Winter term 2023/2024
Lecture: Prof. Dr. Michael Röckner
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Exercises to Introduction to Stochastic Partial Differential Equations II

Sheet 8
Total points: 14
Submission before: Friday, 08.12.2023, 12:00 noon

Problem 1 (Sobolev functions revisited). (4 Points)

(i) Let I be an open interval in $\mathbb{R}$. Prove that for each $f \in H^1(I)$ one can find a version which is (absolutely) continuous on I. Furthermore, show that $H_0^1(I) \subset L^\infty(I)$ continuously.

(ii) Find a counterexample for both of the assertions in (i) in the multidimensional case.

Problem 2. (4 Points)

Use inequality (5.1.16), Young's inequality and the interpolation inequality

$$||u||_{L^4}^2 \leqslant 2\sqrt{2}||u||_{L^2}^{\frac{1}{2}}||\nabla u||_{L^2}^{\frac{3}{2}}, \quad \forall u \in H_0^{1,2}(\Lambda)$$

to prove assertion (2) of Lemma 5.1.6, i.e. show that for $d = 3$ (and $f \in L^\infty(\mathbb{R};\mathbb{R}^d)$)

$$2_{V^*}\langle A(u) - A(v), u - v\rangle_V \leqslant -||u-v||_V^2 + (C + C||v||_V^4)||u-v||_H^2$$

holds for some constant $C \in (0,\infty)$ and all $u, v \in V$.

Problem 3 (Sums of Lebesgue spaces). (2+2 Points)

(i) Let Λ be an open bounded domain in $\mathbb{R}^d$. Let $g \in L^d(\Lambda) + L^\infty(\Lambda)$ and $\varepsilon \in (0,\infty)$. Show that

$$\tilde{\alpha} := \inf\left\{\beta \in (0,\infty) : ||\mathbb{1}_{\{|g|^2>\beta\}}g||_{L^d} \leqslant \varepsilon\right\}$$

is finite (cf. Lemma 5.1.7).

(ii) Show that $L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d) \subset L^q(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ for all $1 \leqslant q \leqslant p \leqslant \infty$.

Definition. Let X, Y be two Hilbert spaces and $A : D(A) \subset X \to Y$ a linear operator with domain $D(A)$ which is dense in X. For every $y \in Y$ we define

$$f_y : D(A) \to \mathbb{R}, x \mapsto \langle Ax, y\rangle_Y.$$

This map is only defined on $D(A)$, but can (since $D(A)$ is dense) be extended to the whole space X. If f_y is continuous by the Riesz representation theorem (here we use completeness!) for each $y \in Y$ there exists a unique $z_y \in X$ such that

$$f_y(x) = \langle x, z_y\rangle_X,$$

We now define the adjoint of A by $A^*(y) := z_y$ with

$$D(A^*) := \{y \in Y \mid f_y : D(A) \subset X \to \mathbb{R} \text{ is continuous}\}.$$

Problem 4. (2 Points)

Let X, Y be two Hilbert spaces and $A : D(A) \subset X \to Y$ densely defined, bounded linear operator. Show that $D(A^*) = Y$ and A^* is bounded with $||A|| = ||A^*||$.