Exercises for Functional Analysis

Exercise 5
Submission date: Friday, 21.05.2021
Digital submission via the E-Learning site of the tutorial

Exercise 1.
Let $p \in [1, \infty[$.
a) Prove that $C_c^0(\mathbb{R}^n)$ *is dense in* $L^p(\mathbb{R}^n)$. *(2 Points)*
b) Conclude that $L^p(\mathbb{R}^n)$ *is separable.* *(2 Points)*
Hint for b): You can conclude this from Example 2.5.3 and Lemma 2.4.

Exercise 2.
Let X, Y *be compact metric spaces and* $f \in C(X \times Y)$. *Prove that for all* $\varepsilon > 0$ *there exist* $n \in \mathbb{N}$ *and* $a_1, ..., a_n \in C(X)$ *as well as* $b_1, ..., b_n \in C(Y)$ *with*

$$\sup_{x \in X, y \in Y} \left| f(x,y) - \sum_{k=1}^{n} a_k(x) b_k(y) \right| \leqslant \varepsilon.$$

(4 Points)

Hint: Approximation Theorem of M.H. Stone

Exercise 3.
Let $h \in C([0,1])$ *be a strictly increasing function. Prove that for all* $f \in C([0,1])$ *and all* $\varepsilon > 0$ *there exist* $n \in \mathbb{N}$ *and constants* $a_1, ..., a_n \in \mathbb{R}$ *with*

$$\sup_{x \in [0,1]} \left| f(x) - \sum_{k=0}^{n} a_k h(x)^k \right| \leqslant \varepsilon.$$

(4 Points)

Exercise 4.
Let $f \colon [0, 2\pi] \to \mathbb{R}$ *be a continuous function with* $f(0) = f(2\pi)$. *Prove that for all* $\varepsilon > 0$ *there exist* $n \in \mathbb{N}$ *and constants* $a_1, ..., a_n \in \mathbb{R}$ *as well as* $b_1, ..., b_n \in \mathbb{R}$ *such that*

$$\sup_{x \in [0, 2\pi]} |f(x) - p(x)| \leqslant \varepsilon$$

holds with

$$p(x) := a_0 + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx).$$

(4 Points)