Winter term 2022/2023
Prof. Dr. Michael Röckner and Dr. Marco Rehmeier

Exercises to Stochastic Analysis

Sheet 7
Total points: 16
Submission before: Friday, 02.12.2022, 12:00 noon

([Parts of] Exercises marked with "" are additional exercises.)*

Problem 1 (Review: Construction of stochastic integrals). (8 Points)

The construction of stochastic integrals is one of the most important aspects of basic stochastic analysis and a cornerstone of this lecture. You should try to develop a thorough technical and intuitive (!) understanding for the construction and the properties it entails. Stochastic integrals are fundamental for stochastic differential equations, which are one of the main objects studied in stochastic analysis research both in pure and applies disciplines.

Review Chaper 2.3., make a summary of the steps towards the general construction of $H.M$, and try to get a feeling for the "big picture" and what the essential steps are. You do not have to copy whole proofs, but you might want to point out important steps within the main proofs. A suggestion for the structure of the summary is as follows:

(i) **The space $\mathcal{M}^2$:** Why this class of integrators? What are the advantages of the particular choice of $||\cdot||_{\mathcal{M}^2}$? Which properties does $\mathcal{M}^2$ and its elements have and how do they help within the construction?

(ii) **Elementary integrands:** Recall the definition, try to build an intuition for it. What is essential to obtain that $H.M$ is a martingale? Also, recall the crucial Ito-isometry.

(iii) **Extension via isometry:** What happens in the passage after the proof of Lemma 2.3.20, in particular: why does one need to show that P_M is a measure? How does one extend the definition from the previous part to $\bar{\mathcal{E}}^{\mathcal{M}}$ via the Ito-isometry? To this end, recall what you know about the unique extension of a uniformly continuous map on a subset of a metric space to its closure, cf. ex.2.

(iv) **Class of admissible integrands:** Once $H.M$ is defined for $H \in \bar{\mathcal{E}}^{\mathcal{M}}$, the important question is: How large is this space in general? Under which important assumptions on M is this space even larger?

(v) **Localization:** How does one extend the notion of $H.M$ to $M \in \mathcal{M}^2_{\text{loc}}$?

Note: This question is very important, since Brownian motion B does not belong to $\mathcal{M}^2$ - but this is by far the most frequently used integrator!

Problem 2 (Side questions to ex.1). (4 Points)

This exercise contains two technical auxiliary points to the above summary exercise.

(i) Concerning part (i) of ex.1, show the following: For $M \in \mathcal{M}^2$, the measure on $\mathbb{R}$ associated to $t \mapsto \langle M \rangle_t(\omega)$ is finite for a.a. $\omega \in \Omega$.

(ii) Let (X, d) be a metric space, $A \subseteq X$ and $F : A \to \mathbb{R}$. Prove: If F is uniformly continuous, then there is a unique continuous extension of F to the closure of A in X. In addition, if X is a linear space, A a linear subspace and F linear on A, then the unique extension of F to the closure of A in X is linear as well.

Problem 3 (Measurability wrt. σ-algebra of predictable rectangles). (4 Points)

Let τ_n be a partition of $[0, \infty)$, $(\mathcal{F}_t)_{t \geqslant 0}$ a filtration on Ω and set

$$\mathcal{P}_n := \sigma\big(]t, t'] \times A_t : t, t' \in \tau_n \text{ and } t' \text{ is the successor of } t \text{ in } \tau_n \text{ and } A_t \in \mathcal{F}_t\big).$$

Prove: $H : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is $\mathcal{P}_n$-measurable and bounded if and only if there are $h_{t_i} : \Omega \to \mathbb{R}$ such that h_{t_i} is bounded and $\mathcal{F}_{t_i}$measurable, and $c \in \mathbb{R}_+$ so that

$$H_t(\omega) = \sum_{t_i \in \tau_n} \left(h_{t_i}(\omega) \mathbb{1}_{]t_i, t_{i+1}]}(t) \right) + c.$$