Classifying representations through local homology and cosupport

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Triangles in Engaku-ji at Kamakura

A motivating problem: vanishing of Hom

Fix a triangulated category T with suspension $\Sigma\colon \mathsf{T} \xrightarrow{\sim} \mathsf{T}$.

Problem

Given two objects X, Y, find invariants to decide when

$$\operatorname{Hom}^*_{\mathsf{T}}(X, Y) = \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathsf{T}}(X, \Sigma^n Y) = 0.$$

In this talk I discuss an approach which is based on joint work with

Dave Benson and Srikanth Iyengar.

A simple example: quiver representations

Fix a quiver Q of Dynkin type Δ and a field k.

- $\mathsf{T} = \mathsf{D}^b(\operatorname{mod} kQ)$
- NC_Δ = the lattice of non-crossing partitions of type Δ

Proposition

There is a map $\sigma \colon \mathsf{T} \to \mathsf{NC}_\Delta$ *such that for all* X, Y *in* T

$$\operatorname{Hom}^*_{\mathsf{T}}(X, Y) = 0 \iff \sigma(X) \cap \sigma(Y) = \varnothing.$$

Idea:

- $\operatorname{Ker}\operatorname{Hom}^*_{\mathsf{T}}(X, -)$ and $\operatorname{Ker}\operatorname{Hom}^*_{\mathsf{T}}(-, Y)$ form thick subcategories of T.
- Work of Ingalls-Thomas and Brüning yields a bijection

$$\{\text{thick subcategories of } \mathsf{T}\} \xrightarrow{\sim} \mathsf{NC}_\Delta\,.$$

Vanishing of Hom: support and cosupport

Let R be a graded commutative noetherian ring and T an R-linear compactly generated triangulated category.

We assign to X in T

- the support $\operatorname{supp}_R X \subseteq \operatorname{Spec} R$, and
- the cosupport $\operatorname{cosupp}_R X \subseteq \operatorname{Spec} R$,

where $\operatorname{Spec} R =$ set of homogeneous prime ideals.

Theorem (Benson-Iyengar-K, 2010)

The following conditions on T are equivalent.

- T *is stratified by R.*
- *For all objects X, Y in T one has*

$$\operatorname{Hom}^*_{\mathsf{T}}(X, Y) = 0 \quad \Longleftrightarrow \quad \operatorname{supp}_R X \cap \operatorname{cosupp}_R Y = \varnothing.$$

Stratified triangulated categories

About stratification:

- T is stratified by R if for each $\mathfrak{p} \in \operatorname{Spec} R$ the essential image of the local cohomoloy functor $\Gamma_{\mathfrak{p}} \colon \mathsf{T} \to \mathsf{T}$ is a minimal localizing subcategory of T.
- The derived category $\mathsf{D}(\operatorname{Mod} A)$ of a commutative noetherian ring A is stratified by A [Neeman, 1992].
- The stable module $\operatorname{StMod} kG$ of a finite group is stratified by its cohomology ring $H^*(G, k)$ [Benson-Iyengar-K, 2008].

About support: The notion generalizes the one

- for commutative noetherian rings [Foxby, 1979], and
- for group representations [Benson-Carlson-Rickard, 1996].

About cosupport: Not much seems to be known ...

Cosupport: some basic facts

Fix an R-linear compactly generated triangulated category T.

Basic properties:

- For an exact triangle $X \to Y \to Z \to \Sigma X$,

$$\operatorname{cosupp}_R \Sigma X = \operatorname{cosupp}_R X \subseteq \operatorname{cosupp}_R Y \cup \operatorname{cosupp}_R Z.$$

- $\operatorname{cosupp}_R(\prod_i X_i) = \bigcup_i \operatorname{cosupp}_R X_i$.
- $\operatorname{Max}(\operatorname{cosupp}_G X) = \operatorname{Max}(\operatorname{supp}_G X)$.
- $\operatorname{cosupp}_R X = \varnothing$ if and only if $X = 0$.

Notation: $\operatorname{Max} \mathcal{U} = \{\mathfrak{p} \in \mathcal{U} \mid \mathfrak{p} \subseteq \mathfrak{q} \in \mathcal{U} \implies \mathfrak{p} = \mathfrak{q}\}$.

Example: representations of finite groups

The setup:

- G a finite group (for simplicity: a p-group)
- kG = the group algebra over a field k
- $\operatorname{StMod} kG$ = the stable category of kG-modules
- $H^*(G,k) = \operatorname{Ext}^*_{kG}(k,k)$ = the group cohomology

Some facts:

- $\operatorname{StMod} kG$ is a compactly generated triangulated category.
- The ring $H^*(G,k)$ acts on $\operatorname{StMod} kG$ via homomorphisms

$$H^*(G,k) \longrightarrow \underline{\operatorname{End}}^*_{kG}(X), \quad \phi \mapsto \phi \otimes_k X.$$

- $\operatorname{StMod} kG$ is stratified by $H^*(G,k)$.

Finite groups: support and cosupport

$V_G =$ the set of non-maximal homog. prime ideals of $H^*(G, k)$
$\kappa_{\mathfrak{p}} =$ the Rickard idempotent kG-module for $\mathfrak{p} \in V_G$

Definition

Fix a kG-module X.

- $\operatorname{supp}_G X = \{\mathfrak{p} \in V_G \mid \kappa_{\mathfrak{p}} \otimes_k X \neq \text{proj.}\}$
- $\operatorname{cosupp}_G X = \{\mathfrak{p} \in V_G \mid \operatorname{Hom}_k(\kappa_{\mathfrak{p}}, X) \neq \text{proj.}\}$

Some facts:

- $\operatorname{cosupp}_G \operatorname{Hom}_k(X, k) = \operatorname{supp}_G X$.
- $\operatorname{cosupp}_G X = \operatorname{supp}_G X$, if X is finite dimensional.

Example: the Klein four group

Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and k a field of characteristic 2.

- $kG \cong k[x_1, x_2]/(x_1^2, x_2^2)$ and $H^*(G, k) \cong k[\xi_1, \xi_2]$ with $|\xi_i| = 1$.
- kG/soc is stably equivalent to the Kronecker algebra.
- The non-zero primes in V_G parametrize the (homogeneous) tubes of the AR-quiver of mod kG.
- Let $0 \neq \mathfrak{p} \in V_G$. The modules with support $\{\mathfrak{p}\}$ are precisely the non-zero modules in the direct limit closure of the corresponding tube.
- $\kappa_{\mathfrak{p}}$ is the Pruefer module $R(\mathfrak{p}, \infty) = \bigcup_{n \geq 1} R(\mathfrak{p}, n)$ for $\mathfrak{p} \neq 0$.
- The modules with support $\{0\}$ are precisely the direct sums of copies of the generic module κ_0.
- $\mathrm{supp}_G\, \kappa_{\mathfrak{p}} = \{\mathfrak{p}\}$ and $\mathrm{cosupp}_G\, \kappa_{\mathfrak{p}} = \{\mathfrak{p}, 0\}$ for $\mathfrak{p} \in V_G$.

Example: commutative noetherian rings

A = a commutative noetherian ring
$\mathrm{D}(\operatorname{Mod} A)$ = the derived category of the category of A-modules
$k(\mathfrak{p})$ = the residue field $A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$ for a prime ideal $\mathfrak{p}$

Definition

Fix a complex X of A-modules.

- $\operatorname{supp}_A X = \{\mathfrak{p} \in \operatorname{Spec} A \mid k(\mathfrak{p}) \otimes^{\mathbf{L}}_A X \neq 0\}$
- $\operatorname{cosupp}_A X = \{\mathfrak{p} \in \operatorname{Spec} A \mid \mathbf{R}\mathrm{Hom}_A(k(\mathfrak{p}), X) \neq 0\}$

Some examples:

- $\operatorname{supp}_A X = \{\mathfrak{p} \in \operatorname{Spec} A \mid (H^* A)_{\mathfrak{p}} \neq 0\}$ for $X \in \mathrm{D}^b(\operatorname{mod} A)$.
- Let $A = \mathbb{Z}$. Then $\operatorname{cosupp}_A X = \operatorname{supp}_A X$ for $X \in \mathrm{D}^b(\operatorname{mod} A)$.
- Let $(A, \mathfrak{m})$ be complete local. Then $\operatorname{cosupp}_A A = \{\mathfrak{m}\}$.

Localizing and colocalizing subcategories

Fix a triangulated category T with set-indexed (co)products.

Definition

A triangulated subcategory $\mathsf{C} \subseteq \mathsf{T}$ is called

- localizing if C is closed under taking all coproducts,
- colocalizing if C is closed under taking all products.

Notation: For any class $\mathsf{S} \subseteq \mathsf{T}$ write

- $\operatorname{Loc}(\mathsf{S})$ = the smallest localizing subcategory containing S
- $\operatorname{Coloc}(\mathsf{S})$ = the smallest colocalizing subcategory containing S

Examples:

- $\operatorname{Ker}\operatorname{Hom}^*_{\mathsf{T}}(-, Y)$ is localizing for each $Y \in \mathsf{T}$.
- $\operatorname{Ker}\operatorname{Hom}^*_{\mathsf{T}}(X, -)$ is colocalizing for each $X \in \mathsf{T}$.

Classifying colocalizing subcategories

Theorem (Benson-Iyengar-K, 2010)

Let G be a finite p-group and k a field. The assignment

$$V_G \supseteq \mathcal{U} \longmapsto \mathsf{Coloc}(\{\kappa_\mathfrak{p} \mid \mathfrak{p} \in \mathcal{U}\}) \subseteq \mathsf{StMod}\, kG$$

induces a bijection between

- *the collection of subsets of V_G, and*
- *the collection of colocalizing subcategories of $\mathsf{StMod}\, kG$.*

The inverse map sends $\mathsf{C} \subseteq \mathsf{StMod}\, kG$ to $\bigcup_{X \in \mathsf{C}} \mathrm{cosupp}_G\, X$.

Corollary

The assignment $\mathsf{C} \mapsto \mathsf{C}^\perp$ induces a bijection between

- *the collection of localizing subcategories of $\mathsf{StMod}\, kG$, and*
- *the collection of colocalizing subcategories of $\mathsf{StMod}\, kG$.*

Concluding remarks

This classification of colocalizing subcategories

- is surprising because products tend to be complicated,
- is inspired by a similar classification for $\mathrm{D}(\operatorname{Mod} A)$ where A is commutative noetherian [Neeman, 2009],
- is based on local homology functors (= right adjoints of local cohomology functors),
- implies the classification of localizing subcategories:

$$\text{costratification} \quad \Longrightarrow \quad \text{stratification},$$

- implies for kG-modules X, Y:

$$\operatorname{Coloc}(X) \subseteq \operatorname{Coloc}(Y) \quad \Longleftrightarrow \quad \operatorname{cosupp}_G X \subseteq \operatorname{cosupp}_G Y.$$

Not a triangle but a triple at Oberwolfach