional constant multiple in front of t. The condition (10.41) clearly implies (10.33). Hence, both upper and lower bounds of p_t^Φ follow from the corresponding estimates of Lemma 10.17. ∎

Example 10.21 As in Example 10.16, assume that $V(r) = \mathrm{const}\, r^{\alpha}$ for large r, so that $m(r) = \frac{\alpha-1}{r}$. Consider the function $\Phi(r)$ such that

$$\Phi(r) = \frac{b}{r^{\gamma}} \quad \text{for large } r,$$

where $0 < \gamma < 2$ and $b > 0$. Then the function

$$\Psi(r) = m' + \frac{1}{2}m^2 + 2\Phi = \frac{(\alpha-1)(\alpha-3)}{2r^2} + \frac{2b}{r^{\gamma}} \sim \frac{2b}{r^{\gamma}}$$

obviously satisfies (10.35) and (10.41). It follows from (10.40) that, for large t, $\mathcal{R}(t) \simeq t^{\frac{2}{2+\gamma}}$ and

$$p_t^{\Phi}(o,o) \asymp C \exp\left(-ct^{\frac{2-\gamma}{2+\gamma}}\right).$$

In the case $M = \mathbb{R}$ this result follows from [214].

Note also that the functions $\widetilde{m}$ and h from the proof of Theorem 10.20 are estimated in this case as follows: $\widetilde{m}(r) \simeq r^{-\gamma/2}$ and $h(r) \asymp C\exp\left(cr^{1-\gamma/2}\right)$ for large r.

10.6 Appendix: behavior of harmonic functions at ∞

We prove here the following lemma, which was used in the proof of Theorem 10.10 (similar results can be found in [177]).

Lemma 10.22 ([84]) *Let (M,μ) be a geodesically complete noncompact weighted manifold. Assume that (M,μ) satisfies the Harnack inequality and has relatively connected annuli. Then any non-negative harmonic function $u(x)$ in a neighborhood of infinity in M has a limit as $x \to \infty$, and the limit is finite provided (M,μ) is non-parabolic.*

Proof. Let o be the origin of M, and denote as before $B_r = B(o,r)$ and set

$$S_r = \partial B_r = \{x \in M : d(x,o) = r\}.$$

By the hypothesis of relative connected annuli, there exists a constant $K > 1$ such that for all r large enough and for all $x, y \in S_r$, the points x and y can be connected by a continuous path γ in $B_{Kr} \setminus B_{K^{-1}r}$.

Let us first prove that if r is large enough then

$$\sup_{S_r} u \leq C \inf_{S_r} u, \tag{10.42}$$

with a constant C that is independent of u and r. Since (M,μ) satisfies the volume doubling property, the ball B_{2Kr} can be covered by at most N balls of radius $\rho := \frac{1}{4}K^{-1}r$ where the constant N depends only on K and on the volume doubling constant. Let x, y be two points on S_r and γ be a path connecting them in $B_{Kr} \setminus B_{K^{-1}r}$. Select those of the balls of the radius ρ that intersect γ. Then we obtain a chain of at most N balls $\{B(z_i,\rho)\}$, which connects x and y and such that any ball $B(z_i,\rho)$ in this chain intersects $B_{Kr} \setminus B_{K^{-1}r}$. In particular, any ball $B(z_i,2\rho)$ is outside $B_{K^{-1}r-2\rho} = B_{2\rho}$ and hence is in the domain of u provided r is large enough. By the Harnack inequality applied to u in $B(z_i,2\rho)$, we obtain

$$\sup_{B(z_i,\rho)} u \leq C \inf_{B(z_i,\rho)} u,$$

whence (10.42) follows by a chaining argument.

Set

$$L=\limsup_{x\to\infty} u(x)$$

and show that in fact $L=\lim_{x\to\infty} u(x)$. Let $x_i\to\infty$ be a sequence of points such that $u(x_i)\to L$ as $i\to\infty$, and set $r_i=|x_i|$. Consider two cases.

Let $L=\infty$. It follows from (10.42) that

$$\inf_{S_{r_i}} u\geq c\sup_{S_{r_i}} u\geq cu(x_i)\to\infty\quad \text{as } i\to\infty.$$

By the minimum principle,

$$\inf_{B_{r_{i+1}}\setminus B_{r_i}} u=\min\left(\inf_{S_{i+1}} u,\inf_{S_i} u\right)\to\infty\quad \text{as } i\to\infty,$$

whence it follows that $u(x)\to\infty$ as $x\to\infty$.

Let $L<\infty$. Fix $\varepsilon>0$ and note that $u(x)<L+\varepsilon$ for large enough $|x|$. Applying (10.42) to the function $L+\varepsilon-u$, we obtain, for large enough i,

$$\sup_{S_{r_i}}(L+\varepsilon-u)\leq C\inf_{S_{r_i}}(L+\varepsilon-u)\leq C(L+\varepsilon-u(x_i))\to C\varepsilon$$

whence, for large enough i,

$$\inf_{S_{r_i}} u\geq L-C\varepsilon.$$

and hence

$$\inf_{S_{r_i}} u\to L\quad \text{as } i\to\infty.$$

It follows from the minimum principle that $\lim_{x\to\infty} u(x)=L$.

Finally, let us show that if (M,μ) is non-parabolic then $L<\infty$. Let $g(x,y)$ be the Green function of (M,μ), which is finite by hypothesis. Since function $v(x):=g(o,x)$ is harmonic and positive in $M\setminus\{o\}$, it has limit at infinity, and this limit is equal to $\inf_M v$ because v is superharmonic on M. Since $v(x)$ is the *minimal* positive fundamental solution of the Laplace operator, this limit is 0. Set $\Omega_a=\{x\in M: v(x)<a\}$ and note that $\partial\Omega_a$ is compact and, for almost all $a>0$, $\partial\Omega_a$ is a smooth hypersurface. Let ν be the normal unit vector field on $\partial\Omega_a$ pointing inside Ω_a, and σ be the boundary area on $\partial\Omega_a$. For any harmonic function f defined in a neighborhood of ∞ set

$$\operatorname{flux} f:=\int_{\partial\Omega_a}\frac{\partial f}{\partial\nu}d\sigma,$$

for large enough a, and notice that by the Green formula, $\operatorname{flux} f$ does not depend on the choice of a.

Assume that $u(x)\to\infty$ as $x\to\infty$ and consider the function

$$w=u+C_1v-C_2,$$

where C_1,C_2 are positive constants. Choose $a>0$ so small that u is defined in Ω_{2a}. Observe that $\operatorname{flux} v=-1$ and hence

$$\operatorname{flux} w=\operatorname{flux} u-C_1.$$

Choosing C_1 large enough, we obtain $\operatorname{flux} w<0$. Next, choose C_2 so big that $w<0$ on $\partial\Omega_a$.

Hence, we have $w<0$ on $\partial\Omega_a$ and $\operatorname{flux} w<0$ whereas $w(x)>0$ for large enough $|x|$ because $u(x)\to\infty$. Consider the domain

$$W=\{x\in\Omega_a: w(x)<0\}.$$

This domain is relatively compact, and its boundary ∂W consists of two disjoint pieces: $\partial\Omega_a$ and $\partial_0 W := \partial W \cap \{w = 0\}$. On $\partial_0 W$ we obviously have $\frac{\partial w}{\partial\nu} \leq 0$ whence, by the Green formula and $\Delta_\mu w = 0$,

$$\operatorname{flux} w = \int_{\partial W} \frac{\partial w}{\partial \nu} d\sigma - \int_{\partial_0 W} \frac{\partial w}{\partial \nu} d\sigma = -\int_W \Delta_\mu w \, d\mu - \int_{\partial_0 W} \frac{\partial w}{\partial \nu} d\sigma \geq 0,$$

thus contradicting to $\operatorname{flux} w < 0$. ■

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