Summer term 2023
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Exercises to Introduction to Stochastic Partial Differential Equations I

Sheet 8
Total points: 10+1*
Submission before: Friday, 02.06.2023, 12:00 noon

Problem 1 (Prove the details). (2 Points)

Let E be a separable Banach space and $(M_t)_{t\in[0,T]}$ be an E-valued stochastic process on a filtered probability space $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\in[0,T]})$ such that $E||M(t)|| < \infty$, for all $t \in [0,T]$. Then $(M(t))_{t\in[0,T]}$ is an $(\mathcal{F}_t)$-martingale (cf. Definition 2.2.4) if and only if $(l(M(t)))_{t\in[0,T]}$ is a real-valued $(\mathcal{F}_t)$-martingale for all $l \in E^*$ (cf. Remark 2.2.5).

Problem 2. (1 extra point for presentation)

Prepare to present the proof of Proposition C.0.5.

Problem 3 (Section 3.1). (2+2 Points)

Consider b and σ as considered in Section 3.1. in the lecture notes.

(i) Prove that both b and σ restricted to $[0,t] \times \mathbb{R}^d \times \Omega$ are $\mathcal{B}([0,t]) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_t$-measurable for every $t \in [0,\infty)$.

(ii) Find b which satisfies the conditions of Theorem 3.1, but is not locally Lipschitz continuous in its x-coordinate (when fixing the other variables).

Problem 4. (1+3 Points)

Let $d \in \mathbb{N}$. Consider the stochastic differential equation on $\mathbb{R}^d$ of the following form.

$$dX(t) = -X(t)dt + dW(t),\ X(0) = x \in \mathbb{R}^d. \tag{SDE}$$

Let $(W(t))_{t\in[0,T]}$ be an $(\mathcal{F}_t)$-Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\in[0,T]})$ satisfying the usual conditions.

(i) Find the stochastic process $(X(t))_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\in[0,T]})$ such that (X, W) is a weak solution to (SDE) and write X as a functional of W and x.

(ii) Use (i) to show that, for every $t \in [0,T]$, the law of $X(t)$ is a Gaussian measure on $\mathbb{R}^d$. Calculate the mean and the covariance matrix.