Winter term 2022/2023
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Exercises to Stochastic Analysis

Sheet 1
Total points: 14+4*
Submission before: Friday, 21.10.2022, 12:00 noon

([Parts of] Exercises marked with "" are additional exercises.)*

Problem 1 (BV functions, cf. Def. 1.1.1). (3 Points)

The definition of bounded variation *(short: BV) may seem a bit technical at first. A helpful intuition might be as follows: consider the graph of a function as the profile of a mountain. Loosely, a function is BV if a hiker will travel only a finite total vertical elevation in finite time along this profile. The following exercise provides a characterization of BV functions in one dimension.*

Prove: A function $f : [0, \infty) \to \mathbb{R}$ is of bounded variation if and only if f can be written as $f = f_1 - f_2$ for two positive, increasing functions f_i, $i \in \{1, 2\}$.

Such a decomposition is not unique, since if $f = f_1 - f_2$ for f_i as in the assertion, then also $f = (f_1 + g) - (f_2 + g)$ for any positive and increasing g.

Hint: For " $\Longrightarrow$ ", you may try $f_1(t) := \mathrm{var}_t f$.

Problem 2 (Quadratic variation of Brownian motion, cf. Thm. 1.1.4). (3 Points)

The notion of the quadratic variation *of a function or a process will be essential throughout this lecture. Note that for a partition τ_n with sufficiently small mesh such that $|X_{t^n_{i+1}} - X_{t^n_i}| < 1$, $\sum_{t^n_i \in \tau_n} |X_{t^n_{i+1}} - X_{t^n_i}|^p$ is decreasing in p. Hence, if X is not BV ($p = 1$), it might still be of finite quadratic variation ($p = 2$). In this sense, the quadratic variation is a refined measurement of the total vertical displacement of a path. The next exercise deals with an important technical claim about the quadratic variation of Brownian paths.*

Prove Theorem 1.1.4. (ii).

Problem 3. (4 Points)

Under the assumptions stated at the beginning of Sect. 1.2. of the notes, prove the claim about the existence of the "$\alpha-$integral" $\alpha - \int_0^t f(X_s) dX_s$ as well as identity (1.2.8).

Problem 4 (You might need Itô's product rule, which you will cover in the lecture on Oct 19). (4 Points)

It is usually not favorable to work with integrals of type $\int Y_s\, dX_s$ or quadratic variations by their plain definition as a limit of sums subject to partitions τ_n with mesh converging to 0 (similarly, in classical Analysis nobody wants to work with Riemann integrals via their definition as the limit of Riemann sums!). Instead, one uses Itô's formula, product rule and elementary properties of the *quadratic variation as a toolbox for calculations. This toolbox can be seen as the stochastic analogue to classical analysis-tools as the chain rule. This analogy coins the term* stochastic analysis. *You should use these tools in the following exercise.*

Let $B : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a continuous Brownian motion with $B_0 = 0$.

(i) Compute $\int_0^t B_s \, dB_s$ pathwise.

(ii) Let $g(t) := tB_t$ and $h(t) := e^{-B_t}$. Compute $\langle g \rangle_t$ and $\langle h \rangle_t$ pathwise (i.e. more precisely, compute $\langle g(\omega) \rangle_t$ and $\langle h(\omega) \rangle_t$ for $g(\omega)(t) := tB_t(\omega)$ and $h(\omega)(t) := e^{-B_t(\omega)}$).

Hint: You may use that any C^1-function is BV.

Problem* 5 (Paley-Wiener integrals, cf. Exa. 1.3.20). (2+2 Points)

One central goal of this lecture is to give meaning to integrals $\int Y_s \, dX_s$, where $X, Y : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ are stochastic processes. The most important case is that of X being a continuous Brownian motion. In this, as well as most other cases, such integrals cannot be defined as pathwise Lebesgue-Stieltjes integrals $\int Y_s(\omega) dX_s(\omega)$, since typical paths of X will not be BV. A way out is the construction of stochastic integrals, *cf. Ch.2. For now, we want to study special cases of pairs (X, Y) for which the integral* can be *defined pathwise. One such case arises from Itô's formula: if the paths of X satisfy the assumptions from the beginning of Sect. 1.2. and $f \in C^1(\mathbb{R})$, then the integral is pathwisely defined for the pair $(X, f(X))$, see Rem. 1.2.10. The case X = continuous Brownian motion is included. In this exercise, we will study another special case, where X = continuous Brownian motion and Y is a* deterministic *function, i.e. it does not depend on ω.*

Let B be a continuous Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ with $B_0(\omega) = 0$ $\mathbb{P}-$a.s.

(i) Let $h \in C([0, 1], \mathbb{R})$ with bounded variation and $h(1) = 0$. Then, by Itô's product rule, we have $\mathbb{P}$-a.s.

$$\int_0^1 h(s) dB_s(\omega) = -\int_0^1 B_s(\omega) dh(s).$$

Prove

$$\mathbb{E}\left[\int_0^1 h(s) dB_s\right] = 0$$

and

$$\mathbb{E}\left[\left(\int_0^1 h(s) dB_s\right)^2\right] = \int_0^1 h(s)^2 ds. \tag{1}$$

Pay specific attention where you use properties of Brownian motion.

(ii) Work out the details of the end of Exa. 1.3.20: How and between which spaces does (1) yield an isometry? How does one use this isometry to extend the definition of $\int_0^1 h(s) dB_s$ to a larger class of integrands h, and what is this larger class?