Winter term 2022/2023
Prof. Dr. Michael Röckner and Dr. Marco Rehmeier

Exercises to Stochastic Analysis

Sheet 3
Total points: 16
Submission before: Friday, 04.11.2022, 12:00 noon

([Parts of] Exercises marked with "" are additional exercises.)*

Problem 1 (Deterministic stopping of Itô integrals, cf. proof of Prop. 1.4.27). (4 Points)

The following exercise is concerned with a simple version of an important technical property of (stochastic) integrals, namely that "stopping times can be pulled inside the integral", cf. Cor. 2.4.37 later. For the moment, however, we confine ourselves to the case of a deterministic time r.

Let $a \in (0, \infty]$ and let $f, g : [0, a) \to \mathbb{R}$ be continuous functions such that for a sequence of partitions $(\tau_n)_{n\in\mathbb{N}}$ of $[0, \infty)$ with the usual conditions*, the integral

$$\int_0^t f(s)dg(s) := \lim_{n\to\infty} \sum_{\substack{t_i^n\in\tau_n,\\ t_i^n\leqslant t}} f(t_i^n)\big(g(t_{i+1}^n) - g(t_i^n)\big)$$

is well-defined. Prove that for any $r \in [0, a)$

$$\int_0^{t\wedge r} f(s)dg(s) = \int_0^t f(s \wedge r)dg(s \wedge r), \quad \forall t \in [0, a),$$

and conclude that (1.4.13) in the proof of Proposition 1.4.27 holds.

Does the above equality remain true for $f(s)$ instead of $f(s \wedge r)$ on the right-hand side?

*i.e. the mesh of τ_n decreases to 0 and the final members $t_{N_n}^n$ of τ_n increase to $+\infty$ as $n \to \infty$.

Problem 2 (Stopping times for non-constant lifetime). (4 Points)

A classical example of stopping times are first entry times of continuous stochastic processes in closed sets. If the "lifetime" of the process depends on the path, things become more subtle, but the result remains true, as you will show in this exercise.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant0}, \mathbb{P})$ be a filtered probability space with the properties stated at the very beginning of Section 1.4., let X be a $\mathbb{R}^d$-valued continuous local martingale up to T, and $C \subseteq \mathbb{R}^d$ a closed set. Show that

$$S : \Omega \to \mathbb{R}, \quad S(\omega) := \inf\{t \in (0, T(\omega)) : X_t \in C)\} \wedge T(\omega)$$

is an $(\mathcal{F}_t)_{t\geqslant0}$-stopping time.

Problem 3 (Itô-calculus for martingales). (4 Points)

Let $f : \mathbb{R} \to \mathbb{R}$ be C^2 such that f' is bounded, let B be a continuous Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $(\mathcal{F}_t)_{t\geqslant0}$ be the right-continuous version of $\big(\sigma(B_u, 0 \leqslant u \leqslant t)\big)_{t\geqslant0}$ such that $\mathcal{F}_0$ contains all $\mathbb{P}$-zero sets.

Determine a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that

$$M_t = f(B_t) + \int_0^t g(B_s)ds, \quad t \geqslant 0,$$

is a continuous (global!) martingale wrt. $(\mathcal{F}_t)_{t\geqslant 0}$.

$(\mathcal{F}_t)_{t\geqslant 0}$ is also called the *usual Brownian filtration* (wrt. B).

Hint: First, choose g such that M is a continuous local martingale (up to $T = \infty$). Then, which tools do you know from Section 1.4. to show that a continuous local martingale is a global martingale?

Problem 4 (Overview: constructed integrals in Ch.1). (4 Points)

In Chapter 2, the actual "stochastic analysis" of this lecture begins. Having an overview for which classes of integrands and integrators we constructed integrals $\int Y dX$ by "pathwise" tools in Chapter 1 will help you to understand to which degree the forthcoming construction is more general and which limitations of Chapter 1 we have to overcome.

Recap Chapter 1 and make an overview: For which pairs (X, Y) can we construct integrals $\int_0^t Y dX$ so far? In each case, how is the integral defined, what do you know about its properties ([local] martingale, continuity, quadratic variation), and why are the respective conditions on X and Y necessary?