Winter term 2021/22
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Exercises to Probability Theory I

Sheet 13
Submission before: Friday, 28.01.2022, 12:00
Digital submission in the tutorial's "Lernraum"

(*Exercises marked with "*" are additional exercises.*)

Problem 52. (Supplement on weak convergence) (2 + 2 points)
Let S be a separable, metric space with metric d and Borel σ-algebra $\mathcal{B}(S)$.

(a) Let $S_0 \subset S$ be a countable, dense subset and set[1]

$$\mathcal{U}_0 := \{B_q(s_0) \mid q \in \mathbb{Q}_+, s_0 \in S_0\}$$
$$\mathcal{U} := \left\{ \bigcup_{n=1}^{m} V_n \mid m \in \mathbb{N}, V_n \in \mathcal{U}_0, 1 \leqslant m \leqslant n \right\}.$$

Show: for every open subset $U \subset S$, there is an isotone sequence $(U_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ with $U = \bigcup_{n=1}^{\infty} U_n$.

(b) Show using (a) that probability measures μ_n on S converge **weakly** to μ if and only if

$$\liminf_{n\to\infty} \mu_n(U) \geqslant \mu(U) \quad \forall U \in \mathcal{U}. \tag{*}$$

Problem 53. (4 points)
Let S be a separable, metric space with metric d and Borel σ-algebra $\mathcal{B}(S)$. Show using Problem 52 (b):

There are functions $g_1, g_2, \ldots \in C_b(S)$, such that probability measures μ_n converge weakly to μ if and only if

$$\lim_{n\to\infty} \int g_k \, d\mu_n = \int g_k \, d\mu \quad \forall k \in \mathbb{N}. \tag{*}$$

Hint: Construct for every $U \in \mathcal{U}$ a sequence of nonnegative continuous functions $(f_j)_{j\in\mathbb{N}}$ that converge monotonely to 1_U.

Problem 54. (Alternative proof of Proposition 2.3.8) (4 points)
Let S be a separable metric space with metric d. Consider independent and identically distributed (i.i.d.) random variables $X_1, X_2, \ldots$ on a probability space $(\Omega, \mathcal{A}, P)$ with values in S and distribution μ. For $\omega \in \Omega$ and $n \in \mathbb{N}$, we define

$$\xi_n(\omega) := \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i(\omega)} \in \mathcal{M}_1(S).$$

Prove Proposition 2.3.8 using Problem 53.

[1] Here $B_q(s_0)$ denotes a ball around s_0 with radius q.

Problem 55. (Example 4.2.11 (ii)) (4 points)

(a) Consider the following random experiment: given n urns, of which every urn contains s black and w white balls. Now we draw from the first urn one random ball and put it in the second urn. Then, in turn, we draw a ball at random from the second urn and put it in the third urn, etc., until finally a ball of the penultimate urn is put into the last urn. Now we draw a ball from the last urn. What is the probability for this ball to be white?

(b) An **equilibrium distribution** for a kernel K is a probability measure μ with the property

$$\mu K = \mu,$$

where

$$\mu K(A_2) := \int K(x_1, A_2) d\mu(x_1).$$

Show that $N\left(0, \frac{\sigma^2}{1-\alpha^2}\right)$ is an equilibrium distribution for the kernel K from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where this kernel is defined by

$$K(x, \cdot) := N(\alpha x, \sigma^2), \quad |\alpha| < 1.$$