Winter term 2023/2024
Lecture: Prof. Dr. Michael Röckner
Exercises: Dr. Sebastian Grube

Exercises to Introduction to Stochastic Partial Differential Equations II

Sheet 5
Total points: 13
Submission before: Friday, 17.11.2023, 12:00 noon

Problem 1. (2+2+2 Points)

Prove the details of Proposition 4.3.5 1. in terms of the following three tasks.

(i) Let $t \in [0,\infty)$. Show that for all bounded $\mathcal{B}(H)$-measurable $F : H \to \mathbb{R}$

$$H \ni x \mapsto E_x(F(\pi_t))$$

is $\mathcal{B}(H)$-measurable. Conclude that the right-hand side in (4.3.9) is $\mathcal{G}_s$-measurable.

(ii) Let $s,t \in [0,\infty)$. Provide the details on how to prove that for all $n \in \mathbb{N}$ and all bounded $\otimes_{i=1}^n \mathcal{B}(H)$-measurable $G : H^n \to \mathbb{R}$ and all $0 \leqslant t_1 < t_2 < \cdots < t_n \leqslant s$

$$E_x(G(\pi_{t_1},\ldots,\pi_{t_n})F(\pi_{t+s})) = \int G(\pi_{t_1}(w),\ldots,\pi_{t_n}(w))E_{\pi_s(w)}(F(\pi_t))P_x(dw).$$

(iii) Prove that (ii) implies (4.3.9).

Problem 2 (cf. (4.3.16) and below)**.** (2+2 Points)

Let $W(t), t \in [0,\infty)$, and $W^{(1)}(t), t \in [0,\infty)$, be two independent cylindrical Wiener processes on a probability space $(\Omega, \mathcal{F}, P)$ with covariance operator $Q = I$. Define

$$\bar{W}(t) := \begin{cases} W(t), & \text{if } t \in [0,\infty), \\ W^{(1)}(t), & \text{if } t \in (-\infty,0] \end{cases}$$

with filtration $\tilde{\mathcal{F}}_t := \bigcap_{s>t} \bar{\mathcal{F}}_s^\circ, t \in \mathbb{R}$, where $\bar{\mathcal{F}}_s^\circ := \sigma(\{\bar{W}(r_2) - \bar{W}(r_1) : r_1, r_2 \in (-\infty, s], r_2 \geqslant r_1\}, \mathcal{N})$ and $\mathcal{N} := \{A \in \mathcal{F} : P(A) = 0\}$.

(i) Is it true that for every $s \in \mathbb{R}$, $\bar{W}(t) - \bar{W}(s), t \geqslant s$, is a cylindrical Wiener process with respect to $(\tilde{\mathcal{F}}_t)_{t \geqslant s}$? Provide reasoning for your answer.

(ii) Can one define a stochastic integral of the type $\int_s^t \Phi(s)d\bar{W}(s)$? Is it possible to do this analogous to Section 2.3? Provide reasoning for your answer.

Problem 3. (3 Points)

Prove the details of Lemma 4.3.8; make the proof 'as obvious as possible'.