Random Tessellations by Means of Cluster Properties and Point Processes

Kai Miguel Matzutt
kai@math.uni-bielefeld.de

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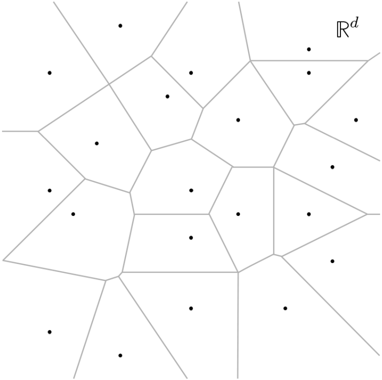
Outline

Motivation

- Very general approach to produce random tessellations . . .
- or other random sets of geometric or discrete objects
- Builds on known point processes in Euclidean space
- Random tessellations (may) have many possible applications inside and outside mathematics

Outline

The Voronoi Tessellation



Properties

- (Random) point sets build the starting position for the tessellation
- Geometric objects (polytopes) are derived from points
- Objects cover the whole space for certain point configurations

What Needs to be Done by the Theory?

Question 1

How can we describe random point sets?

Question 2

What kind of geometric objects are appropriate?

Question 3

How can we combine geometric objects and random point sets in a valid way?

Question 4

Can we create interesting examples?

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Phase Spaces

- Starting with a measurable space $(X, \mathscr{A})$, where
  - X is some set (the space where 'the points' lie in) and
  - $\mathscr{A}$ is a σ-Algebra in X which contains at least all sets of the kind $\{x\}$, $x \in X$.
- For an idea of locality we need the concept of bounded sets, a class $\mathcal{B}(X)$ of subsets of X with certain properties.
- 'Bounded' should not necessarily depend on some metric

Definition

We will call a valid triple $(X, \mathscr{A}, \mathcal{B}(X))$ a phase space.

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Required Properties of Bounded Sets

(B_1) Heredity: is $B \in \mathcal{B}(X)$ and $C \subseteq B$, then also $C \in \mathcal{B}(X)$;

(B_2) $\mathcal{B}(X)$ is stable under finite unions;

(B_3) there exists some sequence $B_1, B_2, \ldots \in \mathcal{B}(X) \cap \mathscr{A} := \mathcal{B}_0(X)$ with $X = \bigcup_{i=1}^{\infty} B_i$;

(B_4) for all $B \in \mathcal{B}(X)$ exists some $B_0 \in \mathcal{B}_0(X)$ with $B \subseteq B_0$.

And we also need some separation axiom:

(C) There exists some countable subclass $\tilde{\mathcal{B}}_0(X)$ of $\mathcal{B}_0(X)$ which separates the points of X in the following sence: for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n$ there exist pairwise disjoint $B_1, \ldots, B_n \in \tilde{\mathcal{B}}_0(X)$ with $x_i \in B_i$ for all $i = 1, \ldots, n$.

Easy Examples of Phase Spaces

- Main Example: $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d), \mathcal{B}(X))$, where $\mathscr{B}(\mathbb{R}^d)$ are the Borel sets with respect to the Euclidean topology and $\mathcal{B}(X)$ are the metrically (Euclidean metric) bounded subsets of $\mathbb{R}^d$. We will call this phase space the Euclidean phase space.
- More general: X is some locally compact topological space with a countable basis. Let $\mathscr{A}$ again be the corresponding Borel sets. As bounded sets we can choose the relative compact sets.

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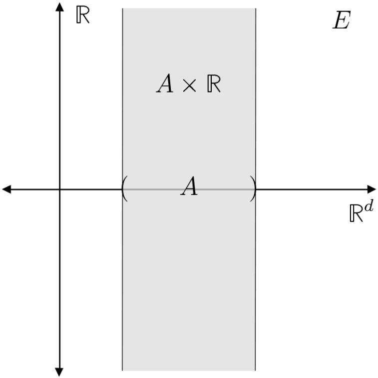
A Slightly Different Example

- Let $X = \mathbb{R}^d \times \mathbb{R}$, $\mathscr{A}$ again the Euclidean Borel sets.
- Then the subsets of cylindrical sets of the kind $A \times \mathbb{R}$, where A is a metrically bounded subset of $\mathbb{R}^d$, are a valid family of bounded sets.



Outline

Locally Finite

Locally Finite Measures

Let $(X, \mathscr{A}, \mathcal{B}(X))$ be a phase space.
Then a measure μ on $(X, \mathscr{A})$ is called locally finite, if all the values $\mu(B)$, $B \in \mathcal{B}_0(X)$, are finite.
The set of all locally finite measures will be denoted by $\mathscr{M}(X)$.

Locally Finite Subsets of X

A subset A of X is called locally finite if card $A \cap B < \infty$ for all $B \in \mathcal{B}(X)$.
Locally finite subsets of X are countable.

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Important Subsets of $\mathscr{M}(X)$

Counting Measures

If $\mu \in \mathscr{M}(X)$ has the property that all values $\mu(B)$, $B \in \mathcal{B}_0(X)$ are (positive) integers, then μ is called counting measure. ($\mu \in \mathscr{M}^{\cdot\cdot}(X)$)

- Can be uniquely expressed in the form $\mu = \sum_{x \in \mu^*} n_x \delta_x$, where μ^* is some locally finite subset of X, called the support of μ, and all $n_x \in \mathbb{N}$.

Simple Counting Measures

A counting measure is called simple if $\mu(\{x\}) \in \{0,1\}$ for all $x \in X$. ($\mu \in \mathscr{M}^{\cdot}(X)$)

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Identification of μ and μ^*

- From now on we will identify a simple counting measure μ with its uniquely determined support μ^*.
- In this sense a counting measure 'is' a sequence of points in X.

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Finite Simple Counting Measures

$\mu \in \mathscr{M}^{\cdot}(X)$ is called finite, if $\mu(X) < \infty$. $(\mu \in \mathscr{M}^{\cdot}_{f}(X))$

Diffuse Measures

A locally finite measure μ is called diffuse or free of atoms if $\mu(\{x\}) = 0$ for all $x \in X$. $(\mu \in \mathscr{M}^{\circ}(X))$

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Counting Variables

The Counting Variables

For $B \in \mathcal{B}_0(X)$ let

$$\begin{aligned} \zeta_B : \quad \mathscr{M}(X) &\longrightarrow \mathbb{R}_0^+ , \\ \mu &\longmapsto \mu(B) . \end{aligned}$$

These functions are called counting variables.

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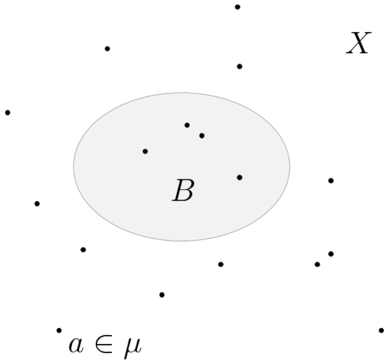

Figure: In this case $\zeta_B(\mu) = 4$

Making a Measurable Space out of $\mathscr{M}(X)$

$\mathscr{F}(X)$

Let $\mathscr{F}(X)$ be the σ-Algebra in $\mathscr{M}(X)$ which is generated by all the counting functions ζ_B, $B \in \mathcal{B}(X)$.

- We now prepare the subclasses of $\mathscr{M}(X)$ with the associated trace σ-algebras of $\mathscr{F}(X)$ on them:

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Random Measures and Point Processes

- We now have several measurable spaces $(\mathscr{M}(X), \mathscr{F}(X))$, $(\mathscr{M}^{\cdot\cdot}(X), \mathscr{F}^{\cdot\cdot}(X))$, $(\mathscr{M}^{\cdot}(X), \mathscr{F}^{\cdot}(X))$, ...

Random Measures

A probability on $(\mathscr{M}(X), \mathscr{F}(X))$ is called a random measure X.

Point Process

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Typical Events

(Recall the counting variables $\zeta_B(\mu) = \mu(B)$, $B \in \mathcal{B}_0(X)$)

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A typical event for a random measure would be of the form

$$\{\mu \in \mathscr{M}(X) \,|\, \zeta_B(\mu) \in [a, b]\} \ , \quad B \in \mathcal{B}_0(X),\ a, b \in \mathbb{R}_0^+$$

(Simple) Point Processes

A typical event for a (simple) point process would be of the form

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The Poisson Point Process

Theorem (Existence and Uniqueness of Poisson Processes)

Let $\rho \in \mathscr{M}(X)$. Then there exists exactly one point process P_ρ in X with the following properties:

1. The counting variables ζ_B, $B \in \mathcal{B}_0(X)$ have the Poisson distribution with parameter $\rho(B)$, that is

$$P_\rho(\{\mu \in \mathscr{M}(X) \,|\, \zeta_B(\mu) = k\}) = \frac{\rho(B)^k}{k!} e^{-\rho(B)}$$

2. If $B_1, \ldots, B_n \in \mathcal{B}_0(X)$, pairwise disjoint, then the counting variables $\zeta_{B_1}, \ldots, \zeta_{B_n}$ are independent with respect to P_ρ.

This process is called the Poisson point process with intensity ρ.
(c.f. Kerstan, Matthes, Mecke: *Infinitely Divisible Point Processes*, 1978, p. 57)

Further Properties of the Poisson Point Process

Simple Point Process

P_ρ is a simple point process, if and only if ρ is diffuse.

Translation Invariant

In case of the Euclidean phase space $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d), \mathcal{B}(X))$ the processes $P_{z\lambda}$, where λ is the Lebesgue measure and z is a positive constant, the Poisson point process is translation invariant. (The translation in $\mathbb{R}^d$ induces a Translation on the counting measures.)

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First Answer

How can we describe random point sets?

Point processes, especially the Poisson point process, 'produce' random point sets.

Outline

Important Phase Spaces

- We will now fix two special phase spaces.

The Phase Space for 'the points'

$(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d))$, the Euclidean phase space

The Phase Space for 'the geometrical objects'

$\left(\mathscr{M}_f(\mathbb{R}^d), \mathscr{F}_f(\mathbb{R}^d), \mathcal{B}\left(\mathscr{M}_f(\mathbb{R}^d)\right)\right)$, finite subsets of $\mathbb{R}^d$.

- The elements of $\mathscr{M}_f(X)$ will be shortly referred to as clusters (in $\mathbb{R}^d$).

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Bounded Sets for $\mathscr{M}_f^{\cdot}(\mathbb{R}^d)$

$\mathcal{B}(\mathscr{M}_f^{\cdot}(\mathbb{R}^d))$

$A \subseteq \mathscr{M}_f^{\cdot}(\mathbb{R}^d)$ belongs to $\mathcal{B}\left(\mathscr{M}_f^{\cdot}(\mathbb{R}^d)\right)$ if it is a subset of at least one of the sets

$$\mathcal{F}_B := \left\{x \in \mathscr{M}_f^{\cdot}\left(\mathbb{R}^d\right) \,\middle|\, x(B) \geq 1\right\}, \quad B \in \mathcal{B}_0(\mathbb{R}^d)$$

Bounded Sets for $\mathscr{M}_f^{\cdot}(\mathbb{R}^d)$ (2)

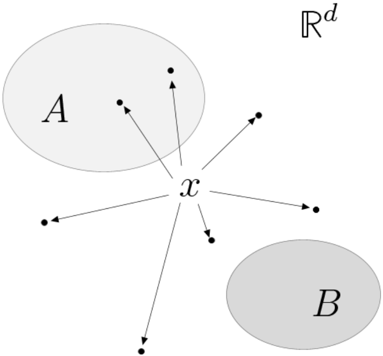

Figure: x is in $\mathcal{F}_A$ but not in $\mathcal{F}_B$

$\mathscr{M}^{\cdot}_f\left(\mathbb{R}^d\right)$

- One can check that by that definition of the bounded sets $\left(\mathscr{M}^{\cdot}_f\left(\mathbb{R}^d\right), \mathscr{F}^{\cdot}_f\left(\mathbb{R}^d\right), \mathcal{B}\left(\mathscr{M}^{\cdot}_f\left(\mathbb{R}^d\right)\right)\right)$ really is a valid phase space.
- Therefore we also have
  - $\mathscr{M}\left(\mathscr{M}^{\cdot}_f\left(\mathbb{R}^d\right)\right), \mathscr{M}^{\cdot\cdot}\left(\mathscr{M}^{\cdot}_f\left(\mathbb{R}^d\right)\right), \ldots$
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Outline

Ideas

- The objects that we have in mind for our tessellations are convex polytopes or even more specific simplices

Basic Concepts

- Take the known phase space $\left(\mathscr{M}_f^{\cdot}\left(\mathbb{R}^d\right), \mathscr{F}_f^{\cdot}\left(\mathbb{R}^d\right), \mathcal{B}\left(\mathscr{M}_f^{\cdot}\left(\mathbb{R}^d\right)\right)\right)$.
- A convex Polytope is uniquely determined by its vertices.
- Identify the finitely many vertices of a polytope with the support of a finite simple counting measure.

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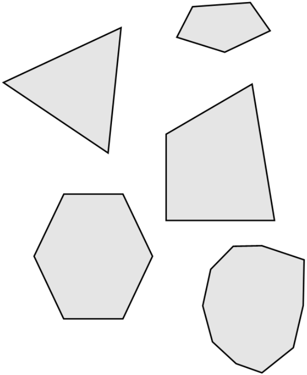
Discrete Polytopes

Discrete Polytope

- $x \in \mathscr{M}_f^{\cdot}(\mathbb{R}^d)$ is called a discrete polytope if its support x^* is the vertice set of a convex polytope.
- The set of all discrete polytopes is denoted by $\mathscr{K}(\mathbb{R}^d)$.
- The convex hull of the support of a discrete polytope is a convex polytope.



Identification of the Vertices

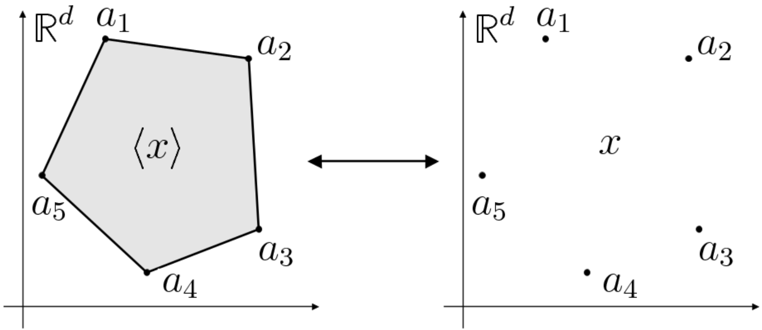
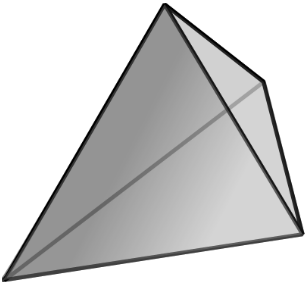


Figure: A polytope $\langle x \rangle$ is identified with its vertices x

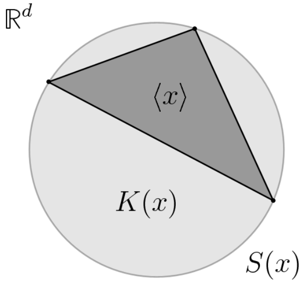
Discrete Simplices

Discrete Simplices

- $x \in \mathscr{M}_f^{\cdot}\left(\mathbb{R}^d\right)$ is called a discrete simplex if its support x^* is an affinely independent set of points.
- The set of all discrete simplices is denoted by $\mathscr{S}\left(\mathbb{R}^d\right)$.
- The convex hull of the support of a discrete simplex is a simplex.
- A discrete simplex is a discrete polytope.



Circumballs



Circumballs

Let $x \in \mathscr{S}\left(\mathbb{R}^d\right)$ be full-dimensional (card $x^* = d$). Then there exists some uniquely determined ball $K(x)$ in $\mathbb{R}^d$, which has all the points of x in its boundary $S(x)$.

- $K(x)$ will be called the circumball of x, $S(x)$ the circumsphere. (We will also denote the center of the circumball with $z(x)$.)

Outline

Locally Finite Tessellations

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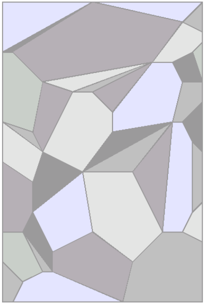
A configuration $\mu \in \mathscr{M}^{\cdot}\left(\mathscr{M}^{\cdot}_{f}\left(\mathbb{R}^d\right)\right)$ is called a locally finite tessellation if

1. $x \in \mu \Rightarrow x \in \mathscr{K}\left(\mathbb{R}^d\right)$ and
2. the corresponding convex polytopes are face to face.

The set of all locally finite tessellations is denoted by $\mathbb{M}(\mathbb{R}^d)$



Locally Finite Tessellations

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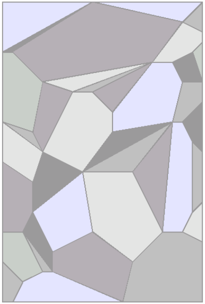

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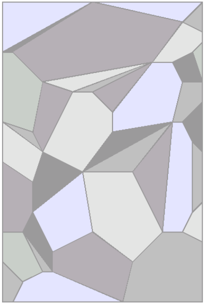

'Locally Finite' in the Context of Tessellations

$\mu \in \mathscr{M}^{\cdot}\left(\mathscr{M}_f^{\cdot}\left(\mathbb{R}^d\right)\right)$

For a locally finite tessellation 'locally finite' means, that only finitely many discrete polytopes of the tessellation have vertices in a given bounded set of $\mathbb{R}^d$.

Face to Face

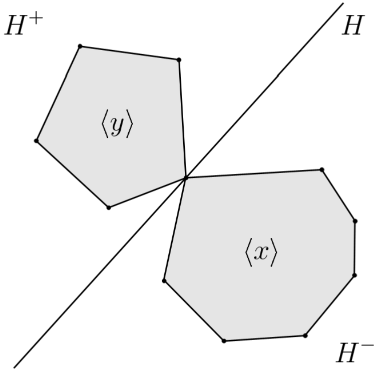

Figure: Polytopes intersect in a vertex

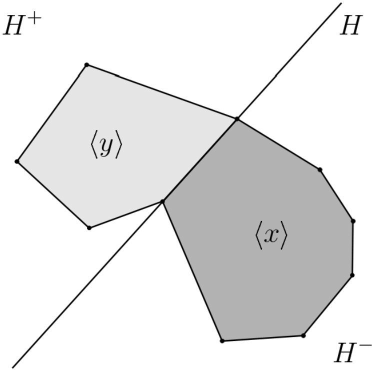

Figure: Polytopes intersect in an edge

Not Face to Face

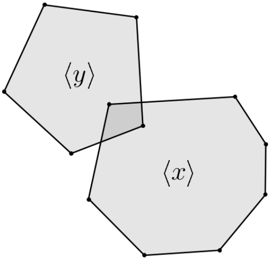

Figure: Polytopes have vertices in the interior of each other

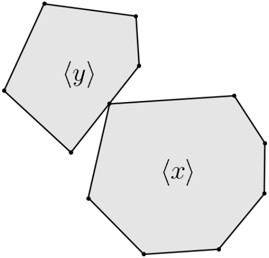

Figure: One polytope has a vertex in the relative interior of an edge of the other one

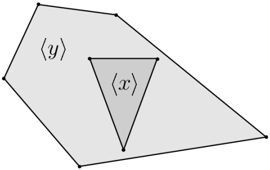

Figure: One polytope lies totally in the interior of the other one

More Specific Tessellations

Complete Tessellations

A locally finite tessellation μ is called complete, if

$$\bigcup_{x \in \mu} \langle x \rangle = \mathbb{R}^d$$

The set of all complete locally finite tessellations is denoted by $\mathbb{M}_v(\mathbb{R}^d)$.

Simplicial Tessellations

If a tessellation has only simplices as elements, it is called simplicial. The set of all simplicial tessellations is denoted by $\mathbb{M}^s(\mathbb{R}^d)$, the complete ones by $\mathbb{M}_v^s(\mathbb{R}^d)$.

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Random Tessellations

Random Tessellation

A simple point process Q in $\mathscr{M}_f^{\cdot}\left(\mathbb{R}^d\right)$ is called random tessellation, if a realization $\mu \in \mathscr{M}^{\cdot}\left(\mathscr{M}_f^{\cdot}\left(\mathbb{R}^d\right)\right)$ is Q-almost-surely a locally finite tessellation.
The terms complete and simplicial are applied analogously to the non random case.

Answer to Question 2

What kind of geometric objects are appropriate?

We can take discrete polytopes/simplices and place them into locally finite tessellations. A random tessellation then gets chance into the game.

Outline

Cluster Properties

Cluster Property

An element D of the product-σ-algebra $\mathscr{F}_f^{\cdot}\left(\mathbb{R}^d\right) \otimes \mathscr{F}^{\cdot}\left(\mathbb{R}^d\right)$ is called cluster property (in $\mathbb{R}^d$).

- The easiest example for a cluster property: $D = \mathscr{M}_f^{\cdot}\left(\mathbb{R}^d\right) \times \mathscr{M}^{\cdot}\left(\mathbb{R}^d\right)$.

(Ideas in this chapter are based on several publications/preprints of Hans Zessin)

Clusters of Type D

- Let D be a cluster property in $\mathbb{R}^d$ and $\eta \in \mathscr{M}^{\cdot}\left(\mathbb{R}^d\right)$

Clusters for η

An element $x \in \mathscr{M}_f^{\cdot}\left(\mathbb{R}^d\right)$ is called a cluster (of type D) *for* η, if $(x, \eta) \in D$.

Clusters in η

An element $x \in \mathscr{M}_f^{\cdot}\left(\mathbb{R}^d\right)$ is called a cluster (of type D) *in* η, if $(x, \eta) \in D$ and $x^* \subseteq \eta^*$.

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The Cluster Counting Function

Cluster Counting Function

$$\begin{aligned} \mathsf{cd}_D : \quad \mathscr{M}^{\cdot}\left(\mathbb{R}^d\right) &\longrightarrow \mathbb{R}_0^+ \cap \{+\infty\} \\ \eta &\longmapsto \sum_{x \in \mathscr{M}_f^{\cdot}\left(\mathbb{R}^d\right),\, x^* \subseteq \eta^*} 1_D(x, \eta) \end{aligned}$$

- cd_D counts the clusters in a given $\eta \in \mathscr{M}^{\cdot}\left(\mathbb{R}^d\right)$.

Locally Finite Cluster Properties

Let D be a cluster property and $\eta \in \mathscr{M}^{\cdot}\left(\mathbb{R}^d\right)$. Define

$$\begin{aligned} D_\eta &:= \left\{x \in \mathscr{M}_f^{\cdot}\left(\mathbb{R}^d\right) \,\middle|\, x \text{ cluster in } \eta\right\}, \\ D_\eta^o &:= \left\{x \in \mathscr{M}_f^{\cdot}\left(\mathbb{R}^d\right) \,\middle|\, x \text{ cluster for } \eta\right\}. \end{aligned}$$

Weak Locally Finite Cluster Properties

A cluster property is called weakly locally finite, if for all $\eta \in \mathscr{M}^{\cdot}\left(\mathbb{R}^d\right)$ the set $D_\eta \subseteq \mathscr{M}_f^{\cdot}\left(\mathbb{R}^d\right)$ is locally finite.

Strong Locally Finite Cluster Properties

A cluster property is called strongly locally finite, if for all $\eta \in \mathscr{M}^{\cdot}\left(\mathbb{R}^d\right)$ the set $D_\eta^o \subseteq \mathscr{M}_f^{\cdot}\left(\mathbb{R}^d\right)$ is locally finite.

The Inner Cluster Function

Inner Cluster Function

Let D be a weak locally finite cluster property.

$$\begin{aligned} \varphi_D : \quad \mathscr{M}^{\cdot}\left(\mathbb{R}^d\right) &\longrightarrow \mathscr{M}^{\cdot}\left(\mathscr{M}_f^{\cdot}\left(\mathbb{R}^d\right)\right), \\ \eta &\longmapsto \sum_{x \in D_\eta} \delta_x . \end{aligned}$$

φ_D is called the inner cluster function.

The Outer Cluster Function

Outer Cluster Function

Let D be a strong locally finite cluster property.

$$\begin{aligned} \psi_D : \quad \mathscr{M}^{\cdot}\left(\mathbb{R}^d\right) &\longrightarrow \mathscr{M}^{\cdot}\left(\mathscr{M}_f^{\cdot}\left(\mathbb{R}^d\right)\right), \\ \eta &\longmapsto \sum_{x \in D_\eta^o} \delta_x \, . \end{aligned}$$

ψ_D is called the outer cluster function.

Inner Cluster Processes

Inner Cluster Process

Let D be a weak locally finite cluster property and P a simple point process in $\mathbb{R}^d$, then $P \circ \varphi_D^{-1}$ is a well defined point process in $\mathscr{M}_f^{\cdot}(\mathbb{R}^d)$, called the inner cluster process for P with respect to D.

Outer Cluster Processes

Outer Cluster Process

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Outline

Stationary Cluster Properties

Stationary Cluster Properties

A cluster property D is called stationary, if for all $e \in \mathbb{R}^d$

$$(x, \eta) \in D \Longleftrightarrow (x + a, \eta + a) \in D\,.$$

The 0-∞-Law of Stochastic Geometry

Theorem (The 0-∞-Law of Stochastic Geometry)

Let P be a translation invariant point process in $\mathbb{R}^d$ and D be a stationary cluster property. Then

$$P\left(\left\{\eta \in \mathscr{M}^{\cdot}\left(\mathbb{R}^d\right) \,\middle|\, 0 < \mathsf{cd}_D\, \eta < +\infty\right\}\right) = 0\,.$$

Some Result Derived from The Theorem

Corollary

Let D be a stationary cluster property and P be a translation invariant simple point process such that

$$P\left(\left\{\eta \in \mathscr{M}^{\cdot}\left(\mathbb{R}^d\right) \mid \mathrm{cd}_D\, \eta \geq 1\right\}\right) > 0\,.$$

- The process

$$P_D = P\left(\cdot \,\middle|\, \left\{\eta \in \mathscr{M}^{\cdot}\left(\mathbb{R}^d\right) \mid \mathrm{cd}_D\, \eta \geq 1\right\}\right)$$

is concentrated on $\left\{\eta \in \mathscr{M}^{\cdot}\left(\mathbb{R}^d\right) \mid \mathrm{cd}_D\, \eta = +\infty\right\}$.

Answer to Question 3

How can we combine geometric objects and random point sets in a valid way?

- With a cluster property we have a connection between our geometric objects and point sets.
- If we take appropiate translation invariant point processes and stationary cluster properties, we even have infinitely many geometric objects in a random realization.

Outline

A Special Cluster Property

Radius Restricted Delaunay Clusters

Let $0 < R < +\infty$. The tuple $(x, \eta) \in \mathscr{M}_f^{\cdot}\left(\mathbb{R}^d\right) \times \mathscr{M}^{\cdot}\left(\mathbb{R}^d\right)$ is in D_R, iff

1. $x \in \mathscr{S}\left(\mathbb{R}^d\right)$,
2. $\operatorname{card} x = d$,
3. The radius of $K(x)$ is smaller than R and
4. $\eta(K(x) \setminus x^*) = 0$ (Delaunay property).

Cluster Property!

Proposition 1

D_R is a stationary weak locally finite cluster property.

- So now we are interested in the clusters in a point configuration η.

Clusters of Type D_R in η

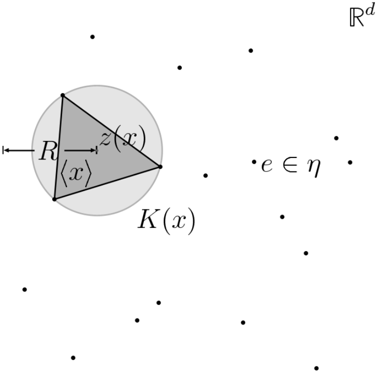

Figure: An example for a cluster of type D_R in a configuration η

No Clusters of Type D_R in η

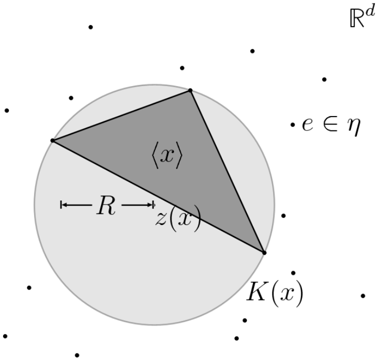

Figure: The circum ball is to big

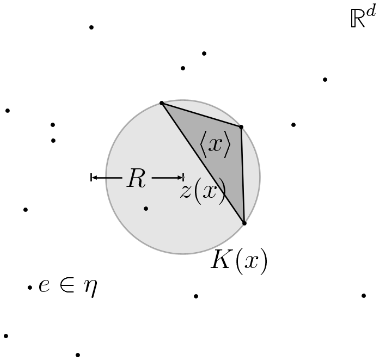

Figure: The configuration η has additional points in the circum ball

First Result

(Recall the inner cluster function $\varphi_D(\eta) = \sum_{x \text{ is a cluster in } \eta} \delta_x$)

Proposition 2

Let $\eta \in \mathscr{M}^{\cdot}\left(\mathbb{R}^d\right)$. Then $\mu := \varphi_{D_R}(\eta)$ is a simplicial locally finite tessellation.

(Idea based on: Boris Delaunay (Delone): *Sur la sphére vide*, Bull. Acad. Sci. URSS VI, Class. Sci. Math. Nat., p. 793-800 (1934))

Sketch of the Proof of Proposition 2

- μ is locally finite, thanks to the weak local finiteness of D_R.
- The elements of μ are simplices per Definition of D_R.
- We only have to check 'face to face' position of the simplices.

Sketch of the Proof of Proposition 2, μ is Face to Face

- Let $x, y \in \mu^*$, $\langle x \rangle \cap \langle y \rangle \neq \emptyset$. Then also $K(x) \cap K(y) \neq \emptyset$.
- We have to consider four cases:
  1. $S(x) \cap S(y) = \emptyset$,
  2. $S(x) \cap S(y)$ is a single point,
  3. $S(x) \cap S(y)$ is a $d-2$-dimensional sphere or
  4. $S(x) = S(y)$.

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Case 1: $S(x) \cap S(y) = \emptyset$

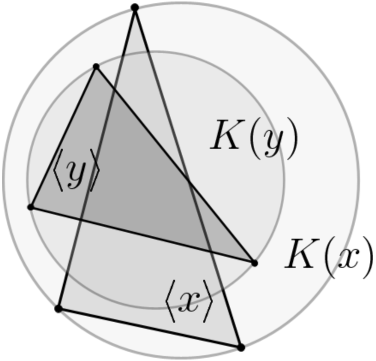

Figure: Case 1 cannot occur because of the Delaunay property

Case 2: $S(x) \cap S(y)$ is a single point

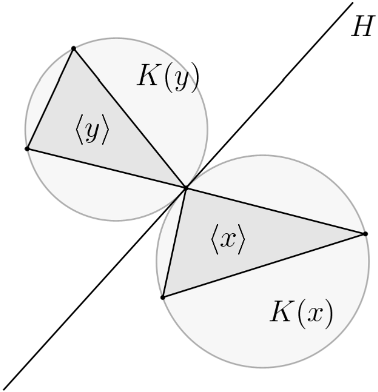

Figure: Case 2 is possible and does not interfere with 'face to face': the polytopes intersect in a vertex

Case 3: $S(x) \cap S(y)$ is a $d-2$ dimensional sphere

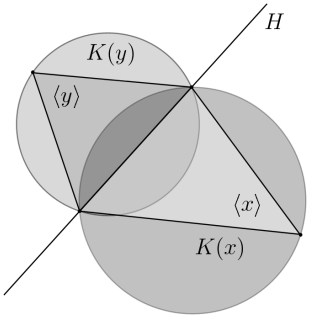

Figure: Case 3 is only possible if it does not interfere with 'face to face': the polytopes intersect in a face

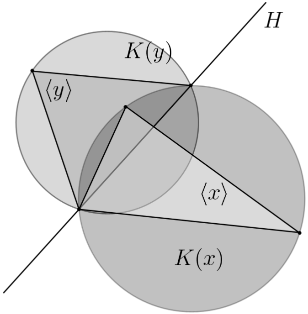

Figure: Case 3 is not possible if it does interfere with 'face to face': the Delaunay property is broken

Case 4: $S(x) = S(y)$

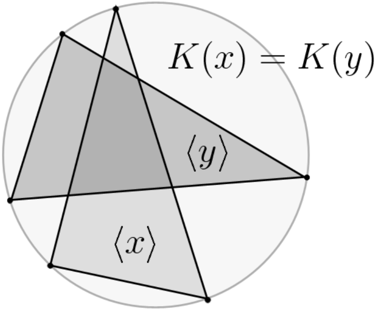

Figure: Case 4 cannot occur because of the Delaunay property

An Appropiate Point Process

The Poisson Point Process and D_R

Let $P = P_{z\lambda}$, $z > 0$, λ the Lebesgue measure in $\mathbb{R}^d$. Then

- P is translation-invariant,
- $P\left(\left\{\eta \in \mathscr{M}^{\cdot}\left(\mathbb{R}^d\right) \,\middle|\, \mathrm{cd}_{D_R}\,\eta \geq 1\right\}\right) > 0$,
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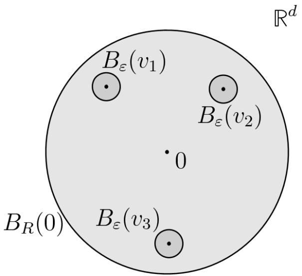

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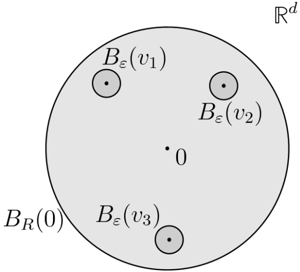

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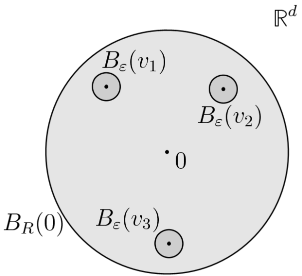
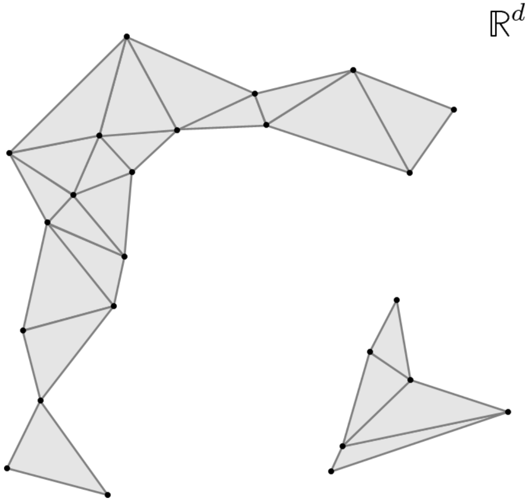


Theorem

Let $P = P_{z\lambda}$. Then $Q_{D_R} = P \circ \varphi_{D_R}^{-1}$ is a random simplicial tessellation. (Follows directly from the previous thoughts and the transformation theorem.)

A Typical Tessellation Produced by Q_{D_R}



Answer to Question 4

Can we create interesting examples?

With Q_{D_R} we have an interesting random simplicial tessellation.

Outline

Q_{D_R} is not complete

Proposition 3

There exists some Q_{D_R}-0-set N such that $\mathbb{M}_v(\mathbb{R}^d) \subset N$

Sketch of the Proof of Proposition 3

Need to show that $P_{z\lambda}$-almost-surely φ_{D_R} produces 'holes'.

Idea

- Make a testing lattice of small ε-balls.
- Check if all balls are covered by simplices of the tessellation.

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The Testing Lattice

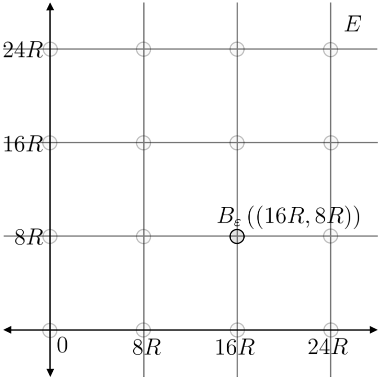

Figure: All ε-balls have to be covered

How to Get Holes

Next Idea

Thanks to the restricted radius of the simplices too big holes in the point configuration will produce holes in the tessellation.

Possible Holes

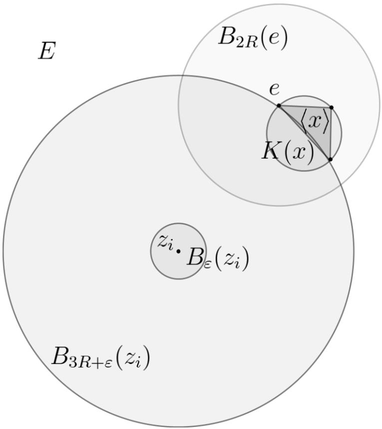

Figure: No points in a big area produce holes

Test Area Around ε-Balls

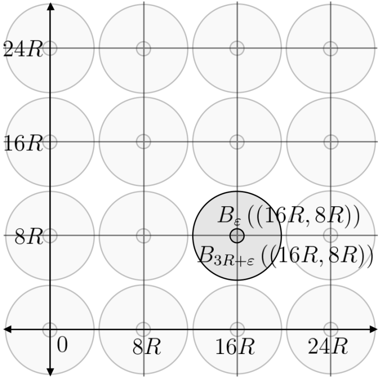

Figure: We have to check the areas around the ε-balls of the lattice

Slightly More Precise

$$
\begin{aligned}
& P_{z\lambda}\left(\left\{\eta \in \mathscr{M}^{\cdot}\left(\mathbb{R}^d\right) \,\middle|\, \bigcap_{x \in \varphi_{D_R}(\eta)} \subset \mathbb{R}^d\right\}\right)\langle x \rangle \\
\leq \quad & P_{z\lambda}\left(\text{Every } \varepsilon\text{-ball is covered by } \varphi_{D_R}(\eta)\right) \\
\leq \quad & P_{z\lambda}\left(\eta \text{ has points in every } (3R+\varepsilon)\text{-ball}\right) \\
= \quad & \prod_{\text{All Lattice Points}} \underbrace{P\begin{pmatrix}\eta \text{ has Points in a Ball of Radius } (3R+\varepsilon) \\ \text{centered at } 0\end{pmatrix}}_{<1} \\
= \quad & 0
\end{aligned}
$$

Thank you for your audience!