Winter term 2021/22
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Exercises to Probability Theory I

Sheet 2
Abgabe: Friday, 29.10.2021, 12:00
Digital submission in the tutorial's "Lernraum"

(*Exercises marked with "*" are additional exercises.*)

Problem 6. (Sylvester's formula / Inclusion-exclusion principle)
Let $(\Omega, \mathcal{A}, P)$ be a probability space, I a finite index set and $A_i \in \mathcal{A}$, $i \in I$ a family of sets from $\mathcal{A}$.

(a) Show that

$$P\left(\bigcup_{i\in I} A_i\right) = \sum_{J\subset I, J\neq\emptyset} (-1)^{|J|-1} \cdot P\left(\bigcap_{j\in J} A_j\right).$$

(4 points)

(b) Show that in the special case $I = \{1, 2, \ldots, n\}$ with $n \in \mathbb{N}$ the following formula holds:

$$P\left(\bigcup_{i\in I} A_i\right) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1\leqslant i_1<\ldots<i_k\leqslant n} P\left(A_{i_1} \cap \ldots \cap A_{i_k}\right).$$

(2 points)

Problem 7. (Discrete probability measures, cf. Satz 1.2.1 (ii))
Let $\Omega \neq \emptyset$ be countable and $\mathcal{A} = \mathcal{P}(\Omega)$. Show:
Every probability measure P on $(\Omega, \mathcal{A})$ is of the form $p(\omega) := P(\{\omega\})$ for all $\omega \in \Omega$; with other words: there is a uniquely determined function $p\colon \Omega \to [0, 1]$ with

$$P(A) = \sum_{\omega\in A} p(\omega) = \sum_{\omega\in A} P(\{\omega\}) \quad \forall\ A \subset \Omega.$$

(4 points)

Problem 8. (Preimage σ-Algebra) Let $(\Omega, \mathcal{A})$, $(\tilde{\Omega}, \tilde{\mathcal{A}})$ be measurable spaces with $\Omega, \tilde{\Omega} \neq \emptyset$. Show that

(a) for a $\tilde{\mathcal{A}}/\mathcal{A}$-measurable map $T\colon \tilde{\Omega} \to \Omega$, the system of sets

$$\sigma(T) := \{T^{-1}(A) \mid A \in \mathcal{A}\}$$

is a σ-algebra over $\tilde{\Omega}$ and (4 points)

(b) This σ-algebra $\sigma(T)$ is the smallest σ-algebra with respect to which T is a $\tilde{\mathcal{A}}/\mathcal{A}$-measurable map. (2 points)