Winter term 2022/2023
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Exercises to Stochastic Analysis

Sheet14
Total points: 8+8*
Submission before: **Thursday, 02.02.2023**, 12:00 noon

([Parts of] Exercises marked with "" are additional exercises.)*

Problem 1 (Strong well-posedness of an SDE). (2+1+1 Points)

(i) Let $\alpha, \sigma > 0$, $\beta \in \mathbb{R}$, and prove, using general theory from Ch.6, that the SDE

$$dX_t = -\alpha(X_t - \beta)dt + \sigma dB_t, \quad t \geqslant 0, \tag{1.1}$$

has a unique strong solution. Why can this result not be obtained by the Zvonkin transformation as stated in Thm.6.2.6?

(ii) Now assume $\alpha, \sigma : \mathbb{R}_+ \to \mathbb{R}_+$ and $\beta : \mathbb{R}_+ \to \mathbb{R}$ are functions of $t \in \mathbb{R}_+$. Which assumptions do you have to make on the t-dependence of α, β and σ such that - based on the theory of Ch.6 - the strong well-posedness from (i) remains true?

(iii) Based on the interpretation that (1.1) roughly means

$$X_{t+h} - X_t \approx -\alpha(X_t - \beta)h + \sigma(B_{t+h} - B_t)$$

for $t \geqslant 0$ and $0 < h \ll 1$, interpret the meaning of the parameters σ, α and β.

The assertions of Problem 2 and 3 are also true for $\mathbb{R}^d$, $d \geqslant 2$, instead of $\mathbb{R}$, but since we treated only one-dimensional SDEs in this lecture, we restrict to the case $d = 1$.

Problem 2 (Solutions of weakly well-posed time-homogeneous SDEs form Markov process). (4 Points)

A very important class of Markov processes arises as solutions to SDEs. Remarkably, even if an SDE is not well-posed, i.e. if there is more than one solution for a given initial condition, it is (under certain assumptions on the coefficients of the SDE) possible to select one particular solution for each initial condition such that the family of these selected solutions forms a Markov process. However, the 'classical' situation is that the SDE is (at least weakly) well-posed, and in this case it is always true that the family of its (weakly unique) solutions constitutes a Markov process, as you will show in this exercise.

Consider the SDE with time-independent (also called *time-homogeneous*) coefficients $b, \sigma : \mathbb{R} \to \mathbb{R}$

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad t \geqslant 0, \tag{2.1}$$

and assume that for every initial condition $X_0 = x$, $x \in \mathbb{R}$, there exists a unique weak solution, and denote by $\mathbb{P}_x$ the path law on $(\Omega, \mathcal{F}) := (C(\mathbb{R}_+, \mathbb{R}), \sigma(\pi_t, t \geqslant 0))$ of the unique weak solution with start in x (where, as usual, π_t denotes the canonical projection from Ω to $\mathbb{R}$ at t).

Prove that $(\Omega, \mathcal{F}, (\pi_t)_{t\geqslant 0}, \{\mathbb{P}_x)_{x\in\mathbb{R}}\}$ is a Markov process wrt. $(\mathcal{F}_t)_{t\geqslant 0}, \mathcal{F}_t := \sigma(\pi_s, 0 \leqslant s \leqslant t)$.

Hint: You may use the following: For each $\mathbb{P}_x$ as above and $s \geqslant 0$, the regular conditional probability kernel p of $\mathbb{P}_x[\,\cdot\,|\mathcal{F}_s]$ from $(\Omega, \mathcal{F}_s)$ to $(\Omega, \mathcal{F})$ is of the form $p(\omega, A) = q(\pi_s(\omega), A)$ for $\omega \in \Omega$, $A \in \mathcal{F}$ and an (up to zero sets) uniquely determined map $q : \mathbb{R} \times \mathcal{F} \to \mathbb{R}$, and for $\mathbb{P}_x \circ \pi_s^{-1}$-a.e. $z \in \mathbb{R}$, the probability measure $q(z, \cdot)$ on $\mathcal{F}$ is the path law of a weak solution to (2.1) with initial condition z.

Problem* 3 (From solutions to SDEs to solutions to FPKEs). (4* Points)

Fokker–Planck–Kolmogorov equations (FPKEs) are differential equations for measures, which are closely related to stochastic analysis and SDEs. For instance, one can show: For every weakly continuous solution $t \mapsto \mu_t$ to (FPKE) *as below, there is a weak solution process X to the corresponding* (SDE) *such that $\mathbb{P} \circ X_t^{-1} = \mu_t$ for all $t \geqslant 0$. This remarkable result is called* superposition principle. *The converse implication is much easier to prove and is treated in this exercise.*

Let ν be a Borel probability measure on $\mathbb{R}$, let (X, B) be a weak solution to the SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = \xi_0, \quad t \geqslant 0, \qquad \text{(SDE)}$$

on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$, where ξ_0 is an $\mathcal{F}_0$-measurable random variable on Ω such that $\mathbb{P} \circ \xi^{-1} = \nu$, and assume $\mathbb{E}[\int_0^t |b|(s, X_s)ds] + \mathbb{E}[\int_0^t |\sigma^2|(s, X_s)ds] < \infty$ for all $t \geqslant 0$.

First, show that the curve of probability measures on $\mathcal{B}(\mathbb{R})$ given by $t \mapsto \mu_t := \mathbb{P} \circ X_t^{-1}$ (also called *one-dimensional time marginals* of X) is weakly continuous (in the sense of measures). Then, using Itô's formula, prove that $t \mapsto \mu_t$ is a *distributional solution* to the so-called *Fokker–Planck–Kolmogorov equation*

$$\partial_t \mu_t = \partial_x^2(a\mu_t) - \partial_x(b\mu_t), \quad t \geqslant 0, \quad \mu_0 = \nu \qquad \text{(FPKE)}$$

(where $a := \sigma^2$), which means that

(i) $\int_0^t \int_{\mathbb{R}} |b|(s, x) + |a|(s, x)|\, d\mu_s(x)ds < \infty$ for all $t \geqslant 0$,

(ii) $\int_{\mathbb{R}} \varphi d\mu_t - \int_{\mathbb{R}} \varphi d\nu = \int_0^t \int_{\mathbb{R}} a(s, x)\varphi''(x) + b(s, x)\varphi'(x)\, d\mu_s(x)ds$ for all $\varphi \in C_c^2(\mathbb{R})$.

Problem* 4 (Approximation of stochastic integrals by 'Riemannian sums'). (4* Points)

Let $B = (B_t)_{t\geqslant 0}$ be an $(\mathcal{F}_t)$-Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$, and let $H : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be product measurable and $(\mathcal{F}_t)$-adapted such that $\mathbb{E}[\int_0^\infty |H|^2 dt] < \infty$ (where $\mathbb{E}[\cdot]$ denotes expectation wrt. $\mathbb{P}$). Then we know that the stochastic integral $\int_0^\cdot H_s dB_s$ is well defined on $[0, \infty) \times \Omega$, and that it can be approximated by a sequence of stochastic integrals $\int_0^\cdot H_s^{(n)} dB_s$, $n \in \mathbb{N}$, where $H^{(n)}$ are elementary functions as in Def.2.3.19. However, it is not clear whether one can also approximate it by stochastic integrals with integrands of type

$$H^{(n)}(t, \omega) = H_{t_i^n}(\omega) \;\forall t \in (t_i^n, t_{i+1}^n] \text{ for some } 0 = t_0^n < \cdots < t_{k_n}^n < \infty.$$

In other words, the question is whether $\int_0^\cdot H_s dB_s$ can be approximated by "Riemannian sums". This exercise gives a positive answer to this question.

Let B and H be as above, let $n \in \mathbb{N}$, and prove the following: There is a partition $0 = t_0^n < \cdots < t_{k_n}^n = n$ of $[0, \infty)$ such that $\max_{0\leqslant i\leqslant k_n-1} |t_{i+1}^n - t_i^n| \leqslant \frac{1}{n}$, and

$$\mathbb{E}\left[\int_0^\infty |\bar{H}_s^{(n)} - H_s|^2 ds\right] \xrightarrow{n\to\infty} 0,$$

where $\bar{H}_t^{(n)} := H_{t_i^n}$ for $t \in (t_i^n, t_{i+1}^n]$, and $\bar{H}_t^{(n)} := 0$ for $t > n$. By showing that $\bar{H}^{(n)}$ is predictable, conclude that $\int_0^t \bar{H}_s^{(n)} dB_s \xrightarrow{n\to\infty} \int_0^t H_s dB_s$ in $L^2(\mathbb{P})$ for all $t \geqslant 0$.