Exercises to Introduction to Stochastic Partial Differential Equations I

Sheet 4
Total points: 16
Submission before: Friday, 05.05.2023, 12:00 noon

Problem 1 (Dominated convergence). (4 Points)

Let $(X, \mathcal{F}, \mu)$ be a complete measure space and $(B, ||\cdot||_B)$ a Banach space. Let $g \in L^1(\mu; \mathbb{R})$, $f_n \in L^1(\mu; B), n \in \mathbb{N}$, and $f : X \to B$ be a function such that $f_n(x) \to f(x)$ for a.e. $x \in X$, as $n \to \infty$, and

$$||f_n(x)||_B \leqslant g(x)$$

for almost every $x \in \Omega$. Then $f \in L^1(\mu; B)$ and

$$\int_X ||f_n - f||_B \; d\mu \to 0, \text{as } n \to \infty.$$

Problem 2 (Prove the details). (1+1+1+1 Points)

(i) Consider the situation of Theorem 2.1.6. Prove in detail why $\sum_{k=1}^{n} \sqrt{\lambda_k} \beta_k e_k$, $k \in \mathbb{N}$, converges in $L^2(\Omega, \mathcal{F}, P; U)$.

(ii) Consider the situation in the *Alternative Proof of Corollary 2.1.7.* Show that $\mu = N(0, Q)$.

(iii) Consider the situation in the proof of Proposition 2.1.10. Provide the details why $\beta_k(t) - \beta_k(s)$ is distributed as $N(0, t - s)$ for all $s < t$.

(iv) Assume $(W(t))_{t\in[0,T]}$ is an $(\mathcal{F}_t)$-Wiener process, i.e. a Wiener process on a probability space $(\Omega, \mathcal{F}, P)$ with respect to a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F})$. Then $(W(t))_{t\in[0,T]}$ is also an $(\mathcal{F}_t^0)$-Wiener process, where

$$\mathcal{F}_t^0 := \sigma(\mathcal{F}_t \cup \mathcal{N}),\ \mathcal{N} := \{A \in \mathcal{F} : P(A) = 0\}.$$

(Compare with the proof of Proposition 2.1.13.)

Problem 3. (4 Points)

Exercise 2.1.8. in the script.

As mentioned in the lecture, there is a general theorem called ***Kuratowski's theorem*** *(see, e.g., [Par67, Corollary 3.3][1]) which can be used to show (iv). But for this exercise, we should not use it.*

Problem 4. (4 Points)

Prove Proposition 2.2.2 in the script.

Hint: Use a monotone class argument.

[1] [Par67] K. R. Parthasarathy. *Probability measures on metric spaces.* Probability and Mathematical Statistics, No. 3. Academic Press, Inc., New York-London, 1967.