Exercises for Functional Analysis

Exercise 11
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Digital submission via the E-Learning site of the tutorial

Exercise 1.
Let H be a Hilbert space, $L \subseteq H$ a linear subspace and $f \in L'$. Prove, with the help of the Riesz Representation Theorem, that there is a unique extension F' on H with $\|F\|_{H'} = \|f\|_{L'}$. *(4 Points)*

Exercise 2.
Let ℓ^∞ be the space of all bounded reel sequences. Using the Hahn–Banach Theorem, prove that there is a linear functional $F \colon \ell^\infty \to \mathbb{R}$ wih the following properties:

$$\liminf_{n\to\infty} x_n \leqslant F(x) \leqslant \limsup_{n\to\infty} x_n \qquad \forall x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty,$$

$$F((x_1, x_2, x_3, ...)) = F((x_2, x_3, x_4, ...)) \qquad \forall x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty.$$

(4 Points)

Hint: Show that $p(x) := \limsup_{n\to\infty} \frac{1}{n}(x_1 + ... + x_n)$ for $x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty$ defines a sub-linear functional on ℓ^∞. Consider the linear subspace $A := \{x = (x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \lim_{n\to\infty} x_n \text{ exists}\}$ and apply the Hahn–Banach Theorem to the linear functional $f \colon A \to \mathbb{R}$, $x \mapsto \lim_{n\to\infty} \frac{1}{n}(x_1 + ... + x_n)$.

Exercise 3.
Prove with the help of the Hahn–Banach Theorem for linear functionals that the mapping $T \colon \ell^1 \to (\ell^\infty)'$, $T(x)(y) = \sum_{n=1}^\infty x_n y_n$ for $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ is isometric, but not surjective. (4 Points)
Hint: Consider on the subspace of all convergent sequences c the linear functional $f(x) = \lim_{n\to\infty} x_n$ and extend f to a continuous linear functional F on ℓ^∞. Show that $F \notin T(\ell^1)$.

Exercise 4.
Let X be a linear, normed space and x, $y \in X$ with $x \neq y$. Prove that there is a functional $f \in X'$ which satisfies

$$f(x) \neq f(y).$$

(4 Points)