SS 2021
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Exercises for Functional Analysis

Exercise 12
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Exercise 1.
Let $\Omega \subseteq \mathbb{R}^n$ *be an open and bounded set and* $V \in L^1(\Omega)$ *with* $V \geqslant 0$. *Prove that* $\mathring{H}_V^{1,2} := \{f \in \mathring{H}^{1,2} \mid \int V f^2 < \infty\}$ *and*

$$(f,g)_{1,2,V} := \int \nabla f \cdot \nabla g + \int (1+V) f \cdot g$$

define a Hilbert space. *(4 Points)*

Exercise 2.
Let $\mathbb{B}$ *be a* σ*-Ring on a set* S *and* $\lambda\colon \mathbb{B} \to \mathbb{R}$ σ*-additive and bounded. Prove with the help of the Hahn Decomposition Theorem that there are* σ*-additive, bounded measures* $\lambda^+, \lambda^-\colon \mathbb{B} \to \mathbb{R}_+$ *with the following properties*

- $\lambda = \lambda^+ - \lambda^-$
- $|\lambda| = \lambda^+ + \lambda^-$
- $\lambda^+(E) = \sup_{A \subseteq E,\ A \in \mathbb{B}} \lambda(A)$ *for all* $E \in \mathbb{B}$
- $\lambda^-(E) = -\inf_{A \subseteq E,\ A \in \mathbb{B}} \lambda(A)$ *for all* $E \in \mathbb{B}$

(4 Points)

Exercise 3.
Let $(X, \mathcal{F}, \lambda)$ *be a measure space, where* λ *is a* σ*-finite measure on* $\mathcal{F}$. *Let* ν_1, ν_2 *be two* σ*-finite measures on* $\mathcal{F}$ *with* $v_1 \ll \lambda$ *and* $v_2 \ll \lambda$. *Set* $\nu := \nu_1 + \nu_2$. *Prove that* $\nu \ll \lambda$ *and that* $\frac{\mathrm{d}\nu}{\mathrm{d}\lambda} = \frac{\mathrm{d}\nu_1}{\mathrm{d}\lambda} + \frac{\mathrm{d}\nu_2}{\mathrm{d}\lambda}$ *holds almost surely on* X. *(4 Points)*

Exercise 4.
Let $X = (0,1]$ *and* λ_1 *the Lebesgue measure on* $\mathbb{R}$. *Let* μ *be a measure on* X *such that* $\mu((0,x]) = 2^x - 1$ *holds. Prove that* $\mu \ll \lambda_1$ *and calculate* $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda_1}$. *(4 Points)*