Winter term 2021/22
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Exercises to Probability Theory I

Sheet 11
Submission before: Friday, 21.01.2022, 12:00
Digital submission in the tutorial's "Lernraum"

(*Exercises marked with "*" are additional exercises.*)

Problem* 42. (with Corollary 1.10.7 and hint[1]) (4 points)
Consider the probability measures μ_n, μ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with associated cumulative distribution functions (CDFs) F_n, F. Show: if $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to μ and F is continuous, then $(F_n)_{n\in\mathbb{N}}$ converges uniformly to F on $\mathbb{R}$.

Problem* 43. (CLP and weak law of large numbers) (1,5 + 1,5 + 1 points)
Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent, square integrable random variables with variances $\sigma_n^2 := \operatorname{var}(X_n) > 0$, which have the CLP. Let $\varepsilon > 0$ and

$$s_n := \left(\sum_{k=1}^{n} \sigma_k^2\right)^{1/2}, \quad \alpha_n := \frac{\varepsilon n}{s_n}.$$

(a) Show that

$$P\left[\left|\frac{1}{n}\sum_{k=1}^{n}(X_k - \mathbb{E}[X_k])\right| < \varepsilon\right] - \frac{1}{\sqrt{2\pi}}\int_{-\alpha_n}^{\alpha_n} \exp\left(-\frac{x^2}{2}\right) dx$$

converges to 0 for $n \to \infty$ (even uniformly in ε).

(b) Conclude that $(X_n)_{n\in\mathbb{N}}$ does not satisfy the **weak law of large numbers** when the sequence $\left(\frac{n}{s_n}\right)_{n\in\mathbb{N}}$ is bounded.

(c) Apply the result of (b) to the situation of Problem 35 in the case $\lambda \geqslant \frac{1}{2}$.

Problem* 44. (Corollary 3.1.8, Proposition 3.1.19, Remark 3.1.16/17) (4 points)
For every absolutely continuous distribution μ on $\mathbb{R}$, we define (in analogy to the discrete case of Definition 3.1.1) the entropy

$$H(\mu) := -\int_{\mathbb{R}} f(x) \log f(x) dx,$$

where f denotes the density of μ. Prove the following statement: Of all absolutely continuous distributions μ on $\mathbb{R}$ that have variance 1, the standard normal distribution $N(0, 1)$ has the greatest entropy[2].

[1] Show first that for every $\varepsilon > 0$ there are finitely many points $x_0 < x_1 < \ldots < x_k$ such that F changes by less than ε on each of the intervals $]-\infty, x_0]$, $]x_i, x_{i+1}]$, $[x_k, \infty[$.

[2] Consider first the case of an absolutely continuous distribution μ with variance 1 and expectation 0.

Problem* 45. (Definition 3.1.5, Proposition 3.2.7) (4 points)
Determine the probability distribution ν on $\mathbb{R}$ that minimises $H(\nu|N(0,1))$ under the constraint/side condition

$$\int x d\nu(x) = m.$$

Problem* 46. (A very recent proof of the strong law of large numbers) (3 points)
Read the following preprint by P. J. Fitzsimmons and reformulate the proof in your own words.

URL: `https://arxiv.org/abs/2111.05766`

Problem* 47. (Further notions of convergence of measures besides weak convergence) (2+2 points)
In this exercise we want to study how different notions of convergence of measures are related, and in which sense weak convergence is "weak". Let $(S, \mathcal{S})$ be a metric space with Borel σ-Algebra $\mathcal{S}$. Consider the following three forms of convergence of probability measures:

(i) $\mu_n \to \mu$ *weakly*, if $\int f d\mu_n \to \int f d\mu \; \forall f \in C_b(S)$;

(ii) $\mu_n \to \mu$ *strongly* (or *setwise*), if $\mu_n(A) \to \mu(A) \; \forall A \in \mathcal{S}$;

(iii) $\mu_n \to \mu$ *in total variation*, if $\|\mu_n - \mu\|_{\mathrm{TV}} := \sup_{A \in \mathcal{S}} |\mu_n(A) - \mu(A)| \to 0$.

Show that

(a) The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) hold, i.e. convergence in total variation implies strong convergence, which implies weak convergence.

(b) Investigate both examples of Example 1.10.2, i.e. $\mu_n = \delta_{1/n}$ and $\nu_n = N(0, 1/n)$, and $\mu = \nu = \delta_0$. Which of the forms of convergence hold in these examples?

We wish you happy holidays!