Summer term 2022
Prof. Dr. Michael Röckner and Dr. Andre Schenke

Exercises to Probability Theory II

Sheet 11
Submission before: Friday, 24.06.2022, 12:00

(*Exercises marked with "*" are additional exercises.*)

Problem 27. Show that the Fourier transform / characteristic function of the 1-dimensional normal distribution is

$$\widehat{N(0,\sigma^2)}(z) = e^{-\frac{1}{2}\sigma^2 z^2},$$

as a generalisation of Proposition 2.5.8 (which also proves Proposition 2.5.8). (4 points)

Hint: Set $F(z) := \widehat{N(0,\sigma^2)}(z)$. First show by splitting the integral that

$$F(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^\infty 2\cos(zx) e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}\, dx.$$

Show by differentiation with respect to z that F satisfies the following differential equation:

$$\begin{cases} F'(z) &= -\sigma^2 z F(z), \\ F(0) &= 1, \end{cases}$$

and show that the solution to this differential equation is $F(z) = e^{-\frac{1}{2}\sigma^2 z^2}$.

Problem 28. (Characterisation of Gaussian random vectors)
Let $\vec{X}$ be an $\mathbb{R}^n$-valued random variable on a probability space $(\Omega, \mathcal{A}, P)$. Show that the following two statements are equivalent:

(i) $\vec{X}$ is Gauss distributed with parameters $\vec{m}$ and A.

(ii) There are a real $n \times n$-matrix W, $\vec{m}' \in \mathbb{R}^n$ and independent, centered normally distributed random variables $Y_j \colon \Omega \to \mathbb{R}$, $1 \leqslant j \leqslant n$ with

$$\vec{X} = W\vec{Y} + \vec{m}'.$$

Show further that in this case $\vec{m}' = \vec{m}$, the mean of $\vec{X}$ and that $A = WDW^t$, where D is a diagonal matrix with entries $\mathrm{var}(Y_1), \dots, \mathrm{var}(Y_n)$. (4 points)

Problem 29. (Another characterisation of Gaussian random vectors)
Let $\vec{X}$ be an $\mathbb{R}^n$-valued random variable on a probability space $(\Omega, \mathcal{A}, P)$ and A be an invertible, real, positive definite, symmetric $n \times n$-matrix, and $\vec{m} \in \mathbb{R}^n$.

Show that the following are equivalent:

(i) $\vec{X}$ is Gaussian with covariance matrix A and mean $\vec{m}$.

(ii) For the law of $\vec{X}$,

$$\vec{X}(P)(dx) = ((2\pi)^n \det(A))^{-1/2} \cdot \exp\left(-\frac{1}{2}(\vec{x} - \vec{m}) \cdot A^{-1}(\vec{x} - \vec{m})\right) \lambda^n(dx),$$

where λ^n denotes the Lebesgue measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. (4 points)