Winter term 2023/2024
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Exercises to Introduction to Stochastic Partial Differential Equations II

Sheet 14
Total points: 12
Submission before: Friday, 01.12.2023, 12:00 noon

Problem 1 (Check the details). (1+1 Points)

Suppose we are in the situation of the proof of Example 5.2.26 (Tamed 3D Navier–Stokes equation).

(i) Prove that the operator $(t, u) \mapsto Au + F(u) - g_N(||u||^2_{L^\infty})u + \tilde{f}(t)$ is hemicontinuous.

Suppose we are in the situation of the proof of Example 5.2.28. (Cahn–Hilliard equation).

(ii) Prove that for all $v \in V_0$ we have

$$_{V^*}\langle A_1(u), v\rangle_V = \langle -\Delta u, \Delta v\rangle_{L^2}.$$

Problem 2. (2+2 Points)

Let $(X, ||\cdot||)$ be a Banach space. Let $A : X \to X$ be a bounded linear operator.

(i) Show that

$$S(t) := e^{tA}$$

is well-defined as a bounded linear operator from X into X for all $t \in \mathbb{R}$.

(ii) Show that $S : \mathbb{R} \to L(X)$ and fulfills the following properties.

(a) $S(0) = I$,

(b) $S(t)S(s) = S(t + s)$ for all $s, t \in \mathbb{R}$,

(c) $||S(t) - I||_{L(X)} \to 0$, as $t \to 0$.

Here I denotes the identity map on X.

Problem 3. (3 Points)

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space. Let $A \in \mathbb{R}^{d\times d}$ be a matrix, $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be bounded, Lipschitz continuous functions and B an $(\mathcal{F}_t)$-Brownian motion. Consider the following SDE

$$dX(t) = AX(t)dt + f(t, X(t))dt + \sigma(t, X(t))dB(t), \quad X(0) = x \in \mathbb{R}^d.$$

We call an $(\mathcal{F}_t)$-adapted, continuous stochastic process $(X(t))_{t\in[0,T]}$ a *classical/mild* solution, if it fulfills $\mathbb{P}$-a.s. for all $t \in [0, T]$ the corresponding equation

$$X(t) = x + A\int_0^t X(s)ds + \int_0^t f(s, X(s))ds + \int_0^t \sigma(s, X(s))dB(s), \qquad \text{(classical solution)}$$

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A} f(s, X(s))ds + \int_0^t e^{(t-s)A}\sigma(s, X(s))dB(s). \qquad \text{(mild solution)}$$

Show that every *classical solution* is a *mild solution* and every *mild solution* is a *classical solution*.

Problem 4 (Chapter 6, below the Remark under Hypothesis M.0). (3 Points)

Let $(H, ||\cdot||)$ be a separable Hilbert space, $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space with normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, $T > 0$, $p \geqslant 2$. Prove that the space $(\mathcal{H}^p(T, H), ||\cdot||_{\mathcal{H}^p})$ is a Banach space. Here, $\mathcal{H}^p(T, H)$ consists of all processes $Y \in L^\infty(0, T; L^2(\Omega; H))$ which have a version $\tilde{Y}$ such that $[0, T] \times \Omega \mapsto \tilde{Y}(t)(\omega) =: Y(t, \omega)$ is predictable and

$$||Y||_{\mathcal{H}^p} := \sup_{t\in[0,T]} (E(||Y(t)||^p))^{\frac{1}{p}} < \infty.$$

Hint: Repeat the proof of Riesz–Fischer and take care of the supremum.

This exercise sheet will be discussed in the tutorial of 'Selected Topics in stochastic Analysis' in the next semester.