SS 2021
M. Röckner

Exercises for Functional Analysis

Exercise 3
Submission date: Friday, 07.05.2021
Digital submission via the E-Learning site of the tutorial

Exercise 1.
Let $\Omega \subseteq \mathbb{R}$ be open and bounded, $m \in \mathbb{N}$, $p \in [1, \infty)$. Let $f \in H^{m,p}(\Omega)$. Prove that $f^{(\alpha)}$ is unique for all $\alpha \in \mathbb{N}$ with $\alpha \leqslant m$.

Let $\tilde{f}^{(\alpha)}$ be a function with the same properties as $f^{(\alpha)}$. First, show that

$$\int \left(\tilde{f}^{(\alpha)} - f^{(\alpha)}\right) \varphi = 0 \qquad \forall \varphi \in C_c^\infty(\Omega)$$

holds. *(2 Points)*

Deduce that $\tilde{f}^{(\alpha)} = f^{(\alpha)}$ m-a.s. holds. *(2 Points)*

Exercise 2 (Lemma 1.1)**.**
Let $\Omega \subseteq \mathbb{R}$ be open and bounded, let $m \in \mathbb{N}$ and $0 < \gamma < 1$. Prove that $C^{m,\gamma}(\bar{\Omega})$ and $C^m(\bar{\Omega})$ are complete spaces. *(4 Points)*

Hint: The general case can be reduced to the case of $C^0(\bar{\Omega})$ and $C^1(\bar{\Omega})$ (see Lemma 1.1).

Exercise 3.
Let $\Omega \subseteq \mathbb{R}$ be open and bounded. Show that $C^1(\Omega)$ is not complete with respect to the norm

$$\|f\|_{1,2} = \left(\int_\Omega |f|^2 + |\nabla f|^2 \, \mathrm{d}m\right)^{\frac{1}{2}}.$$

. *(4 Points)*

Exercise 4.
Let $a < b$, $p \in (1, \infty)$ and $\alpha := 1 - \frac{1}{p}$. Prove that there is a constant $C = C(a, b, p)$ such that for all $f \in C^1([a, b])$ and $x_0 \in [a, b]$

$$\|f\|_{C^{0,\alpha}([a,b])} \leqslant |f(x_0)| + C\|f'\|_{L^p([a,b])}$$

holds. *(2 Points)*

Deduce that every function $f \in H^{1,p}((a, b))$ has one and only one continuous representative $\bar{f} \in C^{0,\alpha}([a, b])$. *(2 Points)*