Summer term 2022
Prof. Dr. Michael Röckner and Dr. Andre Schenke

Exercises to Probability Theory II

Sheet 10
Submission before: Friday, 17.06.2022, 12:00

(*Exercises marked with "*" are additional exercises.*)

Problem 24. (Some applications of the martingale convergence theorem)

(i) Let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be a filtration. Let $A \in \sigma\left(\bigcup_{n\in\mathbb{N}} \mathcal{A}_n\right)$. Show the **0–1 law of Lévy**:

$$\lim_{n\to\infty} P[A \mid \mathcal{A}_n] = 1_A, \quad P\text{-a.s.}$$

(2 points)

(ii) Derive from (i) the **0–1 law of Kolmogorov** (which you know already): let $\mathcal{B}_1, \mathcal{B}_2, \ldots$ be a sequence of independent σ-algebras and $\mathcal{B}_\infty := \bigcap_{n\in\mathbb{N}} \sigma\left(\bigcup_{k\geqslant n} \mathcal{B}_k\right)$ be the tail σ-algreba. Then

$$A \in \mathcal{B}_\infty \quad \Rightarrow \quad P[A] \in \{0,1\}.$$

Hint: Use $\mathcal{A}_n := \sigma\left(\bigcup_{k=1}^n \mathcal{B}_k\right)$. (2 points)

Problem 25. (A generalisation of the Borel–Cantelli lemma)
Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be a filtration and $A_n \in \mathcal{A}_n, n \in \mathbb{N}$. Show the following generalisation of the Borel–Cantelli lemma:

$$\bigcap_{n\in\mathbb{N}} \bigcup_{m\geqslant n} A_m = \left\{ \sum_{n=1}^{\infty} P[A_{n+1} \mid \mathcal{A}_n] = \infty \right\}, \quad P\text{-a.s.}$$

Here, two sets A, B are equal P-a.s. if $P(A\Delta B) = 0$, where Δ denotes the symmetric difference, i.e. $A\Delta B = (A\backslash B) \dot\cup (B\backslash A)$ (cf. Problem 5, where a similar definition was used). (4 points)

Hint: Apply Proposition 8.6.1 to the following $(\mathcal{A}_n)_{n\in\mathbb{N}}$-martingale (?!):

$$X_n := \sum_{k=0}^{n-1} \left(1_{A_{k+1}} - P[A_{k+1} \mid \mathcal{A}_k]\right).$$

Problem 26. (Missing step in the proof of the backwards martingale convergence theorem 8.5.5)
Let $I = -\mathbb{N}_0$ with the usual ordering, i.e. $\ldots < -2 < -1 < 0$. Let $(X_n)_{n\in-\mathbb{N}_0}$ be an $(\mathcal{A}_n)_{n\in-\mathbb{N}_0}$-submartingale. We want to show part (i) of the Proposition 8.5.5. (i), i.e. that the limit

$$X_{-\infty} := \lim_{n\to-\infty} X_n \in \mathbb{R} \cup \{-\infty\}$$

exists P-a.s. You can prove this in the following steps:

(a) Show that the limit

$$X_{-\infty} := \lim_{n\to-\infty} X_n \in \mathbb{R} \cup \{-\infty, +\infty\}$$

exists P-a.s. (2 points)

(b) Show that

$$X_{-\infty} < \infty, \quad P\text{-a.s.}$$

(2 points)

Hint: Obviously, Doob's Upcrossing Inequality also holds for discrete submartingales. Consider the submartingales $(Y_i)_{1 \leqslant i \leqslant n}$ defined by $(Y_0, \ldots, Y_n) := (X_{-n}, \ldots, X_0)$.