Exercises to Probability Theory I

Sheet 7
Submission before: Friday, 03.12.2021, 12:00
Digital submission in the tutorial's "Lernraum"

(*Exercises marked with "*" are additional exercises.*)

Problem 25. (Lemma 2.1.8) (4 points)
Let $(X_n)_{n\in\mathbb{N}}$ and X be real-valued random variables on $(\Omega, \mathcal{A}, P)$. Show that for **independent** random variables $(X_n)_{n\in\mathbb{N}}$ the following holds:

$$\sum_{n=1}^{\infty} P[|X_n - X| > \varepsilon] < \infty \ \forall \varepsilon > 0 \quad \Leftrightarrow \quad X_n \overset{n\to\infty}{\longrightarrow} X \ P\text{-a.s.}$$

Problem 26. (Definition 2.2.1) (4 points)
Check whether the following random variables X_1 and X_2 are Independent (with proofs):

a) Let $(\Omega, \mathcal{A}) = ([0,1]^2, \mathcal{B}([0,1]^2))$, and P be the Lebesgue measure on $[0,1]^2$. For $(x_1, x_2) \in \Omega = [0,1]^2$, set

$$X_1(x_1, x_2) := x_1,$$
$$X_2(x_1, x_2) := x_2.$$

b) Let $(\Omega, \mathcal{A}) = ([0,1], \mathcal{B}([0,1]))$ and P be the Lebesgue measure on $[0,1]$. For $x \in \Omega = [0,1]$, set [1]

$$X_1(x) := \cos(2\pi x),$$
$$X_2(x) := \sin(2\pi x).$$

Problem 27. (4 points)
Let X be a random variable and let $\varepsilon > 0$ be arbitrary. Show that X is integrable if and only if

$$\sum_{n=1}^{\infty} P[|X| > n\varepsilon] < \infty.$$

Problem 28. (using Problem 27, Borel–Cantelli) (4 points)
Let $(X_n)_{n\in\mathbb{N}}$ be indepent, identically distributed random variables on $(\Omega, \mathcal{A}, P)$. Show that X_1 is P-integrable if and only if

$$\lim_{n\to\infty} \frac{|X_n|}{n} = 0, \quad P\text{-a.s.}$$

[1] X_1, X_2 describe the coordinates of a random point that is equidistributed on the unit circle $\mathbb{R}^2$.