Winter term 2021/22
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Exercises to Probability Theory I

Sheet 12
Submission before: Friday, 21.01.2022, 12:00
Digital submission in the tutorial's "Lernraum"

(*Exercises marked with "*" are additional exercises.*)

Problem 48. (for "$\Leftarrow$", use Problem 27) (4 points)
Let X be a real-valued random variable on a probability space $(\Omega, \mathcal{A}, P)$. The **Laplace transform** $Z\colon \mathbb{R} \to (0, \infty]$ of X is defined by

$$Z(\lambda) := \mathbb{E}\left[\exp(\lambda X)\right], \quad \lambda \in \mathbb{R}.$$

Show that the following statements are equivalent:

(i) Z is finite in a neighbourhood of 0 (i.e. on an interval $[-\varepsilon, \varepsilon]$).

(ii) There exist $a < \infty$, $b > 0$ such that

$$P[|X| \geqslant t] \leqslant a \exp(-bt), \quad t \geqslant 0.$$

Problem 49. (Proposition 3.3.1 — calculate rate functions) (1 + 1,5 + 1,5 points)
Let $U(x) = x$. In the situation of Proposition 3.3.1, we know that

$$\lim_{n\to\infty} \frac{1}{n} \log P\left[\frac{1}{n}\sum_{i=1}^{n} X_i \geqslant m\right] = -H(\mu_{\lambda(m)}|\mu) = -I(m)$$

for the **rate function** $I\colon \mathbb{R} \to [0, \infty]$. From Proposition 3.2.7 (ii) we know furthermore that for $m \in m(\mathring{\Lambda})$, we have the representation

$$I(m) = H(\mu_{\lambda(m)}|\mu) = \max_{\tilde{\lambda}\in\mathring{\Lambda}} \left(\tilde{\lambda} m(\lambda(m)) - \log Z(\tilde{\lambda})\right) = \max_{\tilde{\lambda}\in\mathring{\Lambda}} \left(\tilde{\lambda} m - \log Z(\tilde{\lambda})\right).$$

More generally, it holds (so that we can also have $I(m) = \infty$) that I is the **Legendre transform** of $\log Z(\lambda)$:

$$I(m) = \sup_{\lambda\in\mathbb{R}} \left(\lambda m - \log Z(\lambda)\right).$$

Use this formula to calculate the rate function for the following distributions:

(a) Dirac distribution ε_a, $a \in \mathbb{R}$;

(b) $\mathrm{Exp}(\alpha)$, the exponential distribution with parameter $\alpha > 0$.

(c) $N(m_0, \sigma^2)$.

Moreover, check that in each of these cases the following properties are satisfied:

- $I(m) = 0 \Leftrightarrow m = \mathbb{E}X_1$;
- On the set $\{I < \infty\}^\circ$, I is infinitely differentiable;
- If $\{I < \infty\}^\circ \neq \emptyset$, then $I''(\mathbb{E}X_1) = \frac{1}{\sigma^2}$.

(d)* Investigate the above questions also for the Poisson distribution with parameter α.

Problem 50. (1+3 points)
Let S be a separable metric space with metric d. Consider random variables $X_1, X_2, \ldots, X_n$ on a probability space $(\Omega, \mathcal{A}, P)$ with values in S.

(a) Show that

$$d_n((x_1, \ldots, x_n), (y_1, \ldots, y_n)) := \sup_{1 \leqslant i \leqslant n} d(x_i, y_i)$$

is a metric on S^n.

(b) Show that the map

$$\begin{aligned} \varphi \colon \Omega &\to S^n \\ \omega &\mapsto \varphi(\omega) := (X_1(\omega), \ldots, X_n(\omega)) \end{aligned}$$

is measurable with respect to the Borel σ-Algebra $\mathcal{B}(S^n)$ on S^n that is induced by the metric d_n.

Problem 51. (with Problem 50, Remark 3.5.2) (3+1 points)
Let S be a separable metric space with metric d. Consider random variables $X_1, X_2, \ldots, X_n$ on a probability space $(\Omega, \mathcal{A}, P)$ with values in S.

(c) Show that the map

$$\begin{aligned} \chi_n \colon S^n &\to \mathcal{M}_1(S) \\ (x_1, \ldots, x_n) &\mapsto \chi_n(x_1, \ldots, x_n) := \frac{1}{n} \sum_{i=1}^n \varepsilon_{x_i} \end{aligned}$$

is continuous.

(d) Conclude using parts (b) and (c) that

$$\omega \mapsto \xi_n(\omega) := \frac{1}{n} \sum_{i=1}^n \varepsilon_{X_i(\omega)}$$

is measurable with respect to the Borel σ-Algebra $\mathcal{B}(S^n)$ on S^n that is induced by the metric d_n.